AN INEQUALITY ON THE MASS OF THE BOTTOM QUARK

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ABSTRACT

We establish inequalities on the energy levels of two-particle systems interacting via the same potential with different masses and different angular momenta. When this is applied to the J/ψ and T particles, one gets an inequality on the mass difference between the heavy quark and the charmed quark as a function of the masses of the J/ψ , the T and the T' . We find $M_b - m_c > 3.37$ GeV. In the appendix, a concavity property of the energy levels as a function of the angular momenta is discussed. This property also implies $2E_p > E_{1S} + E_{2S}$, which gives for the mean mass of the P states of T a lower limit of 9.76 GeV.

1. INTRODUCTION

The tremendous success of the description of the J/ψ and related particles as a quark-antiquark system, $c\bar{c}$, bound by a confining potential^{1),2)} (the only shadow being the mystery of the pseudoscalar states) seems to propagate to the next heavy new particle, the T , initially discovered by Lederman et al.³⁾, and observed recently by two groups at DESY⁴⁾. The observed leptonic width of the T of 1.3 ± 0.4 keV agrees with the predictions of the various models⁵⁾, in which T is regarded as a $b\bar{b}$ system, b having a charge $1/3$, bound by a potential relatively close to the $c\bar{c}$ potential. This encouraged us to take this model still more seriously. Here we want to look at the over-simplified situation, where the $c\bar{c}$ and $b\bar{b}$ potentials are identical. In this case a number of important results on mass dependent effects have been obtained⁶⁾. Here we want to draw some rigorous conclusions on the comparison between the energy levels of the two systems. Obviously, since the kinetic energy decreases with increasing mass, the binding energy of the T is lower than the binding energy of the J/ψ . However, this is a very crude inequality, and it could become worse for the $t\bar{t}$ system, if it exists, and if the "top quark" turns out to be extremely heavy. What one can try to do in order to compare the energy levels of the two systems is to compensate the loss of kinetic energy when the quark mass is increased by an increase in the rotation energy. This is what we shall do.

2. AN INEQUALITY FOR ARBITRARY POTENTIALS

We consider two systems interacting by the same potential V , but with different reduced masses and different angular momenta:

$$\frac{u''}{m} = \left[V(r) - E(m, \ell) + \frac{\ell(\ell+1)}{mr^2} \right] u \quad (1)$$

$$\frac{v''}{M} = \left[V(r) - E(M, L) + \frac{L(L+1)}{Mr^2} \right] v \quad (2)$$

$E(m, \ell)$ and $E(M, L)$ are the ground state energies of the systems for angular momenta ℓ and L , respectively, which are not necessarily integers. If u is the exact normalized solution for system (1) we can use it as a trial function for the energy of system (2):

$$E(M, L) \leq \int_0^\infty dr \left[\frac{u'^2}{M} + Vu^2 + \frac{L(L+1)}{Mr^2} u^2 \right] \quad (3)$$

Now we use the well-known inequality⁷⁾

$$\int_0^{\infty} dr \frac{u^2}{4r^2} \leq \int_0^{\infty} dr u'^2 \quad (4)$$

valid for any function such that $u(0) = 0$. We get

$$E(M, L) \leq \int_0^{\infty} dr \left\{ \frac{u'^2}{M} (1 + 4\Delta L) + V u^2 + \frac{L(L+1)}{m r^2} u^2 \right\} \quad (5)$$

valid if

$$\Delta L = L(L+1) - \frac{L(L+1)M}{m} \geq 0 \quad (6)$$

If we make the special choice

$$\frac{M}{m} - 1 = 4\Delta L \quad (7)$$

we get

$$E(M, L) \leq E(m, L) \quad (8)$$

In particular, taking $L = 0$, we obtain

$$E(M, L = \frac{1}{2}(\sqrt{\frac{M}{m}} - 1)) \leq E(m, 0) \quad (9)$$

This inequality has the advantage of being completely general. However, for usual potentials it is far from being saturated; the reason being that inequality (4) is saturated by

$$u = r^{\frac{1}{2} + \epsilon} e^{-r} \quad \epsilon \rightarrow 0 \quad (10)$$

which corresponds to a potential having a singularity like $-1/4r^2$ at the origin.

Numerical experiments with known potentials as the Coulomb potential and the harmonic oscillator potential are rather disappointing. However, inequality (9) contains interesting qualitative features. Take, for instance, the Coulomb case. The crude inequality

$$E(M, 0) \leq E(m, 0) \quad (11)$$

becomes meaningless in the limit $M/m \rightarrow \infty$, while (9) gives

$$-k \frac{4M}{(\sqrt{\frac{M}{m}} + 1)^2} \leq -k m \quad (12)$$

so that we get a discrepancy of a factor 4 in the above limit.

For the harmonic oscillator (12) is replaced by

$$\frac{1}{\sqrt{M}} + \frac{1}{2\sqrt{m}} \leq \frac{3}{2} \frac{1}{\sqrt{m}} \quad (13)$$

and hence we get a factor 3 in the limit $M \rightarrow \infty$, which is qualitatively nice.

However, we conclude that (9) may not be very useful in practical applications to the quark-antiquark system, because everybody believes that the $q\bar{q}$ potential is in between the Coulomb and harmonic oscillator potentials.

3. AN INEQUALITY FOR "GOOD" POTENTIALS

Our definition of "good" potentials will be the same as what we have used in the past⁸⁾; for the time being we shall only impose the following conditions:

$$\begin{aligned} \text{i)} \quad & -\infty < \lim_{r \rightarrow 0} (r V(r)) \leq 0 \\ \text{ii)} \quad & \left(\frac{d}{dr}\right)^3 (r^2 V(r)) \geq 0 \end{aligned} \quad (14)$$

Such potentials have the property that the first $\ell = 1$ state lies below the first excited $\ell = 0$ state, as is observed in charmonium.

A class of potentials satisfying (14) is

$$-\int_{-1}^0 d\alpha r^\alpha \rho(\alpha) + \int_0^\infty d\alpha r^\alpha \sigma(\alpha) \quad \rho \geq 0, \sigma \geq 0. \quad (15)$$

A limiting case of (14) is the Coulomb potential $-g/r$. For the two systems (1) and (2) we take $\ell = 0$ and we adjust L so that $E(M, L)$ coincides with $E(m, 0)$ in the attractive Coulomb case, that means we put $L = \sqrt{M/m} - 1$.

Now the strategy will be essentially the same as in Ref. 8). We consider a family of potentials

$$V(r) = -\frac{g}{r} + \lambda^2 V_c(\lambda r) \quad (16)$$

and we try to prove that

$$\Delta E(\lambda) = E(m, 0, \lambda) - E(M, \sqrt{M} - 1, \lambda) \quad (17)$$

remains positive when λ increases from zero and if V_c satisfies conditions (14).

We shall use the so-called Feynman-Hellmann theorem to compute $d/d\lambda \Delta E(\lambda)$, which involves an integral over $u^2 - v^2$, and a preliminary step will be to study the structure of $u^2 - v^2$. Combining (1) and (2) we have (since $\ell = 0$):

$$I(r) = u'v - v'u \Big|_r = - \int_0^r dr' \frac{u v}{r'} \sigma, \quad \sigma(r) = (M-m)rV + \frac{L(L+1)}{r} - Cr \quad (18)$$

Naturally u and v , being ground-state wave functions, are positive for $0 < r < \infty$. We shall also use the fact that (14) implies convexity for $rV(r)$, which implies convexity of $\sigma(r)$. Since σ goes to infinity at $r = 0$ and for $r \rightarrow \infty$, and $I(\infty) = 0$, we conclude that there are exactly two zeros of the integrand in (18). But this implies that $I(r)$ vanishes only once (say at $r = r_0$). If $u = v$ for some $r < r_0$, $u' - v'$ is negative and hence $u - v$ vanishes at most once in $0 < r < r_0$, and similarly for $r_0 < r < \infty$. Since $u \sim r$ and $v \sim r^{L+1}$, $u > v$ for small r . At large distances

$$\begin{aligned} u &\sim V^{-1/4} \exp - \int_R^r dr' \sqrt{2m(V-E)} \\ v &\sim V^{-1/4} \exp - \int_R^r dr' \sqrt{2M(V-E) + \frac{L(L+1)}{r'^2}} \end{aligned} \quad (19)$$

and since $V \rightarrow \infty$ for $r \rightarrow \infty$, again $u > v$. Hence $u^2 - v^2$ vanishes twice and is positive close to $r = 0$.

Next we use the scaling properties of the potential (16) to conclude

$$2 \Delta E = g \frac{d}{dg} \Delta E + \lambda \frac{d}{d\lambda} \Delta E \quad (20)$$

and note that according to the "Feynman-Hellmann" theorem

$$\frac{d}{dg} \Delta E = - \int_0^\infty dr \frac{u^2 - v^2}{r} \quad (21)$$

$$\frac{d}{d\lambda} \Delta E = \int_0^\infty dr \frac{u^2 - v^2}{r} \frac{d}{d\lambda r} [(\lambda r)^2 V_c(\lambda r)] \quad (22)$$

Using convexity of $d/dr(r^2 V_c)$ we can find $A > 0$ such that

$$\tau(r) = \frac{d}{d\lambda r} [(\lambda r)^2 V_c(\lambda r)] + A + B r \quad (23)$$

takes the same sign as $u^2 - v^2$; so we can conclude (using the normalization condition for u^2 and v^2) that

$$\frac{d}{d\lambda} \Delta E \geq A \frac{d}{dg} \Delta E, \quad A > 0 \quad (24)$$

hence

$$2 \Delta E \leq \left(\lambda + \frac{g}{A} \right) \frac{d}{d\lambda} \Delta E \quad (25)$$

Therefore, as long as ΔE is positive for sufficiently small λ it has to be increasing for all λ . For small λ it is permissible to use perturbation theory to calculate $E(m, 0, \lambda)$ and $E(M, L, \lambda)$ even if the perturbation series does not converge. To justify this one can use the general inequality for the ground state energy of a Hamiltonian $H_0 + \Delta H$ ⁹⁾

$$- \frac{(\psi_0(\Delta H)^2 \psi_0) - (\psi_0 \Delta H \psi_0)^2}{E_2 - (\psi_0 | H_0 + \Delta H | \psi_0)} < E - (\psi_0 | H_0 + \Delta H | \psi_0) < 0 \quad (26)$$

where E_2 is the first excited state of the exact Hamiltonian. In our case, the left-hand side of Eq. (26) exists certainly if V_c increases less fast than any exponential. For the perturbation expression of $\Delta E(\lambda)$ we can repeat the above concavity argument and conclude that $\Delta E(\lambda) > 0$ in first order and remains positive for arbitrarily large λ .

So under condition (14) we have

$$E(M, \sqrt{M/m} - 1) \leq E(m, 0) \quad (27)$$

This inequality is saturated by a pure Coulomb attractive potential. For a harmonic oscillator potential one gets

$$\frac{1}{2\sqrt{M}} + \frac{1}{\sqrt{m}} \leq \frac{3}{2\sqrt{m}} \quad (28)$$

which is not too bad. Let us note that one need not start with the Coulomb potential. Starting, for instance, with the linear potential will give better results for a smaller class of potentials. To apply (27) in practice we have to use some interpolation property of the energy as a function of L since $\sqrt{M/m} - 1$ will not be an integer in general. Here again we have to distinguish between general and "good" potentials. For arbitrary potentials we know that the ground state energy with angular momentum L , $E(M, L)$, is a concave function of $L(L+1)$. This means that if $\frac{1}{2}(\sqrt{M/m} - 1)$ lies between two integers L and $L+1$ one can get [starting from Eq. (9)] an inequality of the form

$$\alpha E(M, L+1) + (1-\alpha) E(M, L) \leq E(m, 0)$$

where the weights are easily calculated. However, that kind of inequality is not very stringent, and we should restrict ourselves to "good" potentials satisfying first of all conditions (14), but also satisfying another familiar condition, which, crudely speaking, means that at large distances the potential does not grow faster than r^2 . It is the concavity condition

$$\left(\frac{d}{dr^2}\right)^2 [2V + r \frac{dV}{dr}] \leq 0 \quad (29)$$

A family of potentials compatible with (14) and (29) is

$$-\int_{-1}^0 d\alpha r^\alpha \rho(\alpha) + \int_0^2 d\alpha r^\alpha \sigma(\alpha) \quad \rho \geq 0, \sigma \geq 0 \quad (30)$$

Potentials of the form (30) have the first D state above the second S state. At this point we want to make a conjecture, which is almost a theorem:

Conjecture:

For potentials satisfying (14) and (29), the ground state energy is a concave function of the angular momentum ℓ

$$\left(\frac{d}{d\ell}\right)^2 E(m, \ell) \leq 0 \quad (31)$$

In particular the potentials (30) should have this property. In the appendix we give all the evidence we have in favour of this conjecture at the present time. Let us just notice that for the r^2 potential the energy depends linearly on ℓ , $E = (\ell + 3/2) \cdot \text{const}$, and this is a limiting case. For the $-1/r$ potential it is obviously satisfied. In what follows we regard conjecture (31) as absolute truth.

4. DISCUSSION AND COMPARISON WITH EXPERIMENT

If we want to apply these considerations to the J/ψ and T families we see that the quark mass ratio, with $m_c \simeq 1.3 - 1.6$ and $M_b \simeq 4.7$ will be such that $L = \sqrt{M/m} - 1 \simeq 0.7$ to 0.9 , so L lies between 0 and 1. Using (31) we get

$$\left(2 - \sqrt{\frac{M}{m}}\right) E_\gamma(\ell=0) + \left(\sqrt{\frac{M}{m}} - 1\right) E_\gamma(\ell=1) < E_{J/\psi}(\ell=0) \quad (32)$$

When the P states of the T will have been observed, it will be possible to put experimental numbers inside this inequality. This is not yet the case and we shall have to use a detour to come into contact with experiment. First of all, we can apply the concavity property using the $\ell = 0$ and $\ell = 2$ states

$$\left(3 - \sqrt{\frac{M}{m}}\right) E_\gamma(\ell=0) + \left(\sqrt{\frac{M}{m}} - 1\right) E_\gamma(\ell=2) < 2 E_{J/\psi}(\ell=0) \quad (33)$$

Second, $E_{2S} < E_D$, from (29). With

$$M_T = 2 M_b + E_\gamma, \quad M_{J/\psi} = 2 m_c + E_{J/\psi}$$

we can turn this into an inequality on the quark masses:

$$M_b - m_c \geq \frac{1}{2} \left[M_T - M_{J/\psi} + \Delta \frac{\sqrt{\frac{M_b}{m_c}} - 1}{2} \right] \quad (34)$$

$$\Delta = E_{\gamma'} - E_\gamma$$

the square root on the right-hand side is not very sensitive to the precise quark mass; with

$$M_{J,\psi} = 3.09, \quad M_T = 9.46, \quad \Delta = 0.59$$

we get

$$M_b - m_c \geq 3.37 \quad (35)$$

while the naive estimate, not using the information on the spacing of the T levels, would give

$$M_b - m_c \geq 3.18 \quad (36)$$

It is, of course, difficult to know whether that kind of exercise is really meaningful because one system ($c\bar{c}$) is more affected by relativistic effects, and also because the potential is not exactly the same. However, we believe that in a more exact treatment one thing should remain: that information on the level spacing allows to control, to a certain extent, the kinetic energies.

Let us also notice that once the P states of the T are known, the corrections to the quark mass difference may turn out to be larger. In the charmonium system the mean p state energy is larger than the average of the J/ψ ($\ell = 0$) and ψ'' ($\ell = 2$) energies.

Let us also note that an increase of the mass of the heavy quark will raise the lower bound on the T leptonic width proposed in Ref. 6) and increase the probability of a charge $1/3$ for the heavy quark.

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APPENDIX

Arguments in favour of the conjecture that for potentials satisfying (14) and (29), $E(m, \ell)$ is a concave function of ℓ .

- 1) It is true for $V = r^2$ and $V = -1/r$.
- 2) It has been checked numerically for $V = r^\alpha$, $0 < \alpha < 2$, $0 \leq \ell \leq 3$ (see the figure).
- 3) Virial theorem argument for

$$V = \int_0^2 d\alpha r^\alpha \rho(\alpha), \quad \rho > 0 \quad . \quad (A.1)$$

Feynman-Hellmann gives

$$\frac{dE}{d\ell} = (2\ell+1) \int_0^\infty dr \frac{u^2}{r^2} \quad . \quad (A.2)$$

By inequality (4) we get

$$\langle T \rangle = \int_0^\infty dr \left[\dot{u}^2 + \frac{\ell(\ell+1)}{r^2} u^2 \right] \geq (\ell + \frac{1}{2})^2 \int_0^\infty dr \frac{u^2}{r^2} \quad . \quad (A.3)$$

The virial theorem gives

$$\langle T \rangle = \int_0^\infty dr \frac{r}{2} u^2 \frac{dV}{dr} \quad (A.4)$$

and for potentials of class (A.1)

$$\langle T \rangle \leq \frac{E}{2} \quad (A.5)$$

Therefore we get

$$\frac{dE}{d\ell} \leq \frac{E}{\ell + \frac{1}{2}} \quad (A.6)$$

Hence

$$E(\ell) \leq (2\ell+1) E(0) \quad (A.7)$$

which means that we have a linear upper bound on the growth of E . This, however, does not prevent $E(\ell)$ from being convex for small values of ℓ .

4) For the same family (A.1) we have

$$\frac{E(L)}{2L+3} \leq \frac{E(\ell)}{2\ell+3} \quad \text{if } L \geq \ell \quad (\text{A.8})$$

Proof: consider a potential

$$V(r) = \omega^4 r^2 + \lambda^2 v(\lambda r) \quad (\text{A.9})$$

such that $v(0) = 0$ and that $v(r)$ satisfies (29). Then, using the same technique as in Section 3 we study the variation of

$$\Delta(\lambda) = \frac{E(\ell, \lambda)}{2\ell+3} - \frac{E(L, \lambda)}{2L+3} \quad (\text{A.10})$$

$\Delta(\lambda)$ vanishes at $\lambda = 0$. Scaling gives again

$$2\Delta = \lambda \frac{d\Delta}{d\lambda} + \omega \frac{d\Delta}{d\omega} \quad (\text{A.11})$$

and as before

$$\frac{d}{d\lambda} \Delta = \int_0^\infty dr f(r) \frac{d}{d\lambda} \lambda^2 v(\lambda r), \quad f(r) = \frac{u_\ell^2}{2\ell+1} - \frac{u_L^2}{2L+1} \quad (\text{A.12})$$

$$\frac{d}{d\omega} \Delta = 4\omega^3 \int_0^\infty dr f(r) \cdot r^2 \quad (\text{A.13})$$

It is easy to show that $f(r)$ vanishes only once because $u_\ell' u_L - u_\ell u_L'$ is always negative; $f(r)$ is positive near $r = 0$ and negative at infinity. Then similar as before we combine the assumed concavity property with (A.12) and (A.13) which gives

$$\frac{d}{d\lambda} \Delta \geq c \cdot \frac{d}{d\omega} \Delta, \quad c > 0 \quad (\text{A.14})$$

to prove that $\Delta(\lambda)$ must be positive increasing, which proves (A.8).

The only trouble is that this does not really prove concavity, and that it is a statement which depends on a particular reference energy^{*)}.

5) Perturbation around $V = r^2$. We take

$$V(r) = r^2 + \lambda \Delta v \quad (\text{A.15})$$

where Δv satisfies properties (14) and (29) and consider

$$F(\lambda) = E(l+\Delta l) + E(l-\Delta l) - 2E(l) \quad (\text{A.16})$$

This expression vanishes for $\lambda = 0$. To first order in λ it is

$$F(\lambda) \cong \lambda \int_0^\infty dr \Delta v(r) g(r), \quad g(r) = u_{l+\Delta l}^2 + u_{l-\Delta l}^2 - 2u_l^2 \quad (\text{A.17})$$

$$u_l = N_l r^{l+1} e^{-\frac{r^2}{2}}$$

so the bracket in the integral is

$$r^{2l+2} e^{-r^2} [A r^{2\Delta l} + B r^{-2\Delta l} - 2C], \quad A, B, C > 0 \quad (\text{A.18})$$

It is easy to see that this bracket vanishes twice and is positive near $r = 0$. On the other hand, clearly

$$\int_0^\infty dr g(r) = \int_0^\infty dr r^2 g(r) = 0 \quad (\text{A.19})$$

Combining (A.17)-(A.19) with the concavity of Δv as a function of r^2 , it is easy to prove that $F(\lambda)$ is negative.

6. What is missing for a real proof? Almost nothing, except that if we use our traditional approach we have to know $F(\lambda)$ and its derivative with respect to λ for potentials (A.15) and these expressions contain

*) We would like to thank Sir Rudolf Peierls for stressing this point.

$$u_{e+\Delta e}^2(r) + u_{e-\Delta e}^2(r) - 2 u_e^2(r) \quad (\text{A.20})$$

It is clear that (A.20) is positive for small r and for r large and that it vanishes somewhere because of the normalization condition. However, we have not been able to prove that it vanishes only twice.

From our conjecture one deduces the property, valid separately for the "charmonium" and the "bottonium"

$$E_P \geq \frac{7}{2} (E_{1s} + E_{2s}) \quad (\text{A.21})$$

if the potential satisfies (14) and (29). Indeed we have the chain of inequalities

$$E_P \geq \frac{7}{2} (E_D + E_{1s}) \quad (\text{A.22})$$

from the conjecture and $E_D \geq E_{2s}$, from Ref. 8). We predict therefore that the mean energy of the P states of T will be above 9.76 GeV.

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FIGURE CAPTION

The ground state energy as a function of the angular momentum for potentials $V = r^\alpha/4$.

FIG.1