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RELATING HARD QCD PROCESSES
THROUGH UNIVERSALITY OF MASS SINGULARITIES - II

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A B S T R A C T

Extending previous techniques we obtain at all orders the factorization of mass singularities for every hard QCD process. These appear in a universal factor that can be reabsorbed into the standard parton density. Thus suitable ratios of cross-sections can be computed by a perturbative expansion in the running coupling constant. Moreover, at the leading log level we obtain, after explicit cancellation of infra-red divergences, the scaling violation of operator product expansion.

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1. - INTRODUCTION

In a previous paper ¹⁾ referred hereafter to as I, we proposed a method for relating different hard QCD processes and their scaling violations through the perturbative study of mass singularities. It was found in I that, at the one-loop level, and for any hard, jet-inclusive ²⁾ cross-section, mass singularities appear in universal (i.e., process independent) factors, one for each incoming particle, multiplying the suitably rescaled lowest order cross-section. Assuming this result to hold at all orders for leading logs and combining it with standard renormalization group (RG) techniques, we could argue that every hard, jet-inclusive cross-section can be obtained by folding the lowest order cross-section [evaluated with the effective QCD coupling constant $\alpha_s(Q^2)$] into universal hadronic parton distributions with the form of leading scaling violations predicted by QCD.

The purpose of this paper is twofold. Firstly we compute leading mass divergences and therefore scaling violations at all orders. Our results will indeed show universality and will coincide explicitly with the form of leading scaling violations as obtained in deep inelastic scattering (DIS) by operator product expansion (OPE). Secondly, we want to extend our analysis to non-leading logs, a matter of great phenomenological importance in the present range of energy and momentum transfers. This is made possible through the insight gained by our explicit identification of the QCD diagrams that provide the leading mass divergences.

As in I we use as a basic ingredient of our method the Lee-Nauenberg-Kinoshita (LNK) theorem ^{3),4)} for cancellation of infra-red and mass singularities. In this paper, however, we choose to work in the axial gauge defined by $n \cdot A = 0$, $n^2 = 1$. The advantages of this gauge over the Feynman gauge, used in Paper I, have become recently clear to several authors ⁵⁾ and will not be discussed again here.

The outline of this paper is as follows. In Section 2 we recall notations, results and conjectures made in I on the basis of one-loop calculations. In Section 3 we outline our proof that leading mass singularities, besides being universal, do reproduce at the leading log level and to all orders the OPE results. In Section 4 we turn to non-leading mass singularities for general hard processes. By studying mass singularities of general Feynman diagrams we shall reach the conclusion that these singularities are universally factorized. This means that scaling violations are

universal and can be learned by the analysis of simple processes. Or, in other words, these ratios of hard cross-sections are free from scaling violations other than those implied by the Q^2 dependence of the running coupling constant.

2. - RESULTS AND CONJECTURES FROM I

We start by recalling notations and results from I. The general hard process P under investigation consists, at the elementary parton level of an arbitrary number (in practice 0, 1 or 2), of initial QCD quanta of momenta p_i , of arbitrary number of final jets described by a set q_i and of external currents (e.g., colour-averaged lepton pairs, W mesons, etc). We denote the colour-averaged cross-section for this jet inclusive process by $\sigma_P(\{p_i\}, \{q_i\})$.

On the basis of the explicit lowest order calculations and of indications from higher orders at the leading $\log m^2$ level, it was conjectured in I that the mass singularities in σ_P would factor out to give ^{*} :

$$\sigma_P(\{p_i\}, \{q_i\}) = \int dx_1 \dots dx_n \prod_{i=1}^n F_i(x_i, Q^2/m_i^2) \sigma_P^{(0)}(\{x_i p_i\}, \{q_i\}) \quad (2.1)$$

leading
log m²

where $\sigma_P^{(0)}$ is the lowest order cross-section for the process under consideration which is regular for $m_i^2 \rightarrow 0$ and $F_i(x_i, Q^2/m_i^2)$ is the universal scale breaking function (the structure function of deep inelastic scattering) Q^2 being the common scale of all large invariants in the hard process. For the non-singlet part, the moments of $F(x, Q^2/m^2)$

$$M_n(Q^2/m^2) = \int_0^1 dx x^{n-1} F(x, Q^2/m^2) \quad (2.2)$$

were expected to be given, at the leading $\log m^2$ level by

$$M_n(Q^2/m^2) = (1 - b \alpha_s \log Q^2/m^2)^{-An/2\pi b} \quad (2.3)$$

leading
log m²

^{*}) This is really the form for flavour non-singlet (valence quark) amplitudes. The treatment of other cases in which quarks and gluons can mix will be treated afterwards.

where the A_n are given by

$$A_n = C_F \int_0^1 dx x^{n-1} P(x) = C_F \int_0^1 dx x^{n-1} \left(\frac{1+x^2}{1-x} \right) \equiv C_F \int_0^1 dx \frac{1+x^2}{1-x} (x^{n-1} - 1) \quad (2.4)$$

A_n are the moments of the usual $q \rightarrow q + g$ decay, probability function. In Eq. (2.4) $C_F = \frac{1}{2} N_c (N_c^2 - 1)$ [if $SU(N_c)$ is the gauge group] and $b = (1/12\pi)(11N_c - 2N_F)$ is the lowest order coefficient of the expansion of the Callan-Symanzik β function for N_F types of quark flavours.

The aim of Section 3 will be to prove Eqs. (2.3), (2.4) from a direct study of QCD diagrams in the axial gauge. Before proceeding to that proof, however, we want to recall that, using the RG equation, it was possible to show that Eq. (2.3) implies for leading $\log Q^2$

$$M_n(Q^2/\mu^2, m^2/\mu^2) = \left[\frac{\alpha_s(Q^2)}{\alpha_s(m^2)} \right]^{-\frac{A_n}{2\pi b}} \quad (2.5)$$

where α_s is the running coupling constant at the leading log level, i.e.,

$$\alpha_s(x) = \frac{\alpha_s}{1 + b\alpha_s \ln x/\mu^2} \quad \text{with} \quad \alpha_s = \alpha_s(\mu^2) \quad (2.6)$$

Equation (2.5) has a factorized form which is of course the one expected from the operator product expansion in deep inelastic scattering.

Once universality of leading mass singularities is established, as it is obvious in the axial gauge, the derivation of Eqs. (2.3), (2.4) simply amounts to a check of OPE results by perturbative methods.

3. - LEADING MASS SINGULARITIES TO ALL ORDERS

The aim of this Section is to identify, in the axial gauge, the QCD diagrams that provide leading mass singularities and to derive from them the basic result contained in Eqs. (2.3) and (2.4).

We shall only discuss in detail the case of a single incoming quark whose line is the one that eventually triggers the hard process (the so-called flavour non-singlet, or valence contribution). Generalizations including gluons; the $q\bar{q}$ sea and their mixing, as well as extension to more than one initial QCD quanta, are straightforward and will be briefly discussed at the end of this Section.

At the lowest order, the process under study has the diagrammatic structure shown in Fig. 1, where the blobs represent the Born approximation $\sigma_P^{(0)}$ to the hard process P (e.g., P =deep inelastic electron scattering with a third jet in the final state).

Our derivation of Eqs. (2.3), (2.4) for the case of a single incoming quark consists of three steps, i.e. :

- a) classification of the diagrams giving leading mass singularities ;
- b) cancellation through Ward identities of the mass singularities coming from some virtual corrections ;
- c) use of the LNK theorem to reduce the mass singularities of virtual corrections on the external quark lines to incoming real states and cancellation of infra-red divergences.

At this point, it is a relatively simple matter to sum explicitly all leading logs of m^2 . We now proceed to sketch these three steps leaving some of the technical details to an Appendix.

a) - Classification of the leading diagrams

We shall claim that only the diagrams of Fig. 2 can provide leading mass singularities in the axial gauge. These are planar ladders with dressed vertices and propagators. This result is probably widely believed by now, after the work of several authors ^{5),6)} on physical gauge, but we shall repeat it for completeness. We shall prove this claim by eliminating other type of diagrams as non leading.

First we shall exclude diagrams in which at least an extra line enters the hard blob (e.g., Fig. 3). The intuitive argument to understand this result can be found for instance in Ref. 5) and is just that the vertex for emission of physical (transverse) gluons from a massless quark vanishes as θ , if θ is the angle between the final quark and the gluon. Since large propagators associated with this collinear bremsstrahlung blow up like $(1-\cos \theta)^{-1} \approx \theta^{-2}$, and phase space goes like $\theta d\theta$, one needs two large propagators for each loop in order to produce a divergence. The diagram of Fig. 2 will contain thus angular integrals such as

$$\int_{\theta_{\min} \approx m^2/Q^2} \theta d\theta \cdot \frac{\theta}{\theta^2} \cdot \frac{\theta}{\theta^2} \approx \int_{\theta_{\min}} \frac{d\theta}{\theta} \approx \log Q^2/m^2 \quad (3.1)$$

as compared to the case of Fig. 3 where the angular integration for the gluon entering the hard blob just gives

$$\int \theta d\theta \cdot \frac{\theta}{\theta^2} = \int d\theta = \text{finite} \quad (3.2)$$

We are thus left with generalized ladder diagrams in which the emitted gluons have momenta nearly parallel to p , say within a cone of solid angle δ around p .

Clearly, such diagrams satisfy the Bethe-Salpeter equation of Fig. 4 with a two-particle irreducible (2PI) kernel. One can show, however, that the only piece of the 2PI kernel (Fig. 5) that can build up leading $\log m^2$ terms is the exchange of a single-dressed gluon propagator (Fig. 5a).

The argument uses again crucially the properties of the axial gauge mentioned above. The crossed ladder kernel of Fig. 5b, for instance, can be proved to be free of mass singularities by the following argument. In the limit of zero mass the corresponding skeleton diagram can be only a function of (P_1^2/P_2^2) since, being ultra-violet convergent, it cannot depend on the ultra-violet cut-off Λ . On the other hand, a divergence of the type $\log(P_1^2/P_2^2)$ is ruled out by the fact that, when $P_1^2 \neq 0$, $P_2^2 \rightarrow 0$ the diagram is still finite by the same argument made for the diagram of Fig. 3 through Eqs. (3.1), (3.2). Exactly the same argument applies to the diagram of Fig. 5c and in general to all possible 2PI kernels ⁷⁾. The only possible source of extra $\log P_1^2$, $\log P_2^2$ singularities lies in propagator and vertex correction diagrams since these are the only diagrams which can be ultra-violet

divergent in a renormalizable theory. Indeed the vertex correction of Fig. 6 can be shown to behave as $\log(\Lambda^2/P_1^2)$ if $m^2=0$ and $P_2^2 \ll P_1^2$. It is again a property of the axial gauge that the vertex correction is finite whenever at least one of the external particles is off mass shell. The argument [see Ref. 5] is again based on counting powers of small angles.

The above arguments reduce possible leading kernels to the one of Fig. 5a and therefore leading diagrams to the dressed planar ladders of Fig. 2, as anticipated.

Before proceeding to the actual evaluation of these ladders, however, we will use the simple form of the Ward identity valid in the axial gauge to reduce further on the number of relevant insertions. On the other hand, the arguments made up to here would hold in any physical gauge and not just in the axial gauge.

Use of the Ward identity is made possible by the observation that leading mass singularities come from the kinematical region of integration in which the P_i^2 are strongly ordered ⁶⁾⁸⁾, i.e. :

$$Q^2 \gg P_i^2 ; \quad P_i^2 \gg P_{i+1}^2, (P_i - P_{i+1})^2 \quad (3.3)$$

Indeed the region in which, for instance, $P_2^2 \gg P_1^2$ is kinematically forbidden and the one in which $P_2^2 \simeq P_1^2$ cannot give rise to leading logarithms.

b) - Cancellation of some virtual diagrams via Ward identity and evaluation of the ladders

In the leading diagrams of Fig. 2, let us pair each dressed quark propagator $S_F'(P_i)$ with the dressed vertex next to it down the chain $\Lambda_\mu'(P_i, P_{i+1})$. We shall now show that leading mass singularities due to the dressing cancel among each pair thanks to strong ordering [Eq. (3.3)] and to the simple form of the Ward identity in this gauge. The net effect is that, for leading logs, we can set :

$$S_F'(P_i) \Lambda_\mu'(P_i, P_{i+1}) \rightarrow S_F(P_i) \Lambda_\mu(P_i, P_{i+1}) \quad (3.4)$$

where S_F and Λ are the bare vertices and propagators. Equation (3.4) is illustrated in Fig. 7.

The crucial point is that, as we have previously pointed out, the vertex Λ'_μ , in the relevant strongly-ordered region P_{i+1}^2 , $(P_i - P_{i+1})^2 \ll P_i^2$, has only possible singular terms of the type $\alpha \log(\Lambda^2/P_i^2)$ where Λ is the ultra-violet cut-off. Exactly the same behaviour is expected of the dressed quark propagator $S'_F(P_i^2) \sim \alpha \log \Lambda^2/P_i^2$. Indeed, in the axial gauge, it is known ⁸⁾ that $\log \Lambda$ terms always multiply bare vertices and propagators. Other forms of vertex and propagator involving the vector n defining the gauge, will not contain $\log \Lambda$ and therefore will not contain $\log P_i^2$ either. This implies that they will not contribute to leading logs. Furthermore, in the axial gauge, the Ward identity looks the same as in QED (there are no Faddeev-Popov ghosts!) and give, in standard notations, $Z_1 = Z_2$. We then have, for $P_i^2 \gg P_{i+1}^2$:

$$\Lambda'_\mu(P_i, P_{i+1}) = \gamma_\mu (1 + \alpha_s \zeta_1 \log \Lambda^2/P_i^2) + (\text{finite})_{O(\alpha_s)}$$

$$S'_F(P_i^2) = \frac{1}{\not{P}_i - m} (1 + \alpha_s \zeta_2 \log \Lambda^2/P_i^2) + (\text{finite})_{O(\alpha_s)} \quad (3.5)$$

with the numbers z_1, z_2 related to the $O(\alpha_s)$ renormalization constants Z_1, Z_2 by

$$Z_1^{-1} = (1 + \alpha \zeta_1 \log \Lambda^2/\mu^2) ; \quad Z_2 = (1 + \alpha \zeta_2 \log \Lambda^2/\mu^2)$$

Hence $Z_1 = Z_2$ implies $z_1 + z_2 = 0$. We then conclude that

$$S'_F \Lambda'_\mu = \frac{\gamma_\mu}{\not{P}_i - m} (1 + \alpha (\zeta_1 + \zeta_2) \log \Lambda^2/P_i^2) + \text{finite} = S_F \Lambda_\mu + O(\alpha) \quad (3.6)$$

where the $O(\alpha)$ is non-leading in terms of mass singularities. This is the desired result ^{*)} of Eq. (3.4).

*) A more general but somewhat more formal proof of Eq. (3.6) can be obtained directly from the Ward identity in the limit $P_i^2 \gg P_{i+1}^2$.

At this point, the pair wise cancellation of propagators and vertices has left us with the mass insertions on the last (external) legs, providing an extra factor of Z_2 for the S matrix (i.e., Z_2^2 times Z_2^{-1} from the definition of scattering amplitude) and otherwise we have to compute the ladders of Fig. 8 where only the horizontal rungs are dressed. We do this first starting with the case of one bare rung. In this case a very simple calculation reproduces the well-known ¹⁾ one-loop result

$$\sigma^{1\text{rung}}(p) = \frac{\alpha_s}{2\pi} \log \frac{Q^2}{m^2} \int_0^1 \frac{dz}{z} (1+(1-z)^2) \sigma^{(0)}((1-z)p) \quad (3.7)$$

For the ladder with m bare rungs the result is also very simple :

$$\sigma_P^{(m \text{ bare rungs})}(p, \dots) = \frac{1}{m!} \left(\frac{\alpha_s C_F \log Q^2/m^2}{2\pi} \right)^m \int_0^1 \prod_{i=1}^m \frac{dz_i}{z_i} (1+(1-z_i)^2) \sigma_P^{(0)}((1-z_1)\dots(1-z_m)p, \dots) \quad (3.8)$$

where the $1/m!$ factor (leading to exponentiation for the moments) comes from the strong ordering conditions of Eq. (3.3).

Such exponentiation is not affected by the dressing up of the rungs (strong ordering is still true !) and we can therefore discuss the effect of such a dressing at the one-rung level. Renormalizing the dressed gluon propagator at some Euclidean point $q^2 = -\mu^2$ by

$$\mathcal{D}'_{\mu\nu}(q^2 = -\mu^2) = Z_3(\Lambda^2/\mu^2, \alpha_s) \mathcal{D}_{\mu\nu}(q^2 = -\mu^2) \quad (3.9)$$

where $\mathcal{D}_{\mu\nu}$ is the bare axial gauge propagator, we know that in this gauge

$$Z_3 = 1 + b \alpha_s \log \Lambda^2/\mu^2 \quad (3.10)$$

where b is exactly the β function coefficient of Eq. (2.3) (this is again related to the fact that $Z_1 = Z_2$ in the axial gauge). The logarithmic q^2 dependence in the propagator will be again associated with the Λ dependence and will multiply the bare propagator expression to give, for leading log singularities the effective form

$$\alpha_s^{(0)} \mathcal{D}'_{\mu\nu} = \alpha_{s,\mu} [1 + \alpha_{s,\mu} b \log(-q^2/\mu^2)]^{-1} \mathcal{D}_{\mu\nu} \quad (3.11)$$

where $\alpha_s^{(0)}$ and $\alpha_{s,\mu} = Z_3 \alpha_s^{(0)}$ are the bare and μ renormalized coupling constants, respectively. The explicit calculation, given in the Appendix, then shows that the effect of replacing bare rungs with dressed rungs is simply that of replacing

$$\frac{\alpha_s}{2\pi} \log Q^2/m^2 \quad (3.12a)$$

by

$$-\frac{1}{2\pi b} \log(1 - \alpha_{s,\mu} b \log Q^2/m^2) = \frac{1}{2\pi b} \log \left[\frac{\alpha_s(m^2)}{\alpha_s(Q^2)} \right]_{\substack{\text{leading} \\ \log m^2}} \quad (3.12b)$$

At this point our result, including the extra Z_2 factor from the external leg renormalization, reads :

$$\sigma_P(p, \dots) = \int_0^1 dx F(x, Q^2/\mu^2, m^2/\mu^2) \sigma_P^{(0)}(xp, \dots) \quad (3.13)$$

where

$$F(x, Q^2/\mu^2, m^2/\mu^2) = Z_2 \left[\delta(1-x) + \sum_{m=1}^{\infty} \left(\frac{1}{2\pi b} \log \left(\frac{\alpha_s(m^2)}{\alpha_s(Q^2)} \right) C_F \right)^m \frac{1}{m!} \right. \\ \left. \times \int_0^1 \prod_i f(x_i) dx_i \delta(x - \prod x_i) \right] \quad (3.14)$$

where $f(x) = (1+x^2)/(1-x)$

The moments $M_n(Q^2)$ of F then exponentiate to give

$$M_n(Q^2) = Z_2 \left[\frac{d_s(m^2)}{d_s(Q^2)} \right] A_n^R / 2\pi b \quad (3.15)$$

where

$$A_n^R = C_F \int_0^{1-\Delta E/E} dx f(x) = C_F \int_0^{1-\Delta E/E} dx \frac{1+x^2}{1-x} \quad (3.16)$$

is the real bremsstrahlung contribution to the anomalous dimension A_n and contains an infra-red divergence for $\Delta E \rightarrow 0$ (ΔE here, as in I, is a lower cut-off defining hard gluons). We shall finally show how to evaluate the leading mass divergence of Z_2 and how this amounts to replace A_n^R by the full A_n of Eq. (2.4) leaving us with a perfectly infra-red finite and ΔE independent result.

c) - Computation of virtual corrections via the LNK theorem and cancellation of infra-red divergences

As we said the effect of dressing the last quark propagator gives an extra factor Z_2 . Since Z_2 is a number (independent of Q^2) it can be computed independently of the process under consideration. We shall use the graph of Fig. 9a as a trick to obtain Z_2 via the LNK ³⁾ theorem.

The process of Fig. 9a has no forward going jet in the final state (it is a completely virtual correction to the Born term). By the LNK theorem such a process is made finite if we add to it fictitious processes containing extra incoming quanta to make up a degenerate initial state as shown in Figs. 9b,c.

The non-singular sum of all these diagrams is $\sigma_p^{(0)}$, up to corrections of $O(\alpha_s)$. Notice that the finiteness of the sum follows immediately by using the double-cut diagram technique of Nakanishi ⁴⁾. Contributions in Fig. 9 correspond to pole, two-particle cut, three-particle cut, etc., obtained by cutting the same double cut diagram with a dressed incoming quark propagator.

In analogy with our treatment of real bremsstrahlung, the only contributions to leading logs of m^2 in Figs. 9b,c, etc., come from the planar ladder diagrams of Fig. 10 and from the strongly ordered region of integration (3.3). The identification of the finite sum with $\sigma_P^{(0)}$ has been explicitly shown in Fig. 10, which reads, in formulae

$$Z_2 \left(\sum_{k=0}^{\infty} \sigma_k \right) = \sigma^{(0)} \quad (3.17)$$

where σ_k is the cross-section given by the ladder with k incoming (dressed) rungs. We notice now that, by the LNK theorem, the initial states have all total momentum p . The Born cross-section that factorizes out in each σ_k is therefore always $\sigma^{(0)}(p)$ independently of k . Apart from this fact the calculation of these "incoming ladders" is similar to the previous one and gives

$$\frac{\sigma_k}{\sigma^{(0)}} = \frac{1}{k!} \left[\frac{1}{2\pi b} \log \left(\frac{d_s(m^2)}{d_s(Q^2)} \right) A^v \right]^k \quad (3.18)$$

where

$$A^v = C_f \int_0^{1-\Delta E/E} dx \frac{1+x^2}{1-x} \quad (3.19)$$

We get therefore, from Eq. (3.17) ^{*)} :

$$Z_2 = \left(\sum_k \sigma_k / \sigma^{(0)} \right)^{-1} = \left[\frac{d_s(m^2)}{d_s(Q^2)} \right]^{-A^v/2\pi b} \quad (3.20)$$

This interesting formula for Z_2 coincides with one recently obtained by Frenkel and Taylor ⁵⁾ in the Coulomb gauge if only leading $\alpha \log Q^2/m^2 \times \log \Delta E/E$ terms are kept.

Inserting Eq. (3.20) into Eq. (3.15), we immediately get

^{*)} In reality, Eq. (3.20) introduces in Z_2 a spurious Q^2 dependence which cannot be reliably computed in our method. The correct way to get the full Q^2 dependence of the cross-section uses the RG equation and is discussed in I.

$$M_n(Q^2/\mu^2, m^2/\mu^2) = \left[\frac{d_s(m^2)}{d_s(Q^2)} \right]^{(A_n^R - A_n^V)/2\pi b} = \left(\frac{d_s(m^2)}{d_s(Q^2)} \right)^{A_n/2\pi b} \quad (3.21)$$

with $A_n = A_n^R - A_n^V$ given by Eq. (2.4). As stressed in I, the ΔE dependence drops out completely in A_n , checking explicitly the cancellation of infrared divergences among real and virtual soft bremsstrahlung.

An amusing interpretation of Eq. (3.21) is that both real and virtual corrections are represented by a sum of real processes to be computed in a planar ladder approximation. Each rung of the ladder can be either an outgoing collinear (dressed) gluon or an incoming collinear (dressed) gluon in which latter case its contribution is reversed in sign (from the LNK trick).

Summarizing, we have shown how the well-known OPE expression of leading scaling violations follows from a detailed study of leading mass singularities of QCD diagrams in the axial gauge. The diagrams that actually provide these leading logs are dressed planar ladders with both real and virtual gluons confined inside a narrow (but finite) solid angle cone around the direction of the incoming quark.

Generalization of this result to arbitrary processes involving a single QCD incoming quantum is straightforward. Again, only planar ladders survive, but now each (vertical or horizontal) line in the ladder can be a quark, an antiquark or a gluon.

The result is mixing of $q\bar{q}$ sea and gluons. In other words the coefficients A_n now become matrices A_n^{ab} , to be interpreted as the moments of the probability $P_{a \rightarrow b}(x)$ that parton b comes out of factor a with a fraction x of the original momentum

$$A_n^{ab} = \int_0^1 dx x^{n-1} P_{a \rightarrow b}(x) \quad (3.22)$$

Explicit formulae for the various $P_{a \rightarrow b}(x)$ probabilities can be found in Ref. 9).

In this case a simple result, similar to that of Eq. (3.4) holds for the process independent (but of course n dependent) eigenstates of A_n . For these, Eq. (3.20) holds with A_n replaced by the particular eigenvalue of A_n^{ab} .

The other simple generalization of our results is for processes involving several (in practice two) incoming quanta. One can easily see that, for the same reasons for which gluons entering the hard blobs gave non-leading contributions, now, on top, gluons exchanged between the two incoming lines also give non-leading logs. The fact therefore that leading mass singularities factor out into a product of terms one for each incoming quantum is a trivial consequence of the use of the axial gauge. Universality of leading $\log m^2$ corrections is thus guaranteed.

Yet another extension of our techniques is to less inclusive processes in the final state. If, for instance, a single particle spectrum inside the final jet is looked at, new mass singularities (and scaling violations) originate. The generalization of Eq. (3.13) then becomes

$$\sigma(p, q, \dots) = \int dx dy F(x, Q^2) G(y, Q^2) \sigma^{(0)}(xp, q/y, \dots) \quad (3.23)$$

where $G(y, Q^2)$, the fragmentation function, obeys the same evolution equation that F obeys with A_n^{ab} replaced by A_n^{ba} .

Using and generalizing the results of this paper, it was also possible ¹⁰⁾ to obtain a possible extension of this method to predict arbitrary multiparticle spectrum inside a single QCD jet, with a very simple and elegant algorithm emerging.

A final remark concerns the comparison of our method based on m^2 singularities with other techniques. Several authors ^{6), 11)} have analyzed leading $\log Q^2$ terms. The advantage of mass singularities comes from the fact that, in this gauge, process dependent diagrams with gluons entering the hard blobs are non-leading. The same is not true for $\log Q^2$ terms. On the other hand, we have seen that no information was lost by just looking at m^2 singularities since they imply a well-defined Q^2 dependence via RG equation constraints. In particular, our simpler technique allows us to master infra-red singularities and to control the constant terms in anomalous dimensions.

Frenkel and Taylor ⁵⁾ studied exclusive processes by analyzing leading mass and infra-red singularities. Their results are included in our expressions for virtual corrections. The mass singularity method for inclusive processes has also been used, recently, by Frenkel, Sheiler and Taylor ⁵⁾ in a careful sixth order computation of the Drell-Yan process.

4. - MASS SINGULARITIES AT NON LEADING ORDERS.

We now want to discuss the origin of mass singularities at non-leading orders, concentrating again first on a single quark generated process. We shall do it by observing which are the non-leading effects coming from the dominant diagrams of Fig. 8, and, subsequently, those arising by modification of these leading diagrams. We shall find that all contributions can be grouped in the form ^{*)}

$$\sigma_P(p, \{q_i\}) = \int H^P(xp, \{q_i\}) U(x, \frac{m^2}{Q^2}) dx \quad (4.1)$$

expressed in Fig. 11 where U is a universal function and contains all mass singularities. It contains, as intermediate states, an arbitrary number of collinear quanta where collinearity is defined to be contained in a cone of angle δ around p arbitrarily small but fixed. Moreover, we shall show that U is generated only by leading singularities in all angular integrations.

On the other hand, H^P is a hard process dependent absorptive amplitude free from mass singularities. It will be defined as containing outgoing quanta away from the collinear cone defined by δ . In order to establish this fact we shall systematically investigate the sources of non-leading contributions in diagrams organized as in Section 3. We shall confine the discussion to real bremsstrahlung diagrams. Virtual diagrams which were explicitly shown to eliminate soft gluon singularities in Section 3 will not be considered here.

^{*)} After completion of this work, we have received two papers (R.K. Ellis, H. Georgi, M. Machacek, H.D. Politzer and G.G. Ross, MIT Preprint CTP718, and S.B. Libby and G. Sterman, Stony Brook Preprint ITP-SB-78-41) in which the same factorization property is argued to hold.

Let us start recalling that the leading contributions came from the diagram of Fig. 8 in a collinear configuration and arose from an integration region satisfying the strong ordering of Eq. (3.3). Let us concentrate on the i^{th} integration (over θ_i as discussed in Section 3) and try to identify the origin of non-leading contributions. The i^{th} integration was of the form

$$\int_{\theta_{i+1}}^{\delta} \frac{\theta_i d\theta_i}{(\theta_i^2)^2} \theta_i^2 (\log \theta_i)^{i-1} \rightarrow (\log \theta_{i+1})^i \quad (4.2)$$

where the $\log \theta_i$ come from integrations over θ_j ($j < i$) and the powers of θ were explained with care in Section 3. Non-leading contributions in this i^{th} integration come from two sources : one originates from non-leading angular singularities, i.e., higher powers of θ_i in the integrand of (4.2), the other originates from m^2 terms in the integrand. In this last case, we shall lose the θ_i^2 in the numerator of the integrand in (4.2) : it was indeed generated by the impossibility of having our helicity flip for massless particles. We expect therefore, in place of Eq. (4.2)

$$\frac{m^2}{Q^2} \int_{\theta_{i+1}}^{\delta} \frac{d\theta_i^2}{(\theta_i^2)^2} (\log \theta_i)^{i-1} \propto \frac{m^2}{Q^2} \frac{(\log \theta_{i+1})^{i-1}}{\theta_{i+1}^2} \quad (4.3)$$

This form reproduces itself up to the last (θ_n) integration to give

$$\frac{m^2}{Q^2} \int_{m^2/Q^2}^{\delta} \frac{d\theta_n^2}{(\theta_n^2)^2} (\log \theta_n)^{i-1} \propto \left(\log \frac{m^2}{Q^2} \right)^{i-1} \quad (4.4)$$

We have therefore lost all logarithms from the i^{th} rung up to the last. The other mechanism mentioned before, i.e., non-leading θ_i integration, implies, in place of Eq. (4.2), the form

$$\int_{\theta_{i+1}}^{\delta} \frac{d\theta_i}{\theta_i} (\log \theta_i)^{i-1} \theta_i f(\theta_i) \quad (4.5)$$

where $f(0)$ is regular. The integral is finite for $\theta_{i+1} = 0$ and therefore we lose the whole chain of logarithms for all $j < i$. This claim giving a regular contribution can be lumped into H^P . We notice, moreover,

that the integral of Eq. (4.5) is proportional to δ (fixed but arbitrarily small) and therefore gives rise to a negligible contribution. The same happens if an extra factor θ appears in whatever integration in the chain of Eqs. (4.2) and (4.3) giving rise to non-leading contributions. Summarizing, all the mass singularities from diagram 8 come from leading angular integrations in the collinear configurations and are lumped in U of Eq. (4.1)

Let us now discuss diagram 8 in a non-fully collinear situation, i.e., in which at least a gluon (call it i^{th}) is out of the δ cone. Then p_i^2 is large and therefore all p_j^2 ($j < i$) are bounded by it and all singularities coming from them are lost. A hard non-collinear gluon lumps therefore everything lying between it and the hard process into a mass non-singular amplitude included in H^P of Fig. 11. The same is true if any other intermediate state, coming from the gluon bubbles of Fig. 2, ceases to be collinear.

We turn now our attention to diagrams other than those of Fig. 8. We discuss first the effect of a line that connects the ladder with the hard part of the diagram as exemplified in Fig. 12. The 2PI subdiagram is free from mass singularities and is directly part of H^P . What remains is the insertion into the ladder of 2PI diagrams. For the next to leading, the only possible insertion is that of Fig. 13.

It is a simple matter to see that this insertion originates a factor $\alpha^2 \log m^2$. The integral over the angles θ of the two crossing gluons causes four propagators to vanish in the collinear configuration, two of momenta p_i^2 and the two inside the diagram. The counting of powers seems similar to that of the standard ladder where the two gluons in question do not cross. Nevertheless, the leading angular singularity is softer in the sense that both θ_i must go together to zero in order to generate a singularity from the diagram of Fig. 13. In other words, the end-point singularity is controlled by a single variable and this causes a single power of a logarithm to stem, contrary to the double power coming from the ladder.

This is true for other insertions giving rise to higher non-leading contributions. In other words, the collinear configuration is again universal and can be lumped into U while, again, a non-collinear configuration kills all mass singularities originated from above it.

H^P will contain many different contributions, all free of mass singularities, which we shall again summarize. Let us focus our attention to the emitted quanta (gluon or quark) that represent the border between H and U . All quanta below it are in U and therefore are in the forward cone defined by δ , contribute to the leading angular singularity, and are both emitted and absorbed by quanta within δ . Therefore the one under consideration is the first one found, ascending the diagram from below, which is not of that kind. This means that either : i) it is a hard non-collinear quantum (outside δ), ii) it comes from non-leading angular singularities as should be, for instance, the case for a spin-flip amplitude, iii) it is either emitted or absorbed by a non-collinear quantum.

At this stage, H^P depends on what we call collinear and therefore on the parameter δ defining the aperture of the forward cone. We shall nevertheless dispose of this dependence in comparing different processes.

The dependence on δ will be singular (of the $\log \delta$ form). We notice, however, that, apart from terms which vanish as δ , one can restrict all quanta in H^P to lie outside the forward cone defined by δ .

The afore-mentioned fact that U does not contain a spin flip guarantees that H^P is a forward non-spin-flip absorptive amplitude. Besides it represents the trace over the quark colour indices and therefore has the characteristics of a cross-section. H^P whose properties have been summarized can be computed by an expansion in terms of the running coupling constant $\alpha_s(Q^2)$. As said before, U represents the fragmentation function of a quark of momentum p into another quark of momentum xp and a forward jet (within δ) of momentum $(1-x)p$. We can therefore interpret the integrand of Eq. (4.1) as the cross-section for finding a fraction $1-x$ of the momentum in the forward jet, i.e.,

$$\sigma_P(x, p, \{q_i\}) = H^P(x, p, \{q_i\}, \alpha(Q^2)) U(x, m^2, \alpha(Q^2)) \quad (4.6)$$

where, as usual, $\{q_i\}$ specify the (non-forward) jet momenta of the hard multijet process P .

At leading $\alpha(Q^2) \log m^2$, H^P is the Born cross-section and the moments of U have the form of Eq. (3.21). For processes generated by two (or more) quarks or antiquarks, as Drell-Yan, the preceding arguments give

$$\sigma_P(x_1, x_2, \{q_i\}) = H^P(x, p_1, x_2 p_2, \{q_i\}, d(Q^2)) * U(x_1, m^2, d(Q^2)) * U(x_2, m^2, d(Q^2)) \quad (4.7)$$

where $1-x_1$ and $1-x_2$ are the fraction of momenta in the jets collinear with the two incoming quarks of momenta p_1 and p_2 , respectively. Again H^P is the naïve parton model cross-section.

Let us now apply Eq. (4.6) to a deep inelastic scattering process which is light-cone dominated and for which we can apply a light cone operator product expansion. Considering the moments with respect to x , we know that, for every moment, the expression totally factorizes in a factor which depends on m^2 (the operator matrix element) and a factor which carries the Q^2 dependence (the coefficient). This implies that every moment $U^n(m^2, \alpha(Q^2))$ of $U(x, m^2, \alpha(Q^2))$ factorizes into

$$U^n(m^2, \alpha(Q^2)) = W^n(Q^2/\mu^2, \alpha_s) * V^n(m^2/\mu^2, \alpha_s) \quad (4.8)$$

Of course, U being universal, the factorization of Eq. (4.8) is also universal. Therefore Eq. (4.6) implies that the factorization typical of operator product expansion is a characteristic of all hard processes valid also for those that are not light-cone dominated. And, of course, the mass square dependence is universally factorized out, and, for every incoming hadron, it represents the matrix elements of the twist-two operators appearing in the light-cone expansion. In other words, the m^2 dependence can be fully absorbed in the universal hadron standard function (universal in the sense that it depends on the hadron involved but not on the process to which it participates) as first conjectured by Politzer¹³⁾.

Let us now compare arbitrary processes to deep inelastic scattering. For a single hadron generated process (as a longitudinal ep scattering or an ep scattering with n final jets)

$$\sigma^P(x, p, \{q_i\}) = \sigma^{DIS}(x, Q^2) \frac{H^P(xp)}{H^{DIS}(xp)} \quad (4.9)$$

or for a two-hadron generated process, as Drell-Yan

$$\sigma^P(x_1, x_2, p_1, p_2, \{q_i\}) = \sigma_{(x_1, Q^2)}^{\text{DIS}} \sigma_{(x_2, Q^2)}^{\text{DIS}} \frac{H^P(x_1, p_1, x_2, p_2)}{H_{(x_1, p_1)}^{\text{DIS}} H_{(x_2, p_2)}^{\text{DIS}}} \quad (4.10)$$

so that all hard cross-sections with arbitrary multijet final states can be calculated in terms of deep inelastic scattering cross-sections [i.e., the structure function $F(x, Q^2)$] and ratios of naïve parton model cross-sections. We notice that both these contributions carry scaling violations at the non-leading level. In the case of the ratio of H 's they appear only through $\alpha(Q^2)$.

We said before that H^P depended singularly on the parameter δ that defines what is collinear (the forward jet angle). However, in the ratios of Eq. (4.9) and Eq. (4.10) this singularity compensates - it is again universal - so that in Eqs. (4.9) and (4.10), δ can be set to be arbitrarily small.

Let us finally discuss the dependence on the gauge parameter n which is expected to appear in both real and virtual contributions. On the one hand, cross-sections in Eqs. (4.6) and (4.7) are n independent (gauge invariance of the S matrix). This is certainly true if the processes under consideration would involve on-shell quarks of mass m^2 . For composite hadrons, this will remain true after the off-mass-shell dependence ^{*}) - which in itself is not gauge invariant - is convoluted with the physical hadron bare quark distribution. The gauge dependence of the universal factor U in Eqs. (4.6) and (4.7) must therefore compensate that of H^P . This implies that the ratios of H 's functions appearing in (4.9) and (4.10) are n independent ¹⁴⁾. Also, from Eq. (4.8) V^n as the matrix element of a gauge invariant operator between physical (quark or hadron) states is n independent. Therefore the product of W^n and H^{Pn} [the n th moment of $H^P(x)$] is n independent even if each of the two factors is not. For deep inelastic scattering this corresponds to the statement that only the product of the coefficient and the anomalous dimension exponential in the usual OPE decomposition

^{*}) We thank C. Sachrajda for a discussion concerning this point.

$$M_n(Q^2) = C^n(\alpha_s(Q^2)) \cdot \exp\left[-\int d\lambda \frac{\delta^n}{\beta}\right] \quad (4.11)$$

is gauge invariant.

5. - CONCLUDING REMARKS

The extension of the arguments developed before to the flavour singlet case follows from the pattern already discussed in Section 3 for the leading case. U will become an evolution matrix which, for every momentum n , can be properly diagonalized. The factorization will hold for the eigenstates of the evolution matrix : Eq. (4.7) will be generalized into

$$\begin{aligned} \sigma(xp, \{q_i\}) = & H_I^P(xp, \{q_i\}) U^I(xp, m^2) + \\ & + H_{II}^P(xp, \{q_i\}) U^{II}(xp, m^2) \end{aligned} \quad (5.1)$$

where U^I and U^{II} are the eigenvalues of the evolution matrix : in the language of OPE, the corresponding eigenstates are multiplicatively re-normalizable.

All mass singularities are still contained in U^I and U^{II} . However, by comparison with deep inelastic scattering, one can see that, inside U^I and U^{II} , mass singularities are not factorized. Indeed, U^I and U^{II} , taking the moments of Eq. (3.1) and fixing n , split into a linear combination of two factorized expressions each of them corresponding to the matrix elements of the two independent twist-two flavour singlet operators. In this case, mass singularities will be re-absorbed in two distinct parton distributions : the sea and the gluons. As for the flavour non-singlet case, predictions made by sea and gluon distributions normalized in a process will be free from mass singularities when extended to different processes.

We see therefore that the universality of mass singularities enables us to justify, in the framework of QCD and for all hard processes leading to final jets, a parton model-like result. This implies that cross-sections can be computed by an expansion in $\alpha_s(Q^2)$ - the running coupling constant - folded into scaling violating parton densities whose normalization can be learned in a process and universally used in any other one.

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APPENDIX

In this Appendix, we show that the effect of dressing the gluon propagators in the ladder of Fig. 8 changes Eq. (3.12a) into Eq. (3.12b).

As argued in the text, dressing the gluon propagator at the one-loop level corresponds to replace $\alpha_s^{(0)} D_{\mu\nu}$ with

$$\alpha_s^{(0)} D_{\mu\nu}' = \frac{\alpha_{s,\mu}}{[1 + \alpha_{s,\mu} b \log(-q^2/\mu^2)]} D_{\mu\nu} \quad (\text{A.1})$$

The one-loop integral for the single-dressed rung ladder is then obtained by the replacement :

$$\frac{\alpha_s}{2\pi} \log Q^2/m^2 \rightarrow \frac{1}{2\pi} \int_0^{Q^2} dq^2 \log\left(\frac{Q^2}{q^2 m^2}\right) \frac{\text{Disc}}{2\pi i} \left[\frac{\alpha_{s,\mu}}{(-q^2) [1 + b \alpha_{s,\mu} \log(-q^2/\mu^2)]} \right] \quad (\text{A.2})$$

The easiest way to evaluate this integral is to write it as a contour integral around its right-hand cut (ghost contribution included !) and then close it, using Cauchy, on its left-hand cut (from $-Q^2$ to $-m^2$) throwing away a non-leading contribution from the large circle $|q^2| = Q^2$. One then gets

$$\begin{aligned} \frac{\alpha_s}{2\pi b} \log Q^2/m^2 &\rightarrow \frac{1}{2\pi} \int_{-Q^2}^{-m^2} \frac{dq^2}{(-q^2)} \frac{\alpha_{s,\mu}}{1 + \alpha_{s,\mu} b \log(-q^2/\mu^2)} = \\ &= \frac{1}{2\pi b} \int_{\alpha_{s,\mu} b \log m^2/\mu^2}^{\alpha_{s,\mu} b \log Q^2/\mu^2} \frac{dx}{x} \frac{1}{1+x} = \frac{1}{2\pi b} \log \left(\frac{1 + \alpha_{s,\mu} b \log Q^2/\mu^2}{1 + \alpha_{s,\mu} b \log m^2/\mu^2} \right) = \\ &= \frac{1}{2\pi b} \log \left(\frac{\alpha_s(m^2)}{\alpha_s(Q^2)} \right) \quad (\text{A.3}) \end{aligned}$$

This is the desired result. At order α it coincides, of course, with the result of the bare ladder. At $O(\alpha^2)$ the result has been double checked by computing the explicit fourth order (two-loop) diagram both by the Cauchy trick and by direct computation of the single- and two-particle contributions to the discontinuity of the propagator. Note that, whereas the total contribution of Eq. (A.3) is positive irrespectively of the sign of b , the sign of the $O(\alpha^2)$ correction is controlled by b . In QCD, $b > 0$ and the two-gluon intermediate state gives, from this diagram, a negative contribution to the cross-section.

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FIGURE CAPTIONS

- Figure 1 The lowest order structure of a hard QCD process initiated by a quantum of momentum p .
- Figure 2 Generalized ladder diagrams which provide leading mass singularities in the axial gauge.
- Figure 3 A non-leading diagram in the axial gauge.
- Figure 4 Bethe-Salpeter equation for the generalized ladder diagrams of Fig. 2.
- Figure 5 Structure of two-particle irreducible (2PI) diagrams.
- Figure 6 A vertex correction.
- Figure 7 The Ward identity for leading virtual corrections in the axial gauge.
- Figure 8 Generalized ladder diagrams where only horizontal rungs are dressed.
- Figure 9 LNK theorem for Z_2 .
- Figure 10 Generalized ladder diagrams with all incoming quanta which provide the leading contributions to Z_2 .
- Figure 11 Diagrammatic illustration of the general factorization of mass singularities for an arbitrary hard QCD process P .
- Figure 12 Non-ladder diagram contributing to non-leading mass singularities.
- Figure 13 Insertion of a 2PI subdiagram giving rise to next-to-leading mass singularities.

FIG. 1

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