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SEQUENTIAL JET DECAY DESCRIBED BY AN INÖNÜ-WIGNER CONTRACTION
OF THE LORENTZ GROUP

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A B S T R A C T

Stimulated by the treatment of a sequential decay model for many-particle production with a master equation of convolution type over $SL(2, \mathbb{C})$, we study in detail the Inönü-Wigner contraction to the complex Euclidean group $E(2, \mathbb{C})$ which arises when we take consistently into account the limitation of the transverse momenta.

Our results allow for a clear-cut distinction between longitudinal and transverse dynamics and can be considered as a relativistic generalization of the impact parameter expansion. We believe they can apply to other problems with a similar kinematical structure.

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1. - JET MODELS FOR MANY-PARTICLE PRODUCTION

Many-particle production is often described as occurring through jet formation and subsequent cascade decay into final observed hadrons ^{1),2)}. This description is relevant not only to small $P_{\perp}$ production in hadronic collisions, but also to the high $P_{\perp}$ component and to high energy e^+e^- annihilation.

Jets undergoing a sequential decay with negligible interference effects can be interpreted as highly excited quark systems ³⁾, whereas there are indications in quantum chromodynamics that production occurs indeed via jet formation ⁴⁾. Apart from theoretical speculations there is growing experimental evidence in favour of jets as a universal feature of hadronic production ⁵⁾.

In Refs. 1), 2), a careful treatment was given of a sequential decay model in which a jet was characterized by its invariant mass, its orientation with respect to an arbitrary frame of reference (for instance the laboratory frame) and the probability distribution of its decay products. A master equation was derived which is of convolution type over the Lorentz group [or over the spinor group $SL(2,C)$]. However, taking the limitation of the transverse momenta of the observed particles into account, in Ref. 1) the invariance group $SL(2,C)$ was replaced by a contracted group ⁶⁾ isomorphic to $E(2,C)$, the complex Euclidean group. The master equation was consequently partially diagonalized by projection over a set of representations of $E(2,C)$.

In this paper we fully justify this approach. We first perform the harmonic analysis ⁷⁾ of our master equation using the representations of $SL(2,C)$ belonging to the principal series. The transition to an approximate treatment based on the contracted group $E(2,C)$ is eventually recovered after a careful study of the relation between corresponding infinitesimal generators, unitary representations and group algebras.

Our formalism can be thought as a relativistic generalization of an impact parameter expansion and can, of course, apply to other problems with a similar invariance structure.

From a mathematical viewpoint, the general problem of relating representations of a contracted group to representations of the original group is not completely solved to our knowledge. Under this respect our results could be interesting, as we provide an explicit correspondence through contraction between unitary representations of the two groups, which are complete in the sense of the Plancherel formula ⁷⁾.

In order to make our paper self-contained and to introduce some notations, we briefly summarize the main results obtained in Refs. 1), 2). We call $Q_1, Q_2, \dots, Q_N$ the four-momenta of the jet along the decay chain. We have

$$Q_i^2 = M_i^2, \quad M_i > 0, \quad i = 1, \dots, N, \quad (1.1)$$

and we choose elements $a_1, a_2, \dots$ of $SL(2, C)$ such as

$$Q_i = L(a_i) Q_i^R = L(a_i) (M_i, 0, 0, 0) \quad (1.2)$$

The Lorentz transformation $L(a_i)$ relates one of the rest systems of the i th jet to an arbitrary frame of reference (for instance the laboratory frame). We assume that only one spinless particle of mass m is produced at each step as long as the available invariant energy is larger than a "threshold" value M^* , below which the decay will occur through a different mechanism, presumably without a marked anisotropy. Of course, the model could be generalized in several directions.

If a jet has mass M_i and orientation specified by a_i (we choose the z axis along its symmetry axis), the decay probability into a jet of mass M_{i+1} and orientation given by a_{i+1} is denoted by

$$K(M_i, a_i^{-1} a_{i+1}, M_{i+1}) d^6 a_{i+1} dM_{i+1}, \quad (1.3)$$

$d^6 a$ being the invariant measure over $SL(2, C)$. The probability distribution K is normalized as follows

$$\int K(M, a, M') d^6 a dM' = \theta(M - M^*), \quad (1.4)$$

θ being the usual step function. We call

$$R_i(M_1, a_1^{-1} a_i, M_i) d^6 a_i dM_i \quad (1.5)$$

the probability that the i th jet in the cascade decay has mass M_i and orientation specified by a_i . It obviously satisfies the recursive relation

$$\begin{cases} R_2(M, a, M') = K(M, a, M') , \\ R_{i+1}(M, a, M') = \int K(M, a', M'') R_i(M'', a'^{-1} a, M') \cdot \\ \cdot d^6 a' dM'' , \quad i = 2, 3, \dots \end{cases} \quad (1.6)$$

Summing over i , we get the quantity

$$R(M, a, M') = \sum_{i \geq 2} R_i(M, a, M') , \quad (1.7)$$

where the sum is automatically limited by energy conservation. Then from Eq. (1.6) we derive our basic integral equation

$$R(M, a, M') = K(M, a, M') + \int K(M, a', M'') \cdot R(M'', a'^{-1} a, M') d^6 a' dM'' \quad (1.8)$$

The knowledge of $R(M, a, M')$ allows the calculation of all the inclusive distributions coming from a jet decay ^{1), 2)}.

2. - PARTIAL DIAGONALIZATION OF THE MASTER EQUATION

The harmonic analysis over $SL(2, \mathbb{C})$ involves the irreducible unitary representations belonging to the principal series ⁷⁾.

Owing to energy conservation, $K(a)$ and $R(a)$ are positive measures of compact support over the group manifold ⁸⁾ (here and in the following mass dependences will be often understood^{*}). As a consequence there are no problems in defining the projections

$$\begin{cases} \tilde{K}(\chi) = \int K(a) T_{\chi}(a) d^6 a, \\ \tilde{R}(\chi) = \int R(a) T_{\chi}(a) d^6 a, \end{cases} \quad (2.1)$$

as operator valued functions of χ , where, in the case of the principal series we are considering

$$\chi = \left(\frac{n+i\rho}{2}, \frac{-n+i\rho}{2} \right), \quad (2.2)$$

n integral and ρ real. From the representation property of the unitary operators $T_{\chi}(a)$

$$T_{\chi}(a_1, a_2) = T_{\chi}(a_1) T_{\chi}(a_2) \quad (2.3)$$

one immediately realizes that the projections (2.1) partially diagonalize the integral equation (1.8)

$$\begin{aligned} \tilde{R}(\chi; M, M') &= \tilde{K}(\chi; M, M') + \int \tilde{K}(\chi; M, M'') \cdot \\ &\cdot \tilde{R}(\chi; M'', M') dM'' \end{aligned} \quad (2.4)$$

If we remember energy conservation, we conclude that this "radial" equation is Volterra. Once it has been solved, we recover $R(M, a, M')$ by the inverse formula ⁷⁾

$$R(a) = - (8\pi^4)^{-1} \int T_2 [\tilde{R}(\chi) T_{\chi}^{-1}(a)] c(\chi) d\chi, \quad (2.5)$$

^{*}) We recall that $T_{\chi}(a)$ is a strongly continuous function of a .

$c(\chi)d\chi$ being the Plancherel measure

$$C(\chi) = -\frac{1}{4}(m^2 + p^2) \quad (2.6)$$

3. - THE CONTRACTED GROUP

If we want to introduce the phenomenological information that the transverse momentum of the produced particles is seldom larger than 1 GeV, we have to assume that the probability distribution $K(M, a, M')$ is relevant when a does not differ too much from transformations of the kind

$$a_z(\eta) u_z(\omega) = \begin{vmatrix} e^{\frac{\eta-i\omega}{2}} & 0 \\ 0 & e^{-\frac{\eta-i\omega}{2}} \end{vmatrix} \quad (3.1)$$

We call D the subgroup the typical element of which is exhibited in Eq. (3.1).

Our assumption, when translated into the canonical parameters describing a Lorentz transformation

$$a = u_z(\varphi) u_y(\theta) a_z(\eta) u_z(\omega) u_y(\theta') u_z(\varphi') \quad (3.2)$$

entails the two possibilities

$$\theta \simeq \theta' \simeq 0, \quad (3.3)$$

$$\theta \simeq \theta', \quad \eta \simeq 0, \quad \omega \simeq \pi, \quad (3.4)$$

[we recall that we can choose ¹⁾

$$0 \leq \theta, \theta' \leq \pi/2 \quad (3.5) \square.$$

The possibility (3.4) concerns the soft particle production component that is an essential ingredient for a good phenomenology.

It is convenient to parametrize the elements of $SL(2, \mathbb{C})$ as follows, ϵ being a real quantity,

$$a = \begin{pmatrix} \lambda(1 + \epsilon^2 \mu z) & \epsilon \mu \lambda^{-1} \\ \epsilon \lambda z & \lambda^{-1} \end{pmatrix} . \quad (3.6)$$

We shall eventually take the limit $\epsilon \rightarrow 0$ and keep only first order terms (Inönü-Wigner contraction ⁶⁾). The subgroup with respect to which the contraction is undertaken is D .

Let us recall the algebra of the infinitesimal generators of $SL(2, \mathbb{C})$ ⁹⁾

$$\begin{cases} [H_3, H_{\pm}] = -[F_3, F_{\pm}] = \pm H_{\pm} , \\ [H_3, F_{\pm}] = [F_3, H_{\pm}] = \pm F_{\pm} , \\ [H_+, H_-] = -[F_+, F_-] = 2 H_3 , \\ [H_+, F_-] = [F_+, H_-] = 2 F_3 , \\ [H_{\pm}, F_{\pm}] = 0 . \end{cases} \quad (3.7)$$

In view of the parametrization (3.6), we consider the linear ϵ dependent transformation

$$\begin{cases} \hat{H}_{\pm} = \epsilon H_{\pm} , \\ \hat{H}_3 = H_3 , \\ \hat{F}_{\pm} = \epsilon F_{\pm} , \\ \hat{F}_3 = F_3 . \end{cases} \quad (3.8)$$

The algebra of Eq. (3.7) has a limit when $\epsilon \rightarrow 0$

$$\left\{ \begin{array}{l} [\hat{H}_3, \hat{H}_{\pm}] = -[\hat{F}_3, \hat{F}_{\pm}] = \pm \hat{H}_{\pm} \quad , \\ [\hat{H}_3, \hat{F}_{\pm}] = [\hat{F}_3, \hat{H}_{\pm}] = \pm \hat{F}_{\pm} \quad , \\ [\hat{H}_+, \hat{H}_-] = [\hat{F}_+, \hat{F}_-] = [\hat{H}_+, \hat{F}_-] = [\hat{F}_+, \hat{H}_-] = 0 \quad , \\ [\hat{H}_{\pm \frac{2}{3}}, \hat{F}_{\pm \frac{2}{3}}] = 0 \quad . \end{array} \right. \quad (3.9)$$

As is well known, this algebra gives rise to a new group with the multiplication law

$$(\mu, \gamma, \lambda)(\mu', \gamma', \lambda') = (\mu + \mu'\lambda^2, \gamma + \gamma'\lambda^{-2}, \lambda\lambda') \quad . \quad (3.10)$$

Elements with $\lambda = 1$ belong to the Abelian invariant subgroup with infinitesimal generators $\hat{H}_{\pm}$ and $\hat{F}_{\pm}$, whereas the elements with $\mu = \gamma = 0$ belong to a subgroup isomorphic to D . The group we have obtained by contraction is isomorphic to $E(2, C)$, i.e., the complex Euclidean group in two dimensions.

4. - CONTRACTION OF REPRESENTATIONS

It is well known ⁶⁾ that if we perform the limit $\epsilon \rightarrow 0$ of the infinitesimal generators in a given representation of the original group, we get a representation of the subgroup D . In order to get a faithful representation of the contracted group we shall go to higher and higher representations as ϵ decreases ⁶⁾.

Here and in the sequel we shall use as carrier space of $SL(2, C)$ representations $L_2(Z)$ ⁹⁾, namely the Hilbert space of square summable functions of the complex variable z . On a set dense in $L_2(Z)$ the infinitesimal generators of $SL(2, C)$ take the form

$$\left\{ \begin{aligned} H_+ &= -\frac{\partial}{\partial z} - \bar{z}^2 \frac{\partial}{\partial \bar{z}} + \left(\frac{-n+ip}{2} - 1 \right) \bar{z} , \\ H_- &= z^2 \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}} - \left(\frac{n+ip}{2} - 1 \right) z , \\ H_3 &= -z \frac{\partial}{\partial z} + \bar{z} \frac{\partial}{\partial \bar{z}} + \frac{n}{2} , \\ F_+ &= i \frac{\partial}{\partial z} - i \bar{z}^2 \frac{\partial}{\partial \bar{z}} + i \left(\frac{-n+ip}{2} - 1 \right) \bar{z} , \\ F_- &= -i z^2 \frac{\partial}{\partial z} + i \frac{\partial}{\partial \bar{z}} + i \left(\frac{n+ip}{2} - 1 \right) z , \\ F_3 &= i z \frac{\partial}{\partial z} + i \bar{z} \frac{\partial}{\partial \bar{z}} - \frac{i}{2} (ip - 2) . \end{aligned} \right. \quad (4.1)$$

Let us now consider the following mapping S ^{*)}

$$[S\varphi](z) = |\gamma|^{2-ip} \left(\gamma / \bar{\gamma} \right)^{-\frac{n}{2}} \varphi(\gamma^2) = \frac{1}{\sqrt{2}} \phi(\gamma) \quad (4.2)$$

from $L_2(Z)$ to the Hilbert space $\mathcal{H}(\gamma)$ of the functions of the complex variable γ equipped with the norm

$$\|\phi\|^2 = \int |\phi(\gamma)|^2 \frac{d^2\gamma}{|\gamma|^2} . \quad (4.3)$$

With these prescriptions S is norm preserving ^{**)}

$$\begin{aligned} \|\varphi\|^2 &= \int |\varphi(z)|^2 d^2z = \frac{1}{2} \int |\phi(\gamma)|^2 |\gamma|^{-4} d^2z = \\ &= \int |\phi(\gamma)|^2 |\gamma|^{-2} d^2\gamma = \|\phi\|^2 \end{aligned} \quad (4.4)$$

^{*)} A more general mapping with a dilation of γ

$$[S\varphi](z) = |\xi\gamma|^{2-ip} \left(\xi\gamma / \overline{\xi\gamma} \right)^{-\frac{n}{2}} \varphi(\xi^2\gamma^2) ,$$

ξ being a complex parameter, could also be considered. It is easy to show that different values of ξ give rise to equivalent representations of the contracted group.

^{**)} An extra factor of $\frac{1}{2}$ arises because when γ runs once over the complex plane, z runs over it twice.

We apply the transformation S to the generators of Eq. (4.1). We get

$$\left\{ \begin{aligned} S H_+ S^{-1} &= -\frac{\partial}{\partial \gamma^2} - \bar{\gamma}^4 \frac{\partial}{\partial \bar{\gamma}^2} - \frac{n+ip-2}{4\gamma^2} + \frac{-n+ip-2}{4} \bar{\gamma}^2, \\ S H_- S^{-1} &= \gamma^4 \frac{\partial}{\partial \gamma^2} + \frac{\partial}{\partial \bar{\gamma}^2} + \frac{-n+ip-2}{4\bar{\gamma}^2} - \frac{n+ip-2}{4} \gamma^2, \\ S H_3 S^{-1} &= -\gamma^2 \frac{\partial}{\partial \gamma^2} + \bar{\gamma}^2 \frac{\partial}{\partial \bar{\gamma}^2}, \\ S F_+ S^{-1} &= i \frac{\partial}{\partial \gamma^2} - i \bar{\gamma}^4 \frac{\partial}{\partial \bar{\gamma}^2} + i \frac{n+ip-2}{4\gamma^2} + i \frac{-n+ip-2}{4} \bar{\gamma}^2, \\ S F_- S^{-1} &= -i \gamma^4 \frac{\partial}{\partial \gamma^2} + i \frac{\partial}{\partial \bar{\gamma}^2} + i \frac{-n+ip-2}{4\bar{\gamma}^2} + i \frac{n+ip-2}{4} \gamma^2, \\ S F_3 S^{-1} &= i \gamma^2 \frac{\partial}{\partial \gamma^2} + i \bar{\gamma}^2 \frac{\partial}{\partial \bar{\gamma}^2}. \end{aligned} \right. \quad (4.5)$$

If we put

$$\sigma = \epsilon \frac{p-in}{2}, \quad (4.6)$$

we are ready for performing the contraction limit $\epsilon \rightarrow 0$, σ fixed. Then we notice that our transformation S becomes ϵ dependent and singular in the limit $\epsilon \rightarrow 0$, as it should.

We obtain in a dense set of $\mathcal{H}(\gamma)$ the following expressions for the contracted generators ^{*)}

$$\left\{ \begin{aligned} \hat{H}_+ &= -i \sigma (2\gamma^2)^{-1} + i \bar{\sigma} \bar{\gamma}^2/2, \\ \hat{H}_- &= -i \sigma \gamma^2/2 + i \bar{\sigma} (2\bar{\gamma}^2)^{-1}, \\ \hat{H}_3 &= -\gamma^2 \frac{\partial}{\partial \gamma^2} + \bar{\gamma}^2 \frac{\partial}{\partial \bar{\gamma}^2}, \\ \hat{F}_+ &= -\sigma (2\gamma^2)^{-1} - \bar{\sigma} \bar{\gamma}^2/2, \\ \hat{F}_- &= -\sigma \gamma^2/2 - \bar{\sigma} (2\bar{\gamma}^2)^{-1}, \\ \hat{F}_3 &= i \gamma^2 \frac{\partial}{\partial \gamma^2} + i \bar{\gamma}^2 \frac{\partial}{\partial \bar{\gamma}^2}. \end{aligned} \right. \quad (4.7)$$

^{*)} The convergence is strong in the definition domain of the contracted group generators.

It is a simple exercise to see that the commutation relations (3.9) are satisfied.

The Casimir operators of $SL(2, \mathbb{C})$ are ⁹⁾

$$\begin{cases} I_1 = F_+ F_- + F_- F_+ + 2 F_3^2 - H_+ H_- - H_- H_+ - 2 H_3^2 \\ I_2 = H_+ F_- + H_- F_+ + F_+ H_- + F_- H_+ + 4 H_3 F_3 \end{cases} \quad (4.8)$$

In their definition domains they are multiple of the unit operator

$$\begin{cases} [I_1, \varphi](z) = \left[\frac{\rho^2 - n^2}{2} + 2 \right] \varphi(z) \\ [I_2, \varphi](z) = n \rho \varphi(z) \end{cases} \quad (4.9)$$

In the contraction limit I_1 and I_2 get multiplied by e^2 . As a consequence

$$\begin{cases} [\hat{I}_1, \phi](\gamma) = 2 \operatorname{Re}(\sigma^2) \phi(\gamma) \\ [\hat{I}_2, \phi](\gamma) = -2 \operatorname{Im}(\sigma^2) \phi(\gamma) \end{cases} \quad (4.10)$$

as one can also easily derive from Eq. (4.7).

We now consider finite transformations of the contracted group. The representations can be obtained from the infinitesimal generators using an "exponential" formula ¹⁰⁾. Alternatively we can get them by performing the contraction limit on the corresponding representations of the original group. We shall follow, for the sake of completeness, this second procedure.

We parametrize again the elements of $SL(2, \mathbb{C})$ according to Eq. (3.6) and use $L_2(\mathbb{Z})$ as carrier space. We have ⁷⁾

$$[T_\chi(a) \varphi](z) = \lambda^{1 - \frac{n+ip}{2}} \frac{1}{\lambda^{1 - \frac{n+ip}{2}}} (1 + \epsilon_\mu z)^{\frac{n+ip}{2} - 1}.$$

$$\begin{aligned}
 & \cdot \overline{(1 + \epsilon \mu z)}^{-\frac{n+ip}{2} - 1} \varphi \left[\lambda^2 z \left(\epsilon z z^{-1} + (1 + \epsilon \mu z)^{-1} \right) \right] = \\
 & = \left[(1 + \epsilon \mu z) (1 + \epsilon z z^{-1}) \right]^{\frac{n+ip-2}{4}} \overline{\left[(1 + \epsilon \mu z) (1 + \epsilon z z^{-1}) \right]^{\frac{n+ip-2}{4}}} \\
 & \cdot (z z'^{-1})^{\frac{n+ip-2}{4}} \overline{(z z'^{-1})^{\frac{n+ip-2}{4}}} \varphi(z'), \quad (4.11)
 \end{aligned}$$

where we have put

$$z' = \lambda^2 z \left[\epsilon z z^{-1} + (1 + \epsilon \mu z)^{-1} \right] \quad (4.12)$$

and dropped terms $O(\epsilon^2)$.

We now perform the mapping S from $L_2(Z)$ to $\mathcal{H}(\gamma)$. The transformed generators are

$$\begin{aligned}
 [S T_\gamma(a) S^{-1} \phi](\gamma) &= \left[(1 + \epsilon \mu \gamma^2) (1 + \epsilon z \gamma^{-2}) \right]^{\frac{n+ip-2}{4}} \\
 \overline{\left[(1 + \epsilon \mu \gamma^2) (1 + \epsilon z \gamma^{-2}) \right]^{\frac{n+ip-2}{4}}} &\phi(\gamma'), \quad (4.13)
 \end{aligned}$$

where

$$z' = \gamma'^2. \quad (4.14)$$

Recalling Eq. (4.6), we perform the limit $\epsilon \rightarrow 0$

$$[\hat{T}_\sigma(\mu, z, \lambda) \phi](\gamma) = \exp \{ i \operatorname{Re} [\sigma (\mu \gamma^2 + z \gamma^{-2})] \} \phi(\lambda \gamma) \quad (4.15)$$

In the next section we show that the unitary representations of $E(2,0)$ we have just obtained are a "complete" set in the sense of the Plancherel formula.

5. - THE GROUP ALGEBRAS

In Section 2 we have seen that the integral equation of our model was partially diagonalized by projection on the unitary representations of the principal series. This amounts to considering the representations of the group algebra ^{7),9)}. If we parametrize $a \in \text{SL}(2, \mathbb{C})$ according to Eq. (3.6), the invariant measure is

$$d^6 a = \epsilon^4 |\lambda|^{-2} d^2 \lambda d^2 \mu d^2 z \quad (5.1)$$

We perform the transformation S on Eq. (2.1) and get

$$S \tilde{K}(x) S^{-1} = \int K(a) S T_x(a) S^{-1} \epsilon^4 |\lambda|^{-2} d^2 \lambda d^2 \mu d^2 z. \quad (5.2)$$

Now we assume that, as ϵ becomes smaller and smaller, $K(a)$ gets more and more peaked on elements belonging to the subgroup D . In other words $K(a)$ is to be considered an ϵ dependent measure ^{*)} with

$$\lim_{\epsilon \rightarrow 0} \epsilon^4 K(a) = \hat{K}(\mu, z, \lambda), \quad (5.3)$$

$\hat{K}(\mu, z, \lambda)$ being a measure with finite total "mass". The contraction limit of Eq. (5.2) then gives the Fourier transform of an element of the group algebra of $E(2, \mathbb{C})$

$$\tilde{K}(\sigma) = \int \hat{K}(\mu, z, \lambda) \hat{T}_\sigma(\mu, z, \lambda) |\lambda|^{-2} d^2 \lambda d^2 \mu d^2 z \quad (5.4)$$

as an operator acting on elements of the space $\mathcal{H}(\gamma)$. Explicitly we have the kernel

$$\begin{aligned} [\tilde{K}(\sigma) \phi](\gamma) &= \int \hat{K}(\sigma; \gamma, \gamma') |\gamma'|^{-2} d^2 \gamma' \phi(\gamma') = \\ &= \int \hat{K}(\mu, z, \gamma' \gamma^{-1}) \exp\{i \text{Re}[\sigma(\mu \gamma^2 + z \gamma^{-2})]\} \phi(\gamma') d^2 \mu d^2 z |\gamma'|^{-2} d^2 \gamma'. \end{aligned} \quad (5.5)$$

*) In realistic models $K(M, a, M')$ gets more and more peaked on elements of D at larger and larger values of M . Therefore ϵ is to be thought as a function of M and the limit $\epsilon \rightarrow 0$ should be performed, strictly speaking, together with the limit $M \rightarrow \infty$ [see the example in Ref. 2].

We are led in this way to a projected integral equation over $E(2,C)$ unitary representations

$$\tilde{R}(\sigma) = \tilde{K}(\sigma) + \tilde{K}(\sigma) \tilde{R}(\sigma) \quad (5.6)$$

(mass and γ dependences are understood).

Equation (5.6) cannot be further simplified by relying on symmetry properties. One needs at this point dynamical assumptions concerning the probability distribution $\hat{K}$. The main advantage of Eq. (5.6) with respect to Eq. (2.4) is that longitudinal and transverse dynamics are now treated on a different footing, the quantity σ being the natural relativistic generalization of the impact parameter. In Ref. 2) we have shown that simple inputs in Eq. (5.6) can indeed reproduce the basic phenomenological features of hadronic production (Feynman scaling, sea-gull effect, short-range correlations).

Equation (5.4) can be inverted in a straightforward way

$$\hat{K}(\mu, \gamma, \lambda) = (2\pi^4)^{-1} \int |\sigma|^2 d^2\sigma |\gamma|^2 d^2\gamma \cdot \tilde{K}(\sigma; \gamma, \lambda\gamma) \exp \left\{ -i \operatorname{Re}[\sigma(\mu\gamma^2 + \gamma\gamma^{-2})] \right\} \quad (5.7)$$

This result can also be obtained by "contracting" the inverse formula for $SL(2,C)$ in Eq. (2.5). We start by writing an explicit form of Eq. (2.5) making use of the z representation ⁷⁾

$$K(a) = (32\pi^4)^{-1} \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} (m^2 + p^2) dp \int \tilde{K}(\chi; z, z') d^2z \cdot \lambda^{\frac{n+ip+z}{2}} \bar{\lambda}^{\frac{-n+ip+z}{2}} (1 + \epsilon\mu z)^{-\frac{n+ip+z}{2}} (1 + \epsilon\mu z)^{\frac{-n+ip+z}{2}}, \quad (5.8)$$

where a has been parametrized according to Eq. (3.6), $\tilde{K}(\chi; z, z')$ is the kernel corresponding to $\tilde{K}(\chi)$ and z' is given by Eq. (4.12).

The transformation S leads us to an expression based on the "y - representation", according to the formula

$$K(a) = (32\pi^4)^{-1} \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} (n^2 + p^2) dp \int [S \tilde{K}(x) S^{-1}](x, x') \cdot |x|^{-2} d^2 x \left[(1 + \epsilon \mu x^2)(1 + \epsilon z x^{-2}) \right]^{\frac{-n+ip+z}{4}} \left[(1 + \epsilon \mu x^2)(1 + \epsilon z x^{-2}) \right]^{\frac{-n+ip+z}{4}}. \quad (5.9)$$

Recalling now Eqs. (4.6), (5.3), by performing the limit $\epsilon \rightarrow 0$, we easily recover Eq. (5.7).

Summarizing we see that the harmonic analysis over $E(2, C)$ is done using just those unitary representations which are obtained from the principal series of $SL(2, C)$. We believe that a similar correspondence under contraction between those unitary representations which are needed in the Fourier analysis, can be found for many other groups.

6. - THE ONE-PARTICLE INCLUSIVE DISTRIBUTION AND THE REPRESENTATIONS ON THE HYPERBOLOID

If we denote by $G_1(M, P)(2P_0)^{-1} d^3 P$ the probability that one particle with four-momentum P is produced in a single decay of a jet of mass M in its rest frame, the one-particle inclusive distribution $G(M, P)$ obeys the integral equation ¹⁾

$$G(M, P) = G_1(M, P) + \int K(M, a, M') G(M', L(\bar{a}') P) d^6 a dM', \quad (6.1)$$

the solution of which is

$$G(M, P) = G_1(M, P) + \int R(M, a, M') G_1(M', L(\bar{a}') P) d^6 a dM'. \quad (6.2)$$

Higher order inclusive distributions can be defined by generalizing the above procedure ²⁾.

The one-particle inclusive distribution has its support over the future sheet of the timelike hyperboloid $P^2 = m^2$. As this hyperboloid is a homogeneous space with respect to $SL(2, \mathbb{C})$, the one-particle inclusive distribution induces a measure over $SL(2, \mathbb{C})$ ^{*)}

$$G(M, P) = \Gamma(M, \hat{a}) , \quad (6.3)$$

where

$$P = L(\hat{a}) \hat{P} = L(\hat{a}) (m, 0, 0, 0) , \quad (6.4)$$

which obeys the covariance constraint

$$\Gamma(M, \hat{a}) = \Gamma(M, \hat{a}u) , \quad u \in SU(2) , \quad (6.5)$$

$SU(2)$ being, as is well known, the stability subgroup of the timelike hyperboloid.

In an analogous way we define

$$\Gamma_1(M, \hat{a}) = G_1(M, P) . \quad (6.6)$$

Under the Lorentz transformation a they transform according to the left-regular representation

$$[U(a) \Gamma](\hat{a}) = \Gamma(a^{-1} \hat{a}) \quad (6.7)$$

(mass dependence is understood).

Equations (6.1) and (6.2) become respectively

^{*)} Again the support is compact due to energy conservation and to the fact that $m > 0$.

$$\Gamma(\hat{a}) = \Gamma_1(\hat{a}) + \int K(a) \Gamma(a^{-1}\hat{a}) d^6a \quad (6.8)$$

and

$$\Gamma(\hat{a}) = \Gamma_1(\hat{a}) + \int R(a) \Gamma_1(a^{-1}\hat{a}) d^6a \quad (6.9)$$

If we now project these equations over the unitary representations of the principal series of $SL(2, \mathbb{C})$, we easily realize that, owing to the covariance constraint (6.5), only values of χ with $n=0$ do contribute

$$\chi = \left(\frac{i\rho}{2}, \frac{i\rho}{2} \right) \quad (6.10)$$

This result is a consequence of our produced particle being spinless. We get

$$\tilde{\Gamma}(\chi; M) = \tilde{\Gamma}_1(\chi; M) + \int \tilde{R}(\chi; M, M') dM' \tilde{\Gamma}_1(\chi; M') \quad (6.11)$$

The projection can be inverted using Eq. (2.5) and remembering that only terms with $n=0$ contribute in this case.

In order to make use of $E(2, \mathbb{C})$ representations we proceed as follows. We introduce an auxiliary ϵ dependent measure $g_1(a)$, which, as ϵ becomes smaller and smaller, get concentrated on elements of D *) [see Eq. (5.3)], and put

$$\Gamma_1(a) = \int g_1(au) d^3u \quad (6.12)$$

Equation (6.5) is obviously satisfied, whereas some arbitrariness is left in $g_1(a)$. If we recall the parametrization

*) See footnote on p. 12.

$$a = k u, \quad u \in SU(2) \quad (6.13)$$

and

$$k = \begin{pmatrix} \lambda & \epsilon \mu \lambda^{-1} \\ 0 & \lambda^{-1} \end{pmatrix} \in K, \quad (6.14)$$

a possible choice for $\zeta_1(a)$ is

$$\zeta_1(a) = \Gamma_1(k) \delta^{(3)}(u), \quad (6.15)$$

namely a measure concentrated over the subgroup K . Then we immediately realize that $\Gamma_1(a)$ is an ϵ dependent measure over K , which, as ϵ becomes smaller and smaller, gets more and more peaked on elements of D . Now, from Eq. (6.12)

$$\tilde{\Gamma}_1(x) = \tilde{\zeta}_1(x) \mathcal{Z}(x), \quad (6.16)$$

$\mathcal{Z}(x)$ being a projection operator averaging over $SU(2)$

$$\mathcal{Z}(x) = \int T_x(u) d^3u. \quad (6.17)$$

It is easy to see that

$$\mathcal{Z}(x) = 0 \quad \text{if} \quad u \neq 0. \quad (6.18)$$

Equation (6.11) becomes

$$\tilde{F}(x) = [\tilde{\zeta}_1(x) + \tilde{R}(x) \tilde{\zeta}_1(x)] \mathcal{Z}(x) = \tilde{\zeta}_1(x) \mathcal{Z}(x). \quad (6.19)$$

The equation

$$\tilde{\zeta}_1(x) = \tilde{\zeta}_1(x) + \tilde{R}(x) \tilde{\zeta}_1(x) \quad (6.20)$$

can be contracted in the same way as Eq. (2.4), giving

$$\tilde{\hat{g}}(\sigma) = \tilde{\hat{g}}_1(\sigma) + \tilde{R}(\sigma) \tilde{\hat{g}}_1(\sigma) \quad (6.21)$$

By the inverse transform Eq. (5.7) we can obtain the quantity $\hat{g}(\mu, \lambda)$ which is to be thought as an approximate expression for $\hat{g}(a)$, according to Eq. (5.3). $\Gamma(a)$ is finally recovered by averaging over $SU(2)$
 $\Gamma(a) = \int \hat{g}(au) d^3u^*$.

7. - FINAL REMARKS

In conclusion we see that a natural way of describing the kinematics of a sequential jet decay into hadrons is to associate to each jet a Lorentz frame. The dynamical equation of the decay then becomes of convolution type over the Lorentz group. We can exploit this symmetry eliminating the "angular" variables by projection over a complete set of irreducible representations of $SL(2, C)$ and get a "radial" equation, which contains the relevant dynamical information. However, in this way, longitudinal and transverse dynamics are treated on an equal basis as far as symmetry properties are concerned.

If we want to introduce the phenomenological information that the transverse momenta of the produced particles are seldom larger than 1 GeV, we can approximately replace covariance under $SL(2, C)$ with covariance under $E(2, C)$, the complex Euclidean group in two dimensions. We have to remember the physical meaning of conditions (3.3) and (3.4). Whereas condition (3.3) is obvious, condition (3.4) concerns large angle soft particle production ($\eta \simeq 0$). However, the decay must occur without changing too much the direction of the jet axes ($\theta \sim \theta', \omega \sim \pi$) in order to prevent the production of fast particles at large angles in subsequent decays. In other words, in order to keep memory of the initial privileged direction, a correlation is necessary between the jet axes before and after the decay.

*) We remark that a similar treatment could apply to homogeneous spaces with a non-compact stability group (the future light-cone for instance, which would be relevant to the zero-mass case). Some projections, like $\hat{F}(x)$, $\hat{Z}(x)$..., should not be considered in this case.

Phenomenological applications have been discussed in Ref. 2), where we have shown that simple inputs can indeed reproduce the basic features of high energy hadron production. In this paper we have rather been concerned with a careful study of the Inönü-Wigner contraction to clarify the limiting procedure from a treatment based on $SL(2,C)$ representations to a treatment using the contracted group. We have proved that our procedure is fully consistent if the input decay probability is more and more peaked on diagonal elements of $SL(2,C)$ in the large energy limit. As a by-product we have shown that the eigenvalues of the $SL(2,C)$ Casimir operators bear the same relation to the representation index of $E(2,C)$ as the angular momentum does to the usual impact parameter variable. As a consequence our formalism is to be thought as the natural relativistic generalization of an impact parameter expansion.

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