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MÖBIUS INVARIANCE AND CLASSICAL SOLUTIONS OF SU_2 GAUGE THEORY

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ABSTRACT

We use quaternionic Möbius transformations to construct classical solutions of pure $SU(2)$ gauge theory which correspond to two generalized merons and to two "excited" merons. We discuss how such transformations can yield distinct gauge invariant quantities and point out the limitations for generating further solutions by these techniques.

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1. INTRODUCTION

The use of conformal invariance to generate new solutions of certain classical field theories is a well-established tool in theoretical physics. In this paper we address ourselves to the problem of analyzing the physical distinction between certain conformally related solutions of a free SU_2 Yang-Mills theory¹⁾. It has been shown that a conformal transformation can reposition the maximum of a single instanton and rescale its strength²⁾. The solution consequently remains in a five parameter class. However, for the meron solution³⁾ a similar transformation yields the so-called two-meron (or meron-antimeron) solution which is generally regarded as a distinct solution. In order to be precise about the nature of different solutions we shall adopt the gauge invariant action density as a criterion for testing equivalence. For the meron solutions therefore we shall be concerned with the singularity structure of this density. We shall show that in our formulation, the application of a conformal transformation in four-dimensional Euclidean space induced by a quaternionic Möbius transformation introduces new singularities into certain SU_2 gauge potentials. The action density, however, contains at most two singular points, including possibly the point at infinity. We suggest that the introduction of an action density singularity into the finite region of R^4 from the point at infinity produces a new, physically distinct solution⁴⁾.

Singularities introduced by the Möbius transformation should be distinguished from those introduced by singular gauge transformations. We shall adopt the viewpoint in this paper that such gauge induced singular functions yield a vanishing field everywhere except for isolated points and consequently are classically indistinguishable from vacuum configurations. They may be used to display certain solutions in different gauges in which the winding number of a singularity can be calculated.

After defining our quaternionic Möbius transformations we shall first analyze an $O(4)$ symmetric class of solutions that encompass both the meron and the instanton⁵⁾. We clearly exhibit by a choice of parameters how the two meron solution emerges as distinct from the transformed instanton. Furthermore, we shall draw attention to the generalized two-meron solution, an elliptic function with two singularities. Although many of the results of this section are implicit in a number of previous references¹⁾⁻³⁾ we believe that the formulation in terms of a quaternionic co-ordinate facilitates the discussion and simplifies the generation of transformed solutions. In the following section we apply our methods to the "excited" meron found in Ref. 6). The result is a new solution that we term an "excited" two-meron solution. Finally we comment on the prospects for multi-meron configurations.

2. QUATERNIONIC MÖBIUS INVARIANCE

We associate with each point of R^4 a quaternionic co-ordinate q by the rule

$$q = x_0 - i \sigma \cdot x \quad (1)$$

so that the metric of Euclidean space may be written

$$ds^2 = dq d\bar{q} \equiv |dq|^2 \quad (2)$$

where

$$\bar{q} = x_0 + i \sigma \cdot x$$

An $O(4)$ symmetric vacuum will be described by the quaternionic valued function

$$\Omega_\mu(q) = \bar{Q} \partial_\mu Q \equiv -i \Omega_{j\mu}(q) \sigma_j \quad (3)$$

$j = 1, 2, 3$

where we write $Q = q/|q|$. In terms of these variables an $O(4)$ symmetric class of solutions to the SU_2 Yang-Mills equation is given by

$$A_\mu(q) = f(|q|) \Omega_\mu(q) \quad (4)$$

where

$$f(|q|) = \frac{1}{2} + b \operatorname{sn}\left\{a \ln(|q|^2/|q_0|^2) \middle| \frac{a}{b}\right\} \quad (5)$$

$b^2 + a^2 = \frac{1}{2}$ and $\operatorname{sn}(z|m)$ is the Jacobi elliptic function of the order of m . We shall concentrate here on this solution which encompasses in a compact way several known solutions. It has been explicitly written in terms of the quaternionic co-ordinate q on which the transformations below will operate. When $b = 0$ we recover the one-meron solution and when $a = \frac{1}{2}$, $b = \pm \frac{1}{2}$ we have the instanton and anti-instanton solutions, respectively. A gauge transformation of (4) with the element Q produces

$$A_\mu^Q = (f(|q|) - 1) \partial_\mu Q \bar{Q} \quad (6)$$

When $b = 0$ this has been termed the antimeron solution. We now consider how to construct "new" solutions from the basic solution (4). We define a quaternionic Möbius transformation by

$$q' = (aq + b)(cq + d)^{-1}$$

or

$$q' = q_1 q_2^{-1} \quad (7)$$

where a, b, c, d are constant quaternions. This transformation induces a fifteen parameter conformal transformation in Euclidean space. Explicitly one finds

$$ds^2 = \left\{ \frac{|a\bar{c}'d - b|^2 |c|^2}{|cq + d|^2} \right\} |d\bar{q}|^2 \quad (8)$$

If $A_\mu(q)$ describes a solution in this metric then the invariance of the equations of motion under conformal transformations ensures that $A_\mu(q')$ will yield a solution in the conformally related space with metric

$$ds^2 = |dq'|^2 \quad (9)$$

Since

$$\begin{aligned} \Omega_\mu(q') &= \frac{\bar{q}'}{|q'|} \partial_\mu \left(\frac{q'}{|q'|} \right) = Q_2 \bar{Q}_1 \partial_\mu (q_1 \bar{Q}_2) \\ &= Q_2 (\Omega_\mu(q_1) - \Omega_\mu(q_2)) \bar{Q}_2 \end{aligned} \quad (10)$$

where $Q_n = q_n/|q_n|$, the solution (4) generates by conformal transformation the solution

$$A_\mu(q') = f(|q'|) Q_2 (\Omega_\mu(q_1) - \Omega_\mu(q_2)) \bar{Q}_2 \quad (11)$$

For $b = 0$ (i.e., $f = \frac{1}{2}$) this has been termed the meron-antimeron solution. With the element Q_2 this may be gauged to the form

$$A_{\mu}^{Q_2}(q') = f(|q'|) \Omega_{\mu}(q_1) + (1-f|q'|) \Omega_{\mu}(q_2) \quad (12)$$

For $b = 0$ we have the so-called two-meron solution. The term antimeron clearly does not have the same topological significance as the term anti-instanton. The meron is gauge equivalent to the antimeron. Furthermore, if the parameters are chosen in the two meron solution so that the singularities coincide, then a vacuum results; the meron-meron solution is gauge equivalent to the meron-antimeron solution. However, for the instanton configuration ($a = \frac{1}{2}$, $b = \frac{1}{2}$) we readily check that for some parameter γ

$$A_{\mu}(q') = \frac{|q_1|^2}{|q_1|^2 + \gamma|q_2|^2} \Omega_{\mu}(q_1) + \frac{\gamma|q_2|^2}{|q_1|^2 + \gamma|q_2|^2} \Omega_{\mu}(q_2) \quad (13)$$

so that we have simply shifted the position and strength of the original solution. If we examine the conformal transformation of the action density $\mathcal{S}$ for the theory then we find that in general for the solution (4)

$$\mathcal{S} = 12 \left\{ 2f^2(1-f)^2 + a^2b^2 - \frac{1}{16} \right\} \frac{|c|^4 |ac^{-1}d - b|^4}{|q_1|^4 |q_2|^4} d^4x \quad (14)$$

In the meron-antimeron case this gives

$$\mathcal{S} = \frac{3}{4} \frac{|c|^2 |ac^{-1}d - b|^4}{|aq + b|^4 |cq + d|^4} d^4x \quad (15)$$

whereas in the instanton case we obtain

$$\mathcal{S} = \frac{24 \gamma^2 d^4x}{(|q_1|^2 + \gamma|q_2|^2)^4} \quad (16)$$

It is clear that in the latter case the singularities introduced by the conformal transformations have been cancelled by the form of the transformed instanton

solution. However, no such cancellation occurs for the general case and the action density is singular at the points

$$\begin{aligned} q &= a^{-1}b \\ q &= c^{-1}d \end{aligned} \tag{17}$$

If we perform the successive quaternionic Möbius transformations

$$\begin{aligned} q' &= (aq + b)(cq + d)^{-1} \\ q'' &= (eq' + f)(gq' + h)^{-1} \end{aligned} \tag{18}$$

then even though the constant elements do not commute in general, we have a group structure. The composite transformation is

$$q'' = (pq + t)(rq + s)^{-1}$$

where

$$\begin{pmatrix} p & t \\ r & s \end{pmatrix} = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{19}$$

Consequently successive quaternionic Möbius transformations (even if mixed with singular gauge transformations) will not alter the singularity structure of these densities. Only the parameters of the resulting transformations will change.

3. EXCITED MERON SOLUTIONS

We now turn our attention to the generation of a new solution that does not follow from the form given in Eq. (5). In Ref. 6) it was shown that the potential

$$\begin{aligned} A_{\mu 1} &= 3 \Omega_{1\mu}(q) \\ A_{\mu 2} &= i \Omega_{2\mu}(q) \\ A_{\mu 3} &= i \Omega_{3\mu}(q) \end{aligned} \tag{20}$$

was a complex solution with real action. In a Euclidean space such solutions are singular with action density

$$\mathcal{S} = \frac{32}{|q|^4} d^4 x \quad (21)$$

If we now perform the Möbius transformation on this solution we generate a new solution. If we choose for simplicity the transformation

$$q' = (q - D)(q + D)^{-1} \quad (22)$$

where D is a real number, then

$$\begin{aligned} A_{\mu_1}(q') &= 3(W_{\mu_1} R_1 + W_{\mu_2} S_1 + W_{\mu_3} T_1) \\ A_{\mu_2}(q') &= i(W_{\mu_1} R_2 + W_{\mu_2} S_2 + W_{\mu_3} T_2) \\ A_{\mu_3}(q') &= i(W_{\mu_1} R_3 + W_{\mu_2} S_3 + W_{\mu_3} T_3) \end{aligned} \quad (23)$$

where

$$\begin{pmatrix} \underline{R} \\ \underline{S} \\ \underline{T} \end{pmatrix} = \frac{1}{r_1^2} \begin{pmatrix} t^2 + x^2 - y^2 - z^2 + 2Dt + D^2, 2(tz + xy) + 2Dz, 2(xz - ty) - 2Dy \\ 2(xy - zt) - 2Dz, t^2 - x^2 + y^2 - z^2 + 2Dt + D^2, 2(zy + xt) - 2Dx \\ 2(ty + xz) + 2Dy, 2(yz - xt) - 2Dx, t^2 - x^2 - y^2 + z^2 + 2Dt + D^2 \end{pmatrix}$$

and

$$\begin{pmatrix} W_{1\mu} \\ W_{2\mu} \\ W_{3\mu} \end{pmatrix} = \begin{pmatrix} xa_{12}, ta_{12} - Db_{12}, za_{12}, ya_{12} \\ ya_{12}, za_{12}, ta_{12} - Db_{12}, xa_{12} \\ za_{12}, ya_{12}, xa_{12}, ta_{12} - Db_{12} \end{pmatrix}$$

$$a_{12} = \frac{1}{r_2^2} - \frac{1}{r_1^2}$$

$$b_{12} = \frac{1}{r_2^2} + \frac{1}{r_1^2}, \quad r_n = |q_n|^2$$

The action density explicitly displays the singularities introduced by the conformal transformation

$$\mathcal{L} = \frac{128 D^4 d^4 x}{|q + D|^4 |q - D|^4} \quad (24)$$

Since the action densities (21) and (24) are larger than those corresponding to the meron and two meron solutions, respectively, we have chosen to call the solutions (20) and (23) "excited" meron and "excited" two meron solutions.

For those who prefer the $\eta_{\mu\nu}^j$ notation¹⁾ we quote the form of the potential in this language. The excited meron and two meron solutions are given as

$$A_{\mu\nu} = c_n \gamma_{\mu\nu} \frac{x^\nu}{x^2} \quad (25)$$

and

$$A_{\mu\nu} = c_n \left\{ (y_0^2 - y^2) \gamma_{\mu\nu} + 2 \varepsilon_{n\kappa} y_\kappa \gamma_{\mu\nu} y_0 + 2 y_n \gamma_{\kappa\mu\nu} y_\kappa \right\} \left(\frac{z^\nu}{z^2} - \frac{y^\nu}{y^2} \right) \quad (26)$$

respectively, where

$$\begin{aligned} z^\nu &= (x, y, z, t - D) \\ y^\nu &= (x, y, z, t + D) \\ c_n &= (3, i, i) \end{aligned}$$

4. CONCLUSION

We have shown by explicit calculation that conformal transformations of certain singular solutions may be said to produce new solutions. However, for the solutions considered here we are limited in this way to generating configurations

with at most two singularities in the action density. Although we have demonstrated this limitation with a specific class of solutions the result will follow for any $O(4)$ symmetric solution.

It would appear that in order to generate solutions with more than two singularities in the action density (multi-merons) one must go beyond the Möbius transformations discussed here. For example, a transformation that is conformal in a two-dimensional subspace of R^4 will induce (in general a non-conformal) transformation in R^4 . Such a mechanism appears as the basis of the configurations discussed in Ref. 7). However, the explicit construction of such multi-singular solutions remains to be given.

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