SOMETHAT VIRTUAL PIONS

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ABSTRACT

The fleeting existence of pions inside of nuclei is responsible for the modification of nucleon properties in decay, capture, photo-processes, magnetic moments and the like. The physical conditions of these virtual pions differ strongly from those of real pion, which can affect both their interactions and their behavior inside of nuclei. These lectures discuss physical situations in which the interaction of the pions is expected to be very close to that of real pions, but in which the binding effects will manifest themselves fully. The purpose is to spotlight the essential physics of virtual pions. In the first lecture the effect of binding is illustrated by the line-shape of an absorbing (pionic) atom. The standard Breit-Wigner shape leads to logical inconsistencies of probability conservation far from the resonance energy. It is shown that a natural damping mechanism is introduced by the fact that the centre of the nucleus can only feel pions from a distance of the size of Yukawa field associated with the not quite real pion. In this situation, it is also possible to give a mathematically exact solution to the line-shape. In the second lecture a much more central problem is discussed, namely capture at large energy transfers. In this situation capture kinematically resembles capture. It is then useful to shift the normal viewpoint and consider the muon as surrounded by a cloud of pions, which will interact with the nucleus, when brought into contact with it: the muon becomes the source of the pion field and we see the strong interaction of the muon. In this way the muon is capable of mimicking all pionic effects. This only describes part of capture. The remaining part can be studied via the leaking of the axial current by the pionic field (PCAC hypothesis). It is shown that when a natural locality hypothesis is introduced that the axial current can be determined completely in terms of the pions, in such a way that a very strong parallelism occurs between the e.m. induced polarization vectors in a dielectric, and the pionic polarization vector in the nuclear medium. Formulated in this fashion capture can be uniquely determined as a pionic phenomenon at all momentum transfers: capture becomes exactly equivalent to capture for virtual pions.
INTRODUCTION

One major reason for the strong interest our group has taken in the interaction of low-energy pions with nuclei for over a decade has been our firm belief that this is the key to a clear physical picture of the nature of the bound nuclear pion field.

We have now reached the point at which this hope can be substantiated. The present development is based crucially on our description of the $\pi$-nuclear interactions as due to local $s$- and $p$-wave pion interactions in the nuclear medium (Refs. 1 and 2). This description permits a close analogy to polarization phenomena in a dielectric; the pionic field induces dipoles in the medium just as in electric field induces dipoles in a dielectric. In particular, one finds naturally that the pion in this medium sees a local effective field, which for $p$-wave pions gives rise to the pionic Lorentz-Lorenz effect. This fruitful analogy with dielectrics can be developed further for weak interaction phenomena. M. Ericson shows in her lectures (Ref. 3) how the polarization effect of the pions leads to a modified axial coupling constant in $\beta$-decay: the apparent strength of the axial dipole is renormalized by induced axial dipoles just as an electric dipole medium is renormalized in its apparent strength by the induced electric dipole (Ref. 4). In that particular case it has been possible to retrieve her results using standard graph techniques, but the physical picture is then lost (Ref. 5).

Nearly all evidence for mesonic structure in nuclei and nuclear exchange currents come from properties of nuclear ground states or from transitions between nuclear states of very low excitation. This corresponds to pions so strongly bound that it compensates for their entire rest mass energy: $\omega_\pi = 0$. The way for extracting mesonic effects is always the following: assume the nucleus to consist only of nucleons. If the predictions on this basis can be made accurate enough and there still remains discrepancies, these are ascribed to meson effects. The problem of this method is the smallness of the exchange effects: even in selected cases it is on the level of 10% or below. This raises a severe methodological problem: since the effects are always deduced as a difference between two large numbers, they cannot be determined more accurately than our basic knowledge of the theoretical one. In the best cases, like the deuteron, this poses no severe problem, and we are confident that exchange current effects are observed. In most other cases we must rely very strongly on microscopic models of nuclear wave functions; there is nearly always doubts on the accuracy on the level of pionic effects. Quite apart from this difficulty in reliably extracting mesonic effects from data, nearly all data available, whether from weak or electromagnetic interactions, occur for $\omega_\pi = 0$. This means that they are all obtained under very similar conditions so that there is little variation in their interaction properties. In particular, there is no good experimental check that the extrapolation of pionic interactions at $\omega_\pi \gtrsim m_\pi$ to the region $\omega_\pi = 0$ are reliable, although there are good theoretical guide lines.

It seems clear to me that this situation cries out for a study of nuclear exchange and pionic effects under conditions radically different from those for strongly bound pions. Ideally we would like to see cases in which the meson effects are very dominant. This can be achieved for weakly bound pions, i.e. virtual pions with $\omega_\pi < m_\pi$, but only moderately smaller. This has two major advantages. First, the pionic effects are now very dominant and no small perturbation: there is absolutely no risk of confusing the pionic effect with that of an imperfectly known wave function. Secondly, the extrapolation of
the well understood physical interaction into this region is a relatively small one. This permits one to concentrate entirely on the question of how the exchange is generated and how non-trivial binding effects develop as a function of nuclear size.

My lectures will therefore deal with the properties of weakly bound pions. I will draw liberally from extensive unpublished material of our group, which presently is being prepared for publication. When we started these studies, we rapidly found ourselves on unfamiliar ground with some initial surprises. In this new situation, we found it most instructive to consider also weakly bound pions in atomic physics so as to clarify our thinking. I will present some of this material as a background.

My lectures will be divided into two parts. First I will discuss the shape of the 1s line of a pionic atom (Ref. 6). The reason for this is that a broad 1s line provides the opportunity of explicitly studying how the absorption of a pion bound in the Coulomb field varies as a function of binding, and why it varies. The second topic we refer jokingly to as 'the strong interactions of the muon' (Ref. 7). We deliberately study the $\mu$-capture at very large energy transfer for which the kinematics of $\mu$-capture and $\pi$-capture become extremely similar. This first brings up the question of how the pion field is generated by the muon as a source and how the pion then is absorbed in various situations. This part is an interesting illustration of how a weak pseudoscalar interaction behaves in an extended system; the physics depends quite importantly on the nuclear size. As a second non-trivial part of the discussion one finds that the muon induces exchange currents in the nucleus. These can be found from the PCAC hypothesis, but only if one introduces an additional new locality assumption. Such an assumption will be found to be quite natural. After this we find that it is possible to reduce the entire problem of pionic effects to an ordinary Schrödinger type problem. A very interesting by-product of these considerations is that we find quite naturally how the axial current acquires induced contributions from the local s- and p-wave pion interaction in the medium.

The problem of the line shape of the pionic atom has been studied in collaboration with L. Hambro, while the $\mu$-capture investigation has been developed together with J. Bernabeu and C. Jarlskog.

LINE SHAPE OF A HADRONIC ATOM

A pionic atom consists of a negative $\pi$ bound in the nuclear Coulomb field. Contrary to the electron in an ordinary atom, the $\pi^-$ feels not only the electric Coulomb force but also the strong interaction with the nucleus at the centre of the atom. A fundamental difference between a pionic atom and an ordinary atom is the absorption of the pion; even the 1s state is unstable and broadened. The absorption can be quite strong. For nuclei of $A \approx 20$ the width is (5-10)% of the 2p-1s transition energy. In this region the characteristic 2p-1s energy is $\approx 300$ keV, which is a very small value compared to the 140 MeV liberated by pion description (see Fig. 1).

\[ \begin{align*}
&\text{2p} \\
&\begin{array}{c}
&\text{300 keV} \\
&\text{1s} \\
&\Gamma
\end{array} \\
&\text{140 MeV} \\
&\text{absorption}
\end{align*} \]

Fig. 1
An inconsistency in the standard line shape
A routine way to describe the shape of the broadened line is to approximate it by a Breit-Wigner shape characteristic for El radiation:

\[ I(\omega) \propto \frac{\omega^3}{(\omega - \omega_0)^2 + (\Gamma/2)^2} \]  

(1)

This raises the following problem. It is quite possible to have very large X-ray energies with \( \omega >> \omega_0 \) and still have \( \omega << \) the maximally possible energy of \( \approx 140 \text{ MeV} \). Now the form (1) for the line shape \( I(\omega) \) has a minimum already at \( \omega = 3\omega_0 \), after which the minimum there will be a linear increase for large energies \( \omega \) (see Fig. 2). The minimum at \( 3\omega_0 \) is not very deep. For a width of 10\% of \( \omega_0 \), it is more than 1\% of the peak value. The consequence of this is that this line shape predicts a large amount of high energy X-rays extending as a background over large energy intervals. In fact the strength of this tail is so strong that if we normalize to the peak value of the resonance, then the integrated line shape corresponds to more than one 2p-1s transition/pion! This violates probability conservation.

![Graph of I(\omega) vs \omega]

Fig. 2

Clearly there must be a damping mechanism and Eq. (1) must be wrong. What is the origin of the damping? Purely from the fact that the Bohr radius \( a_B >> R \), the nuclear radius, one suspects that the damping effect is atomic, at least primarily. We will now discuss how it comes about.

Development of a steady state wave function
For a stable 1s state and a negligible radiative width in the 2p state, the 2p-1s transition energy \( \omega_0 \) is well defined. Suppose a wave packet is confined to the 2p state and transitions to an initially empty 1s state start at \( t = 0 \). The "line shape" for the 2p-1s line is for an unretarded El transition

\[ I(\omega) \propto \omega^3 \left| \int \phi_{1s}(x) (\hat{E} \cdot \hat{x}) \phi_{2p}(x) dx \right|^2 \delta(\omega - \omega_0). \]  

(2)

Since the population of the 1s state increases steadily in time, the 1s amplitude will increase indefinitely.

When absorption is introduced into the 1s state, a balance will develop between the radiative filling rate from the 2p state and the removal rate by absorption.
At this point it is therefore abundantly clear how the linear divergence in $\omega$ of the line shape naturally is replaced by the shape $\omega^{-1}$, which damps for large $\omega$. This effect was the immediate consequence of the decreasing wave function at the origin.

The line shape $\propto \omega^{-1}$ has still the problem that its integrated value $\int d\omega/\omega + \ln \Omega$ still diverges for large $\Omega$ (we always assume that we are not limited by phase space which will be the ultimate limitation). This is a much less serious divergence than before, but even this divergence is regulated, this time by the strong interactions.

In addition to the previous scales in the problem which provoked convergence when the propagation length $k^{-1} \rightarrow 0$, there is also a nuclear scale in this problem. The nuclear scale begins to play a role whenever

$$0 < \frac{\kappa |A_0|}{c} \approx 1 \quad \text{or} \quad \kappa R \approx 1.$$ 

In the limit of $\kappa R$ very small, one can show that $\text{Im} A_0$ should be replaced by $\text{Im}(A_0/(1 + \kappa A_0)) = \text{Im} A_0/|1 + \kappa A_0|^2$. This condition is simply the unitarity of the T-matrix for a point interaction. The consequence of this is that for $|A_0| >> 1$ there will be an additional inverse power in the line shape which will converge as $\omega^{-2}$. Before drawing the more general conclusion from this atomic problem it is very interesting to note that the generated wave $\phi_c(x)$ is independent of strong interactions outside resonance and apart from very large energies. As a consequence the line shape is independent of absorption strength and universal until the onset of unitarity. On the other hand, the line shape depends crucially on the initial state, simply because the source function is different.

As general lessons there are particularly two points to be retained:

1. The presence of an "external driving source" is extremely important for the wave function of a virtual pion and it permits the usual regularity conditions for the wave to be fulfilled at any energy.

2. The physics of the virtual particle is strongly dependent on the physical scale parameters of the problem. In the atomic case the physics changes strongly first as $|\omega - \omega_0|$ becomes larger than $\Gamma$, so that the resonance region is left. Then it changes once more as $\kappa a_0 > 1$ when Coulomb effects become negligible and finally it changes as $|A_0| > 1$ as strong interactions introduce convergence. There is in fact a fourth physical region $\kappa R > 1$, when the propagation takes place entirely inside the nucleus, but this is outside our model.

We will see in the following that scale parameters play an extremely important role also in the physics of exchange current phenomena.

$\mu$-CAPTURE AT LARGE ENERGY TRANSFER  
(OR "STRONGLY INTERACTING MUONS")

Reminder on general properties of $\mu$-capture (Ref. 8)

**Normal $\mu$-capture.** The $\mu$-capture in nuclei takes place from the $1s$ state of the muonic atom. The wave function of the muon is to a good approximation constant over nuclear dimensions, particularly, for light and medium weight elements. The weak absorption process is the usual V-A interaction for the vector and axial currents.
The $\mu$-capture process on the free proton $\mu^- + p \rightarrow n + \nu$ has essentially the kinematics $q_0 \approx 0$; $|q| \approx m_\mu = 106$ MeV/c, i.e. $q^2 = -m_\mu^2$. The neutrino carries off nearly all the muon rest mass energy and the neutron is given only a small recoil energy.

Nuclear $\mu$-capture to low excited states ($E^* \ll 106$ MeV) has nearly the same kinematics as for the nucleon with $|q| \approx 100$ MeV/c. Since the kinematics corresponds to a quasi-free nucleon process, the conditions are excellent for a description in terms of the impulse approximation (Primakoff theory). For $E^* \leq 30$ MeV we can qualitatively understand nuclear $\mu$-capture on this basis sufficiently well to use it for information on nuclear structure.

Breakdown of impulse picture for large $E^*$. For energy transfers larger than about 30 MeV the impulse approximation rapidly breaks down. This region is nearly entirely unexplored both experimentally and theoretically. That is why I have marked this region of "terra incognita" on the figure (Fig. 3) as it was marked on medieval maps, 'hic sunt dragones' (there are dragons here).

![Graph](image)

**Fig. 3**

There are a few qualitative experimental indications (Ref. 8):

a) high-energy neutrons have been obtained with energies up to 60 MeV. They are about $\%_2$ to $\%_5$ of the total capture rate;

b) high-energy protons have been observed up to 40 MeV and are roughly 10% of the high-energy neutrons;

c) in the limited region of observations both energy spectra are approximately linear on a logarithmic scale.

We can immediately conclude: The high-energy protons cannot result directly from the single nucleon process $\mu^- + p \rightarrow n + \nu$ and are outside the simple impulse approximation.
The axial current: The time component $A_0$

By far the most interesting objects in Eq. (15) from our view-point are the axial matrix elements $A_0$ and $A$. I will now demonstrate how both of these are intimately related to pions processes. An important although in itself insufficient ingredient in this connection, is the partial conservation of the axial current (PCAC), which in essence states that the axial current leaks, but that the leak is the pion field.

Reminder on PCAC. The conservation of electric charge means that the corresponding 4-current is conserved and has the consequence that the electron charge $e$ is a universal unit. Similarly the conservation of the vector current (CVC) has the important consequence that the weak vector charge $g_V$ is a universal quantity unchanged by strong interactions.

Can a similar relation hold for the axial current? The answer is no for two simple reasons. First $g_A = 1.25$ and not to unity for the nucleon, differing from the leptonic value. It misses unity but not by very much. Strict conservation (CAC) is therefore impossible. Even worse, a conserved axial current has the unpleasant feature of making the pion stable. To arrange the pion decay, the assumption is that there is something very special about the pion as compared to other mesons, simply because it is the lightest pseudoscalar meson by quite a margin. In addition the divergence $\partial_\nu A^{\nu}_\mu(x) \neq 0$ is a pseudoscalar of isospin 1 so it has exactly the pion quantum numbers.

The PCAC assumption (Ref. 9) is that the divergence is proportional to the pion field $\phi_\pi(x)$ with a universal constant $f_\pi$

$$
\BoxPCAC: \partial_\mu A^{\mu}_{\mu}(x) = i f_\pi m_\pi^2 \phi_\pi(x)
$$

(17)

In Eq. (17) we have used a symbolic notation also for the pion field $\phi_\pi(x)$ which represents the transition field $\langle i | \phi^{(-)}_\pi(x) | f \rangle$ between initial and final nuclear states; also otherwise we will use notation for the pion field equivalent to those of Eq. (16). The constant $f_\pi$ is given by the pion decay rate.

**Application of PCAC to $\mu$-capture at $\omega_\nu = 0$.** For $\omega_\nu = 0$ we have $\partial/\partial t = i q_0$ and the PCAC relation reads

$$
i q_0 A_0(x) - \nabla \cdot A(x) = i f_\pi m_\pi^2 \phi_\pi(x)
$$

(18)

By integration over all space the exact space divergence $\nabla \cdot A(x)$ yields zero, so that

$$
A_0 = \frac{f_\pi m_\pi^2}{q_0} \phi_\pi
$$

(19)

and

$$
\sum_{\text{final states}} |A_0|^2 = \frac{f_\pi^2 m_\pi^4}{q_0^2} \sum_{\text{final states}} |\phi_\pi|^2
$$

(20)
One can therefore directly express the time component of axial $\nu$-capture in terms of pions with $\omega_\pi = m_\mu$. In addition, since $\phi_\pi \equiv \int \phi_\pi(x) dx$, these pions must be $s$-wave pions, for all other $l$-values are suppressed by integration over all space.

There is a simple interpretation of Eq. (20). It corresponds exactly to the muons having produced a source for pions. Since the muon wave function is constant in our approximation, the source is the same everywhere, with the constant arranged to unity. The pion produced interacts with the nucleus and gives partial transitions. For a pion with $\omega_\pi \geq m_\mu$ the sum over the partial transitions corresponds to the optical theorem. A very similar relation is valid below threshold but we must first remove the pion pole in Eq. (20):

$$\sum_{\text{final states}} |\phi_\pi|^2 = \sum_f \frac{|\langle \chi | T_{\pi \nu}^{(\omega_\pi = m_\mu)} | f \rangle|^2}{(m_\pi^2 - m_\mu^2)^2} \equiv \frac{J_m \langle \chi | T_{\pi \nu}^{(\omega_\pi = m_\mu)} | \chi \rangle}{(m_\pi^2 - m_\mu^2)^2} \equiv (21)$$

Therefore, as long as one is close enough to the physical pions, it is sufficient to know the elastic interaction of the pion with an absorptive interaction. It is not necessary to carry out the complicated sum over the many open final channels.

**Solution for small nucleus $K R < 1$.** In the limit of a small nucleus, the $\pi$-nucleus $s$-wave interaction is described by the complex scattering length $a_0$. The arguments are quite similar to those used before in the discussions of the atomic line shape.

As threshold the pion wave has the form

$$\phi_\pi(x) = 1 + \frac{a_0}{x} \quad \text{(22)}$$

The muon generates an incoming wave which is the solution to the inhomogeneous wave equation below threshold

$$\left[ \frac{\nabla^2}{m_\pi^2} - (m_\pi^2 - m_\mu^2) \right] \phi_{in}(x) = \lambda = 1 \quad \text{(23)}$$

or $\phi_{in} = -x^{-2}$ with $\kappa^2 = m_\pi^2 - m_\mu^2$. The full solution must contain the scattered wave $e^{-\kappa x}/x$ and in addition have the same behaviour for $x \to 0$ as Eq. (22). Consequently

$$\phi(x) = -x^{-2} \left( 1 + \frac{a_0}{1 + \kappa a_0} \frac{e^{-\kappa x}}{x} \right) \quad \text{(24)}$$

We therefore see that $\text{Im} T_0(\omega_\pi = m_\mu) = \text{Im} a_0 / |1 + \kappa a_0|^2$. The conclusion is therefore that the $A_0$-component of the $\nu$-capture is given by
\[
\sum_{\text{final states}} S_{f}(x) \cdot S_{f}^{*}(x') \propto \text{Im } \alpha(x) \delta(x-x')
\]
\[
\sum_{\text{final states}} S_{i}(x) \cdot S_{i}^{*}(x') \propto \text{Im } \alpha(x) \delta(x-x').
\]

We have deliberately surrounded the \(\delta\)-functions by quotation marks, since they are not strict \(\delta\)-functions. It is only when the pion has a "sufficiently" long wavelength that we can make the point approximation. Pionic absorption in nuclei is believed to be associated with distances of order 0.5 \(\text{fm} \ll 2 \text{fm}\), the characteristic distance of muonic pions. Let us now consider the \(\sum |A|^2\).

Since the pion field will be constant in the medium, we have
\[
\sum_{\text{final state}} |A|^2 \propto \sum_{\text{final states}} |S(x)|^2 = \int dxdx' \sum_{\text{final states}} S(x) \cdot S^{*}(x')
\]
\[
\propto \int dxdx' \text{Im } \alpha(x) \delta(x-x') = \int \text{Im } \alpha(x) dx.
\]

Consequently, the absorption rate per unit volume of the muon is \(\propto \text{Im } \alpha\). This means that it absorbs just as a \(p\)-wave pion in Born approximation. One notes that, as in the \(s\)-wave case, a heavy nucleus absorbs by volume absorption and not by surface absorption.

To summarize this point, in the lepton current \(l_{\mu} = (1, q)\) the time part \((1)\) generates \(s\)-wave pions and the space part \((q)\) generates \(p\) waves, the latter by a contact interaction with the nuclear matter.

CONCLUSION

In the present lectures we have emphasized very strongly the advantages of looking at the nuclear pionic currents as generated by external perturbations or sources. In the specific illustrations we have chosen, both for the atomic line shape and for large energy transfer \(\mu\)-capture, this permits to consider the problem of virtual pions nearly as Schrödinger problem. Still, this was not the whole story. In the muonic case, the sources in the Schrödinger problem depended not only the external perturbation. In the axial current the muon field also generated contact sources in the matter associated with local \(p\)-wave pion interactions. In fact, it is exactly these contact terms which have the largest resemblance to the usual pion exchange currents in nuclei. The reason is that these pions are generated somewhere in the medium, say at a nucleon, and then re-scatter in the medium until they are absorbed, i.e. they are exchanged between initial and final point in the nucleus. In contrast to these we also found the contributions from the muon as a pion source by its virtual decay into (\(\pi\nu\) ): these terms correspond to the pseudoscalar interaction of pions in weak interactions.

The entire discussion was concentrated to the region in which the (\(\pi\mu\)) mass difference is so small that the description of pions by the same optical
potential as at threshold will be valid. We have not discussed the limitations of this picture. It should be emphasized however that this is not a very important point. The problems we have discussed are only formally dealing with muons as an illustration. The central aim has all the time been to obtain a clear physical picture of the nature of the axial current and of the physical phenomena that occur as the pion becomes increasingly virtual. It is only as a by-product of this aim that we have obtained a quantitative description of \( \mu \)-capture at large energy transfer. Indeed, it is easy to extend our general arguments to cover all \( \mu \)-capture, even with the normal kinematics. By a reasoning closely akin to the Goldberger-Treiman argument one can immediately conclude that all axial \( \mu \)-capture is quantitatively strongly pion dominated.

The kinematics can vary radically, however, and thus also the physics. We may therefore conclude: All axial \( \mu \)-capture is \( \pi \)-capture with varying kinematics. This statement unifies the view of nuclear \( \mu \)-capture.

The description we have given of \( \mu \)-capture seems to us very natural and appealing, and it makes quantitative predictions at large energy transfer. In the final analysis, of course, the real test is in experimental evidence, which at present is nearly totally missing.

The present picture has wider implications. Indeed, M. Ericsson has presented you a closely similar viewpoint for the renormalization of \( g_A \) in \( \beta \)-decay. We also believe that the present viewpoint should be useful for discussing electromagnetic pion exchange phenomena in nuclei. There is a good possibility that we can obtain a very clear physical picture of all pionic exchange effects under very varying physical conditions.

REFERENCES

3. M. Ericsson, in this volume.
Application to μ-capture at $\omega_N = 0$: Point nucleus. We have previously discussed the $A^0$ contributions to this case. From Eq. (33) and the fact that $A = \int A(x) dx = \int A^0(x) dx = A^0 \left[ \text{since } \int \nabla \phi(x) dx = 0 \right]$

$$\sum_{\text{final states}} |A|^2 \approx f_n^2 \sum_{\text{final states}} |S|^2.$$  \hspace{1cm} (36)

We derived earlier how the sum over partial s-wave transitions gave the imaginary part of the s-wave amplitude also below threshold. For a point nucleus this amplitude was given in terms of the complex $\lambda = 0$ scattering length by Eqs. (21) and (25). A very similar argument gives now the sum over the $A$ terms proportional to

$$\sum_{\text{final states}} |A|^2 \propto J_m \frac{T_{\lambda=1}(\omega_\mu=m_\mu)}{|1 + \kappa^2 a_\lambda|^2}. \hspace{1cm} (37)$$

This means that we have unitarized the p-wave scattering volume $a_\lambda$, defined by $k^3 \cot \delta_k = a_\lambda^{-1}$ for $k = i\kappa$ with $\kappa^2 = m_p^2 - m_\mu^2$. It is very important to note that even as $m_\mu \to m_\pi$ there is no pole term in Eqs. (36) and (37). One may note that in the usual Goldberger-Treiman relation it is frequent to let $m_\pi \to 0$ and to note that no pole term appears. In the present case such a procedure would be meaningless: the limit $m_\pi \to 0$ would completely modify the whole absorption process. The correct limit for a small nucleon system is $m_\mu \to m_\pi$, instead, if the purpose is to exhibit the absence of a pole. For most nuclei the preservation of the physics forces us to renounce even this extrapolation.

Application to an extended nucleus: General case

In order to treat the extended nucleus, we consider the interaction of a pion in the nuclear medium similar to the elastic case (26). However, since we consider the transition field $\phi(x)$ from an initial to final state, it is suitable to introduce the transition source $s(x) + i\left[\nabla \cdot \phi(x)\right]$. The inhomogeneous wave equation is

$$(\nabla^2 + \omega_\mu^2 - m_\mu^2)\phi(x) = s(x) + g(x)\phi(x) + \nabla \cdot \left[ iS(x) + \phi(x) \nabla \phi(x) \right].$$  \hspace{1cm} (38)

The right-hand side is the total source, which decomposes itself into a local p-wave term and a local s-wave term. Contrary to the previous case of a point nucleus there are now source terms which describe the rescattering of the pion in the medium. These are induced source terms. We now apply the locality principle LAC to Eq. (38) at a particular point. We then conclude that $A^0_\mu(x)$ is
\[
\begin{align*}
\mathcal{A}_x(x) &= f_n \left[ S(x) - i \gamma(\alpha(x)) \nabla \phi(x) \right], \\
\mathcal{A}_o(x) &= \frac{f_n}{q_o} \left[ S(x) + q(\gamma(x)) \phi(x) \right].
\end{align*}
\] (39)

Apart from the direct transition terms \( s(x), S(x) \) there now appears induced axial currents due to the axial polarization of the nuclear medium. M. Ericson discusses in detail in her lectures (Ref. 3) how the axial polarization vector has a very direct correspondence in the electric polarizability with Maxwell's displacement vector of a dielectric. Her discussion of the renormalization of \( g_A \) is carried out directly in these terms for \( \omega_n \approx 0 \) and contributions from \( p \)-wave pion interactions only.

In the present case we find the induced axial polarization vector \( P(x), P_o(x) \) to have the components
\[
\begin{align*}
P(x) &= i \gamma(\alpha(x)) \nabla \phi(x), \\
P_o(x) &= q(\gamma(x)) \phi(x).
\end{align*}
\] (40)

Including the pole term of the pion the total axial current in the nucleus is
\[
\begin{align*}
\mathcal{A}_x(x) &= f_n \left[ S(x) - i \nabla \phi(x) + P(x) \right], \\
\mathcal{A}_o(x) &= \frac{f_n}{\omega_n} \left[ S(x) - \alpha_n \phi(x) + P_o(x) \right].
\end{align*}
\] (41)

These expressions are valid also for \( |q| \neq 0 \) and \( q_o \neq m_p \). From this point on, the further discussion is, in principle, well-determined, since all the properties of the axial current are completely determined by the pion interactions. In fact, the central content of Eq. (41) can be stated as follows:

To every muon-capture reaction by the axial current, there is the corresponding pionic reaction. The matrix element for the pion determines the muonic one. Consequently, we expect a qualitative counterpart of pionic phenomena, like 2-nucleon absorption, in large energy transfer muonic processes, and these are quantitatively predicted, at least in principle.

Capture rate by \( \Delta \) terms in nuclear matter: \( \omega_n = 0 \). The optical theorem can be expressed locally in the nuclear medium for the transition source: the absorptive part of the potential is the sum of the local absorptive transitions to final states.

Such relations are valid also for physical pions, and they are implicit in the normal use of absorptive parameters in the optical potential. More exactly
It is therefore highly likely that the large energy transfer region needs a radical change in theoretical approach.

**Kinematics of maximal energy transfer.** So as to obtain the purest possible example of large energy transfer, consider the case of μ-capture with no energy going into the neutrino. For \( \omega' = 0 \) we have

\[
q_o = 106 \text{ MeV} \quad (\approx m_\mu) \approx E^*; |q| = 0,
\]

which is to say that \( q^2 = + m_\mu^2 \). To an elementary particle physicist the difference \( q^2 = \pm m_\mu^2 \) may appear rather minor. For nuclei it means a major change of physics. A momentum transfer \( |q| \approx 100 \text{ MeV}/c \) to a nucleus changes only a form factor, and for light elements it does not even depend very strongly on the nuclear radius, as is well known from electron scattering. An energy transfer of 100 MeV to a nucleus makes great damage and it is quite a traumatic event: it is physically extremely important what amount of energy \( (q_0) \) is transferred to the nucleus. This point becomes clearer if we simply compare the transfer of energy-momentum for the absorption of a stopping pion to that of maximal energy transfer in μ-capture

\[
\begin{align*}
q_0^\pi &= 139 \text{ MeV}; |q| = 0 \\
q_0^\mu &= 106 \text{ MeV}; |q| = 0
\end{align*}
\]

The kinematical similarity is striking. We will in the following show that also the physics is closely linked, although non-trivially so. In order to intuitively visualize part of this connection, we choose the pseudoscalar one-pion exchange between a nucleon and the muon (see Fig. 4).

![Fig. 4](image)

This diagram is usually thought of as the pseudoscalar pion contribution to the nucleon axial current, which then is coupled to the muon. A corresponding pseudoscalar contribution also occurs in the maximal energy transfer μ-capture on a nucleus (see Fig. 5).
Once more we could try to describe this as a pionic effect in the nuclear axial current. This is, in principle, alright, but it is quite an inefficient way of visualizing the physics. Instead we can look at the upper $\mu\nu\pi$ vertex as describing the virtual decay of the muon into a pion: in free space the muon surrounds itself by a Yukawa field of pions $e^{-\kappa x}/x$ with $\kappa^2 = m^2_\pi - m^2_\mu$ or $\kappa^{-1} \approx 2\text{ fm}$. More exactly, the muon will act as a source for the pion field of the nucleus. This will build up a pion wave function which will be absorbed like a non-physical pion by the nucleus. One should at once note that this will not be a simple pole approximation: only if $\kappa R < 1$ will it be possible to describe this process in terms of the external amplitudes in which an incident pion wave falls in on a nucleus. For most nuclei the pion wave is generated well inside nuclear matter by the muon. This changes the physics as we will see shortly.

General expression for $\mu$-capture at $\omega_\nu = 0$
In order to be specific we consider the general expression for $\mu$-capture per unit excitation energy near $\omega_\nu = 0$. This rate $d\Gamma/dE^*\text{ can symbolically be written}$

$$\frac{d\Gamma}{dE^*} \propto G^2 \frac{\psi_{(0)}^2}{\omega^2} \sum_{\text{final states}} \left\{ |V_{\nu}|^2 + |V_{\bar{\nu}}|^2 + |A_0|^2 + |A_1|^2 \right\}. \quad (15)$$

The notation is $G = 10^{-5}/m^2_p$ = weak coupling constant and $\psi_{(0)}(0)$ is the muon wave at the origin (assumed constant over the nucleus). The central part of Eq. (15) are the symbolic matrix elements $V_0$, $\bar{V}$, ..., from the initial state $i$ to the final state $f$:

$$V_0 \equiv \langle i | V_0(x) | f \rangle \equiv \int V_0(x) \, dx \equiv \int \langle i | V_0(x) | f \rangle \, dx \text{ etc.} \quad (16)$$

We will use these notations indiscriminately in the following. In the limit $\omega_\nu = 0$ the transition rate has a particularly simple form when summed over the lepton spin, since there are four incoherent processes: time-like vector $V_0$ and axial vector $A_0$ and space-like vector $\bar{V}$ and axial vector $A_1$. Further the total rate goes to zero as $\omega^2_\nu$, so we shall consider the coefficient of this term.
of pions of the energy corresponding to the X-ray. The e.m. transition operator acting on the 2p states serves as a driving source for a pion wave. Since the pion wave function does not develop spontaneously as for a bound state, but is produced by a source, it can exist at any energy. The line shape directly reflects the absorption of this (unnormalized) wave function. To explore the consequences quantitatively we will consider a simple, exactly soluble model, which also is quite realistic.

Model for the hadronic atom
Both the charge Ze and the strong interaction are assumed concentrated to a massive point at the centre of the atom. The assumption is therefore that the nuclear radius \( R \ll a_B \), the Bohr radius. Because of the short range only s-waves will have strong interactions; interactions with \( l \neq 0 \) are neglected since the centrifugal barrier will keep them small. In this limit the strong interaction can be described by the pseudopotential

\[
V(x) = -\frac{4\pi}{2m} A_0 \delta^{(0)}(x). \tag{3}
\]

Here \( A_0 \) is the complex scattering length for s-waves. We will take \( |A_0| \ll a_B \), but not necessarily \( |A_0| < R \). In practical cases the inequalities are very well fulfilled. Note, that the scattering length is complex; this means that it is absorptive, so that \( V(x) \) has an absorptive part.

Behaviour close to resonance
The \( \delta \)-function in Eq. (2) is in fact the limit

\[
\delta(\omega - \omega_0) \rightarrow \lim_{\varepsilon \to 0} \frac{1}{\pi} \frac{\varepsilon}{(\omega - \omega_0)^2 + \varepsilon^2}. \tag{4}
\]

For a width \( \Gamma \rightarrow 0 \) in Eq. (1) we see therefore that the \( \delta \)-function corresponds to the limit of a complex energy eigenvalue \( \omega_0 + i \Gamma/2 \) for the resonance energy. In fact the pseudopotential (3) gives the complex energy shift from strong interaction as

\[
\Delta \varepsilon_{1s} = -\frac{4\pi}{2m} A_0 \phi^2_{1s}(0). \tag{5}
\]

In the region near the resonance we have therefore the usual Breit-Wigner line shape

\[
I(\omega) = \frac{\Gamma}{2\pi} \frac{\omega^3}{(\omega - \omega_0)^2 + (\Gamma/2)^2} \left| \int \phi_{1s}(\xi)(\xi \cdot \hat{\xi}) \phi_{2p}(\xi) d\xi \right|^2 \tag{6}
\]

with \( \Gamma = 4\pi/m \text{ Im } A_0 \phi^2_{1s}(0) \). These approximations give the well-known standard result.

Behaviour far from resonance
Outside the delicate resonance region (at which I deliberately oversimplified the discussion) the physics is completely dominated by the external driving source produced by the initial 2p state. The driving source for an unretarded El operator is
\[ j(x) = \omega (\hat{e} \cdot x) \phi(x) \frac{2}{2p}(x) \]  

(7)

This source will now act as the generator for the wave function \( \phi(x) \) in the Coulomb field with Hamiltonian \( H = T + V_C \).

\[ (H - \varepsilon) \phi(x) = \dot{j}(x) \]  

(8)

with \( \varepsilon = \varepsilon_{ap} - \omega \). The solution to this equation can immediately be expressed formally by the Green function \( G_C(x,x') \) for the Coulomb field.

\[ \phi(x) = \int G_C(x,x') \dot{j}(x') dx'. \]  

(9)

In principle there is in addition a scattered wave from the strong interaction at the origin, but we will for the moment neglect it. In the spirit of the pseudopotential approach, we can now express the absorption rate as the integral over the absorptive potential (absorption rate for energy \( \varepsilon \))

\[ \alpha = \int \phi(x)^2 \text{Im} V(x) dx \approx \phi(0)^2 \text{Im} A_0 \]  

(10)

The problem of finding the line shape is therefore simply the problem of finding the wave function at the origin. Now

\[ \phi(0) = \int G_C(0,x') \dot{j}(x') dx' \]  

(11)

where \( G_C(0,x) \) is the Whittaker function. The solution for any \( \varepsilon \) (apart from near resonance) is quite straightforward. We will not discuss the general case however. For \( \omega \gg \omega_0 \) the Coulomb interaction is quite negligible and we can simply replace the Whittaker function by an ordinary Yukawa function:

\[ G_C(0,x) \rightarrow G_0(0,x) = -\frac{1}{4\pi} \frac{e^{-\kappa x}}{x} \]  

(12)

with \( \kappa^2 = -2m\varepsilon \approx 2m\omega \). The meaning of this is quite clear: the initial operation produces a wave packet \( j(x) \). This wave packet spreads like a Yukawa field around every point, if there is no Coulomb interaction. Therefore the wave function at the origin can only sample the source \( j(x) \) inside of a region \( \kappa^{-1} \rightarrow 0 \) around the origin as \( \omega \rightarrow \infty \). Since in the short range region \( \phi_{2p}(x) \approx x \) the integral takes the approximate form

\[ \phi(0) \approx \omega \int_{-\infty}^{\infty} x \cdot \frac{e^{-\kappa x}}{x} x^2 dx \approx \omega \kappa^{-2} \approx \omega^{-1}. \]  

(13)

This means that the probability of finding the pion at the origin decreases as \( \omega^{-2} \) and therefore that the line shape has the form

\[ I(\omega) \xrightarrow{\omega \text{ large}} \omega^{-1}. \]  

(14)
explicit dependence of the physical behaviour on the $\pi-\mu$ mass difference which is the solution of the paradox we found previously.

Our conclusion of the result up to this point is that for $\omega_\gamma = 0$ it is possible to achieve a very detailed understanding of the $A_0$ branch of the $\mu$-capture. It results directly from $\pi$-absorption of a pion wave generated in the sub-threshold region by the muon. We will now turn to the question of the $A$ branch of the capture as well as to the general structure of the axial current at high-energy transfer.

The axial current: General case
Our discussion up to this point has only concerned the $A_0$ term which is very directly related to the pion interaction. The $A$ terms in the capture are more subtle. They are intimately connected to pion exchange phenomena.

The question is thus: what can be said about $A$? The first observation is that PCAC is insufficient to determine $A$. We have already used PCAC in the determination of $A_0$; it is not possible to get two unknowns out of one relation. As the saying goes: you can't draw blood from a stone. I will now argue that $A$ represents the local pion $p$-wave in the medium, like $A_0$ represents the local pion $s$-wave. To make this seem natural we rewrite the PCAC relation.

The restated PCAC relation. We can take out the pion pole from the PCAC Eq. (17) by writing $A_\mu = A_{\text{pole}} + A_\mu^0$, where $A_\mu^0$ is pole free and $A_{\text{pole}} = \text{const} \times \partial_\mu \phi_\pi(x)$. If we simply recall that the usual Klein-Gordon equation for the pion field with a source is

$$\left(\partial_\mu^2 + m_\pi^2\right)\phi_\pi(x) = -j_\pi(x),$$

we obtain immediately the constant by the condition that explicit terms in $\phi_\pi(x)$ should be eliminated.

Restated PCAC

$$\begin{cases}
A_{\mu}^{\text{pole}} = i f_\pi \partial_\mu \phi_\pi(x) \\
\partial_\mu A_\mu^0 = i f_\pi j_\pi(x).
\end{cases}$$

(31)

In order to see how this new relation can be put to work let us assume we have a "point" nucleus with a source function for $s$- and $p$-wave pions

$$j_\pi(x) = s \delta^{(4)}(x) + i (\Sigma \cdot \nabla) \delta^{(4)}(x).$$

(32)

Here $s$ and $\Sigma$ are transition matrices with components $S_{1f}$ and $S_{1f}$, which carry the nucleus from an initial state $i$ to a final state $f$ on pion absorption.

\[ \text{Fig. 6} \]
The question is now how the axial current can be related to the source (32). This is achieved by the following locality hypothesis:

**Locality of the axial current (LAC):** The pole free part of the axial current \( A^0(x) \) is determined by the local pion source.

**Axial current for point system.** If we apply the LAC hypothesis to point-like s- and p-wave source (32), the result is \( (\partial / \partial t = iq_0) \):

\[
A^0_0(x) = \frac{f_n}{q_0} s \bar{\delta}^{00}(x) \\
\tilde{A}^0(x) = \frac{f_n}{q_0} s \bar{\delta}^{13}(x).
\]

Inside of our hypothesis we note that the time-like part \( A^0_0 \) goes with s-wave pions, the space-like part \( \tilde{A}^0 \) goes with p-wave pions.

For a pointlike system the reasonableness of the LAC hypothesis is obvious. It would in fact be disturbing to have the axial current associated with points at which there is no matter. On the other hand, even elementary particles are extended objects. A very strict application of the locality principle would then mean that the axial current is associated with s- and p-wave pions in the particle. We do not have to face this question here. The characteristic distance for \( \mu \)-capture at large energy transfers is 2 fm and so it is at low energy transfers. Therefore we must ask how point-like the hadronic system is on this scale, thereby avoiding the more difficult question. With this physical interpretation of "locality", we will illustrate our viewpoint with a non-relativistic description of the Goldberger-Treiman relation.

The non-relativistic Goldberger-Treiman relation. If we express the non-relativistic axial current at \( q_0 = 0 \) and choose only the pole free part, the expressions for \( \tilde{A}^0(x) \) and the source function are

\[
\begin{align*}
A^0_0(x) &= g_\pi \sigma \bar{\delta}^{00}(x) \\
\tilde{A}^0(x) &= \frac{g_\pi}{2M} (\sigma \cdot \gamma) \bar{\delta}^{00}(x),
\end{align*}
\]

where \( g_\pi \) is the renormalized \( \pi N \) coupling constant. The PCAC relation with the locality condition (LAC) gives according to Eq. (33)

\[
\frac{g_\pi}{2M} = f_n \frac{g_n}{2M}.
\]

This result is the usual Goldberger-Treiman relation (Ref. 10) which is known to be accurate to 5% without form factor effects (Ref. 11). This derivation of the Goldberger-Treiman relation demonstrates that the LAC principle is implicit in the usual applications of PCAC. In the nuclear case the explicit locality statement serves first to separate the contributions to \( A^0_0(x) \) and to \( \tilde{A}^0(x) \) in a clear way, and secondly (which is its most valuable effect) to determine the axial current inside the extended nucleus.
REPOLARIZATION OF MUONS IN MUONIC ATOMS

WITH POLARIZED NUCLEI

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ABSTRACT

The equation of motion for the density matrix of spin $\frac{1}{2}$ and spin 1 particles with contact interaction is solved. Time-averaging shows that 44% (30%) of the nuclear (muonic) polarization is transferred to the muons (nuclei). Generalizations to higher nuclear spins are given.

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The depolarization of captured muons in the atomic cascade has been given considerable attention \(^1\). For muonic atoms with spinless nuclei, the main source of depolarization is spin-orbit coupling, resulting in a loss of about five-sixths of the initial muon polarization \(^2\). If the nucleus has spin the hyperfine contact interaction will give additional depolarization in \(s\) orbits \(^3\). The muon loses part of its polarization to an initially unpolarized nucleus, similar to the Overhauser effect \(^4\). There is an analogous phenomenon in muonium \(^5\).

If the target is initially polarized, the contact interaction can "repolarize" the muons \(^6\). The case for spin \(\frac{1}{2}\) nuclei is similar to muonium \(^5\), so we treat here the case of spin 1 nuclei in detail. We take the muon to be in a \(1s\) orbit and neglect the effects of the finite muon life-time and of possible conversion between the two hyperfine levels \(^7\). Initially the density matrix \(\rho\) of both muons and nuclei is a direct product of density matrices for spin \(\frac{1}{2}\) and spin 1:

\[
\rho = \rho_{\frac{1}{2}} \otimes \rho_1 = \frac{1}{2} \left( \begin{array}{cc} 1 + 2 \vec{a}_0 \cdot \vec{s}_1 & \vec{b}_0 \cdot \vec{s}_1 + 3 \vec{b}_0 \cdot \vec{\tau} \vec{s}_0 \vec{s}_1 \end{array} \right)
\]

where indices 1 and 2 refer to muons and nuclei, respectively. The initial muon and nuclear polarizations are \(\vec{a}_0\) and \(\vec{b}_0\) and \(\vec{b}_0\) is a symmetric, traceless tensor describing the initial tensor polarization (of rank 2) of the spin 1 nuclei. As usual

\[
T_{ij} (\vec{u} \cdot \vec{v}) = \frac{1}{2} (u_i v_j + u_j v_i) - \frac{1}{3} (\vec{u} \cdot \vec{v}) \delta_{ij}
\]

When the contact interaction

\[
H = \omega \cdot \vec{s}_1 \cdot \vec{s}_2
\]

is switched on, the density matrix develops in time and can no longer be factored. Recoupling \(\vec{s}_1\) and \(\vec{s}_2\) as well as \(\vec{a}_1\) and \(\vec{b}_1\) to give symmetric, traceless tensors in the product space of \(\vec{s}_1\) and \(\vec{s}_2\) we obtain a general expression for the density matrix:
\[ \rho(t) = \frac{1}{6} \left\{ 1 + c(t)(\vec{\alpha} \cdot \vec{\beta}) + 2 \vec{c}(t) \cdot \vec{\beta} + \frac{3}{2} c(t)(\vec{\alpha} \cdot \vec{\beta}) + \frac{3}{2} c(t)(\vec{\alpha} \times \vec{\beta}) \right\} + \frac{18}{5} \overline{d}(t) \cdot \left[ \vec{\beta} \cdot \vec{\tau}(\vec{\alpha} \times \vec{\beta}) \right]^{(1)} + 3 \overline{c}(t) \cdot \vec{\tau}(\vec{\alpha} \times \vec{\beta}) \\
+ 3 \vec{c}(t) \cdot \vec{\tau}(\vec{\alpha} \times \vec{\beta}) + 4 \overline{c}(t) \cdot \left[ \vec{\beta} \cdot \vec{\tau}(\vec{\alpha} \times \vec{\beta}) \right]^{(2)} \\
+ 6 \vec{c}(t) \cdot \vec{\tau}(\vec{\alpha} \times \vec{\beta}) + \left[ \vec{\alpha} \cdot \vec{\tau}(\vec{\alpha} \times \vec{\beta}) \right]^{(3)} \right\} \\
(2) \]

i.e., one scalar, four vectors, three rank two tensors and one rank three tensor, giving a total of 35 time-dependent polarization quantities. The boundary conditions are:

\[
\begin{align*}
\vec{c}(0) &= \vec{\alpha}_0 \cdot \vec{\beta}_0, & \vec{d}(0) &= \vec{\alpha}_0, & \vec{d}(0) &= \vec{\beta}_0 \\
\vec{c}(0) &= \vec{\alpha}_0 \times \vec{\beta}_0, & \vec{d}(0) &= \vec{\alpha}_0 \cdot \vec{\beta}_0 \\
\vec{c}(0) &= \vec{\tau}(\vec{\alpha}_0 \cdot \vec{\beta}_0), & \vec{d}(0) &= [\vec{\alpha}_0 \cdot \vec{\beta}_0]^{(2)} \\
\vec{c}(0) &= \left[ \vec{\alpha}_0 \cdot \vec{\tau}(\vec{\alpha}_0 \times \vec{\beta}_0) \right]^{(3)}
\end{align*}
(3) \]

The various symmetric and traceless tensors of rank 1, 2 and 3 are constructed as follows 8)

\[
\begin{align*}
[\vec{u} \cdot \vec{v}]^{(1)}_{ij} &= \frac{c_{ij}}{u} V_{ij} \\
[\vec{u} \cdot \vec{v}]^{(2)}_{ij} &= \frac{1}{2} (E_{imn} u_m V_{nj} + E_{jmn} u_m V_{ni}) \\
[\vec{u} \cdot \vec{v}]^{(3)}_{ijk} &= \frac{1}{3} (u_i V_{jk} + u_j V_{ik} + u_k V_{ij}) \\
&- \frac{3}{15} \left( \delta_{ij} \vec{u} \cdot \vec{v}_k + \delta_{ik} \vec{u} \cdot \vec{v}_j + \delta_{jk} \vec{u} \cdot \vec{v}_i \right)
\end{align*}
\]

if \( \vec{V} \) is traceless and symmetric.

The equations of motion for the polarization quantities follow from the time-dependent Schrödinger equation for \( \rho \) 9)

\[ \]
\[ \frac{d \mathbf{P}}{dt} = i N_p \text{ Tr}( \mathbf{g} [\mathbf{H}, \mathbf{T}] ) \]

where \( \mathbf{P} \) is any of the polarization quantities \( c, \mathbf{a}_1, \ldots, \mathbf{r}^{(3)} \) in Eq. (2) and \( \mathbf{T} \) its associated spin tensor. The normalization \( N_p \) follows from

\[ \mathbf{P} = N_p \text{ Tr}( \mathbf{g} \mathbf{T} ) \]

The 35 equations for the various \( \mathbf{P} \)'s are easy to solve because the commutator \( [\mathbf{H}, \mathbf{T}] \) is a linear combination of tensors of the same rank as \( \mathbf{T} \) itself since \( \mathbf{H} \) is a scalar in the product space of \( \mathbf{s}_1 \) and \( \mathbf{s}_2 \). The trace of a product of two-spin tensors, made from either \( \mathbf{s}_1 \) or from \( \mathbf{s}_2 \) or from both of them \( \text{as the nine tensors in } [2] \text{ is always zero if their ranks (in product space) are different. This can be verified for all cases in this note, using standard trace theorems }^{10} \). Different tensors of the same rank are also orthogonal; the nine tensors in (2) are of course orthogonal. For example:

\[ \text{Tr}\{ (\mathbf{s}_1 \mathbf{s}_2) ; [\mathbf{H} \mathbf{T}(\mathbf{s}_1 \mathbf{s}_2)]_i^j \} = \]

\[ = \epsilon_{imn} \text{ Tr}( s_{im} s_{ip} ) \text{ Tr}( s_{mn} T_{pj} (\mathbf{s}_1 \mathbf{s}_2) ) = 0 \]

since the last trace, involving only \( \mathbf{s}_2 \) operators of different rank, vanishes \(^8\). When we evaluate \( \text{Tr}( \mathbf{g} [\mathbf{H}, \mathbf{T}] ) \), the commutator will pick out only terms with the same rank as \( \mathbf{T} \). But \( \mathbf{T} \) itself will not survive since

\[ \text{Tr}( \mathbf{T} [\mathbf{H}, \mathbf{T}] ) = 0 \]

trivially. There is then a "social hierarchy" in the equations of motion for \( c, \ldots, \mathbf{r}^{(3)} \); the time derivatives of tensor polarizations will only couple to other tensors of the same rank. Therefore \( c \) and \( \mathbf{r}^{(3)} \) are constant since they have nothing to couple to.

We work out the commutators \( [\mathbf{H}, \mathbf{T}] \) and write them in terms of the spin tensors in (2) as well as other tensors involving the rank two tensor \( \mathbf{T}(\mathbf{s}_1 \mathbf{s}_1) \) and the rank three tensor \( \mathbf{r}^{(3)}(\mathbf{s}_2 \mathbf{s}_2 \mathbf{s}_2) \). For the vector polarizations we obtain:
\[ \begin{align*}
\dot{a} &= -2 \dot{r} = \omega z \\
\dot{z} &= -\omega r + \frac{3}{2} \omega \dot{a} - \omega \ddot{a} \\
\dot{\ddot{a}} &= \frac{5}{2} \dddot{a}
\end{align*} \]

which are easily solved subject to the boundary conditions (3):

\[ \ddot{a}(t) = \frac{1}{4} (8 \dddot{a}_0 - 6 (\dddot{r}_0 + \dddot{a}_0)) \cos \frac{3}{2} \omega t - \frac{3}{2} (\dddot{a}_0 \times \dddot{r}_0) \sin \frac{3}{2} \omega t + \dddot{a}_{av} \]  

(5)

and \( \dddot{r}_0 \) is given by \( \dddot{r}_0 = \dddot{r}_0 (a_0^3 - \dddot{a}(t))/2 \). There are similar expressions for \( \dddot{z}_0 \) and \( \dddot{a}_0 \). The last term in Eq. (5), \( \dddot{a}_{av} \), is what we are left with when we take the time-average of \( \dddot{a}(t) \). This is the most relevant quantity, physically, since \( \dddot{a}(t) \) will make of the order of \( 10^5 \) oscillations during the lifetime of the muon. We find

\[ \dddot{a}_{av} = \frac{11}{27} \dddot{a}_0 + \frac{4}{9} (\dddot{r}_0 + 2 \dddot{a}_0) \]  

(6)

This gives the Overhauser effect for spin 1 nuclei. In the absence of any initial nuclear polarization, the muons retain 11/27 of their polarization, and the nuclei are polarized by 8/27 of the polarization the muons had when they entered the 1s orbit.

The muon polarization can never exceed 100%. Take the muons to be 100% polarized initially in the \( z \) direction, \( a_0 = 1 \), and let the nuclei be prepared in a pure state with magnetic quantum number \( +1 \). The initial nuclear density matrix is then:

\[ \rho = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} \]

from which we obtain [see Eq. (1)]:

\( \dddot{r}_0 = \dddot{k} \), \( \dddot{a}_0 = \frac{1}{2} (k \dddot{k} - \frac{1}{2} \dddot{r}) \)
which gives \( \vec{d}_0 = \frac{\vec{k}}{3} \). So when both muons and nuclei are in pure "spin up" states, Eq. (6) gives \( \vec{a}_{av} = \vec{c} \), again 100\% muon polarization.

The equations of motion for \( \vec{b}, \vec{c} \) and \( \vec{e} \) are even simpler than Eq. (4). We find:

\[
\begin{align*}
\dot{\vec{c}} &= -2\vec{c} + 4\omega \vec{e} \\
\dot{\vec{e}} &= \frac{3}{8} \omega (\vec{c} - \vec{t})
\end{align*}
\]

which are trivial to solve. The frequency of the oscillations are again \( 3\omega/2 \) and

\[
\vec{v}_{av} = \frac{1}{3} \left( \vec{v}_0 + 2 \vec{t}(\vec{a}_0 \vec{b}_0) \right)
\]

So we obtain a tensor polarization (alignment) if the nuclei have initial vector polarization even if they have no tensor polarization to start with, provided the muons are polarized.

The corresponding results for spin \( \frac{1}{2} \) nuclei are

\[
\vec{a}_{av} = \vec{b}_{av} = \frac{1}{2} (\vec{a}_0 + \vec{b}_0)
\]

(7)

and the polarization oscillations, similar to Eq. (5), have frequency \( \omega \) and not \( 3\omega/2 \). Spin \( \frac{1}{2} \) nuclei are about 10\% more effective for repolarizing muons than spin 1 nuclei, since they transfer half their polarization.

One can generalize the results (5) and (6) to nuclei with any spin \( I \). There are never more than four vector polarizations, so one gets a system of equations similar to (4), but with coefficients which are complicated functions of \( I \). The re-coupled (time-averaged) results for general spin when \( \vec{a}_0 \) and \( \vec{b}_0 \) are parallel (or anti-parallel) can be obtained by doing an incoherent average over the two hyperfine states:

\[
\langle a \rangle = \sum_{m=-I}^{I} N_+(m) \langle F_+^m|\frac{1}{2} S_3|F_+^m\rangle + \sum_{m=-I}^{I} N_-(m) \langle F_-^m|\frac{1}{2} S_3|F_-^m\rangle
\]
where \( F = I \pm \frac{1}{2} \) and \( \sigma_z \) is a Pauli matrix. The magnetic sub-level populations \( N_m \) have been given in a previous note \(^6\). Rank two and higher nuclear polarizations have been ignored. The result is:

\[
\langle a \rangle = \frac{1}{3} a_0 \left( 1 - \frac{2}{(2I+1)^2} \right) + \frac{4b_0 I}{(2I+1)^3}
\]

which agrees with (6) for \( I = 1 \) and with (7) for \( I = \frac{1}{2} \). For \( I = 0 \) we get no muon depolarization, of course. The result (8) was derived by Überall for \( b_0 = 0 \) \(^3\). The same procedure for \( b \) gives:

\[
\langle b \rangle = \frac{4}{3} a_0 \frac{I+1}{(2I+1)^2} + b_0 \left( 1 - \frac{2}{(2I+1)^2} \right)
\]

This checks with the constancy of \( \vec{a} + 2I\vec{b} \), which follows from the fact that the total spin \( \vec{J} = \vec{s}_1 + \vec{s}_2 \) commutes with \( \mathbf{H} \). In the limit of infinite nuclear polarization the muons retain one third of their original polarization and the nuclear polarization does not change. It is clear from Eq. (8) that spin \( \frac{1}{2} \) nuclei will be the most effective for repolarizing the muons with a polarized target.

Our results are of practical importance for muonic atoms with light nuclei with spin, where the hyperfine conversion is absent or negligible. Applications may be in several areas of physics: (1) Muon spin relaxation (\( \mu \mathrm{SR} \)) with hyperfine coupling. The initial nuclear polarization could be adjusted to give a zero final muon polarization. (2) Parity-violating effects in nuclear muon capture experiments such as the gamma-neutrino correlation, neutron or photon asymmetry, etc. \(^1\) can be enhanced. (3) The coupling of the nuclear polarization to the muon polarization can possibly be used to detect neutral currents \(^{11}\).

We have explicitly solved the equation of motion for the density matrix for spin 1 and spin \( \frac{1}{2} \) particles interacting by a hyperfine contact Hamiltonian. Tensor techniques reduced the equations for the 35 polarization quantities (sometimes referred as Wangmuss-Bloch equations \(^{12}\) in the context of muonium \(^5\)) to, effectively, a set of four equations and a set of three equations. The time-averaged solution gave the polarizations after they have been recoupled by the contact interaction. The explicit time-dependence of the muon polarization, Eq. (5), will be needed to calculate the depolarization in higher \( s \) orbits with lifetimes of the order of the oscillation period \( \omega^{-1} \).
The dynamics treated here, in particular the regeneration of muon polarization, are essential features in any muon capture experiment with a polarized target.

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