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AN $SU(2)$ SYMMETRIC REGGEON FIELD THEORY
WITH A NEGATIVE MASS

J. Bartels and J.J. Scanio *)
CERN -- Geneva

A B S T R A C T

An $SU(2)$ invariant Reggeon field theory (RFT) containing three Reggeon multiplets, one of which has a negative mass, is studied in the classical approximation with zero transverse dimensions. We explain how to introduce a continuous symmetry into RFT and explore under what restrictions our model has stable classical solutions. We find the same ambiguity of having two different types of solutions as in supercritical Pomeron theory. One of them is shown to lead to a constant total cross-section and decreasing cross-sections for quantum number exchanges.

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*) On leave of absence from and address after 1 September 1977 : Department of Physics, University of Cincinnati, OH 45221, U.S.A. Work supported in part by ERDA under contract EY-76-5-02-2978.

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1. INTRODUCTION

Since the formulation of Reggeon field theory, considerable progress has been made towards a satisfactory theory of high-energy scattering. Pomeron field theories have been investigated for different values of the bare intercept: at the critical value slightly above one a theory with all features of a phase transition emerges (scaling laws, long-range correlations), predicting a total cross-section $\sigma_{\text{tot}} \sim (\ln s)^{-\gamma}$ ($\gamma \approx -0.25$). When the intercept is smaller than this critical value, σ_{tot} falls and no RFT is needed (the interaction leads only to finite renormalization effects). Above the critical value (supercritical Pomeron theory) two different solutions have been found¹⁻⁸: one of them belongs to an ordered phase of the theory and for the total cross-section predicts saturation of the Froissart bound [$\sigma_{\text{tot}} \sim (\ln s)^2$], whereas the other one^{9,10} leads to a falling cross-section again.

Once we know to what kind of physical results different values of the bare Pomeron intercept belong to, the most important problem becomes the dynamical explanation of the origin of the Pomeron and the value of its intercept. It is generally believed that the bare Pomeron is built up by production processes (of multiperipheral type, mediated by exchange of elementary or Reggeizing particles). If the first solution to supercritical Pomeron theory is the correct one [with $\sigma_{\text{tot}} \sim (\ln s)^2$] it is easy to obtain a non-falling cross-section: as long as the coupling constants in the underlying production processes are not too small, the bare Pomeron intercept is above the critical one and saturation of the Froissart bound is obtained. In order to obtain the critical theory in this picture, the production vertices, etc., would have to have very special values, and the dynamical reason for this would be completely unknown. If the second solution to the supercritical Pomeron theory is the right one, the situation is even worse: the only way to obtain a non-falling cross-section is to give the underlying parameters these specific values which lead to the critical bare intercept. It is, therefore, possible to obtain a dynamical explanation of a $(\ln s)^2$ -type cross-section, but no mechanism is known that leads to the critical behaviour.

This makes it necessary to investigate other possibilities. An idea that has been around since the first days of Reggeon field theory is that a Pomeron with mass zero might be the result of spontaneous symmetry breaking¹¹⁻¹⁵. In this picture one starts with a RFT that possesses an internal continuous symmetry. Some of the quantum-number-carrying Reggeons have negative masses (intercept above one), and, using analogies from relativistic quantum field theories, one might expect that Goldstone particles emerge. The attractive feature of such a mechanism for generating a Pomeron with intercept one is that the result would depend on the phase structure of the RFT (sign of Reggeon masses) and would not require

specific values of the parameters (Reggeon masses and coupling constants). The recent investigations of supercritical Pomeron theory, however, have proved that the nature of a phase transition in Reggeon field theory is completely different from what it is in a relativistic field theory. The question of spontaneous symmetry breaking in RFT with an internal symmetry, therefore, is still open and needs to be investigated.

In this paper we present first results of such a study. The RFT model that we have chosen to investigate has been suggested¹⁶⁾ by an analysis of the high-energy behaviour of spontaneously broken non-Abelian gauge field theories, and its main features are the following. The internal symmetry is SU(2) (henceforth called isospin T), and there are three input fields: a T = 0 singlet, a T = 1 triplet and a T = 2 quintet. The trajectories of these Reggeons are (Fig. 1):

$$\alpha_0(t) = 1 - \Delta_0 + \alpha'_0 \cdot t \quad \Delta_0, \alpha'_0 > 0; \text{ positive signature (1.1)}$$

$$\alpha_1(t) = 1 - \Delta_1 + \alpha'_1 \cdot t \quad \Delta_1, \alpha'_1 > 0; \text{ negative signature (1.2)}$$

$$\alpha_2(t) = 1 - \Delta_2 + \alpha'_2 \cdot t \quad \Delta_2, \alpha'_2 < 0; \text{ positive signature. (1.3)}$$

These fields interact with each other through triple-Reggeon vertices, which are shown in Fig. 2. This interaction is invariant under SU(2) rotations. The most important feature of this RFT is the fact that one of the quantum-number-carrying fields, the T = 2 Reggeon, has both negative mass and negative slope.

Before entering into a study of this RFT model it will be useful to make a few general remarks. First, in view of the complexity of the model (number of fields and number of coupling constants), it is not obvious that this theory is well defined for all values of its parameters. Except for the sign of the Reggeon masses (and slopes) we, therefore, consider the size of all parameters as variables and search for what range of values the theory is defined. Some of our results will require that the ratio Reggeon mass/triple coupling constant is sufficiently large. Secondly, if our model is to give a realistic description of high-energy physics, the following requirements have to be satisfied:

- i) In order not to violate s-channel unitarity, all j-plane singularities must be at or below $j = 1$, and all slopes must be non-negative.
- ii) The vacuum (T = 0) singularity should be higher than all others, and the singularities of quantum number exchange channels should be monotonically decreasing with increasing isospin (in particular, the T = 1 singularity must be higher than the T = 2 one). Clearly, these requirements are not fulfilled by the input spectrum (1.1)-(1.3), and it must be the interaction between these fields that drastically changes the spectrum.

In our analysis of this RFT we shall, to a large extent, follow the way in which supercritical Pomeron theory has been investigated. Thus our first step will be a study of the classical approximation in zero transverse dimensions. This will give insight into the energy spectrum, disregarding its momentum dependence (in particular, the negative slope of the $T = 2$ Reggeon does not yet come in). Quantum corrections and the influence of transverse degrees of freedom remain the subject of future work. The main difference of our RFT model, compared with supercritical Pomeron theory, is the internal $SU(2)$ symmetry. One of our tasks, therefore, will be to introduce such a continuous symmetry into RFT and to discuss how quantum numbers in the exchange channel appear [actually, in the classical approximation the $SU(2)$ -symmetry eigenvalues are not yet quantized but continuous. We nevertheless shall refer to the isospin value as "quantum number"]. We then study classical solutions and their contribution to the physical scattering amplitudes. Our main results will be the following. We first determine regions in the parameter space (Reggeon mass and coupling constants), for which stable classical solutions exist. If the parameters lie within one of these regions, the situation is rather similar to supercritical Pomeron theory in that we have two kinds of solutions: one is the analogue to the solution of Amati et al.¹⁻⁸⁾, the other corresponds to White's solution^{9,10)}. In this paper, we will mainly be concerned with the first type. The other one will be discussed in a separate paper¹⁷⁾. Secondly, the energy spectrum that emerges from the discussion of these classical solutions satisfies the two requirements mentioned above: the total cross-section (vacuum exchange) is a constant, and the cross-sections for non-zero isospin exchange decrease with energy, the $T = 2$ exchange more strongly than the $T = 1$ one. As to the mechanism which makes the total cross-section constant (Pomeron with intercept one), we find no evidence for spontaneous symmetry breaking or Goldstone particles.

The organization of this paper will be the following. In Section 2 we give the explicit form of our Hamiltonian and define the physical quantities which we want to compute. We then discuss how to define quantum numbers in the t -channel. Calculations will be contained in Sections 3-5. First (Section 3) we discuss the fixed point structure for the vacuum exchange, then (Section 4) the classical solutions and the resulting total cross-sections. Section 5 contains the case of non-zero quantum number, and in Section 6 we give a summary and brief discussion of our results. A few more detailed calculations are put into three Appendices.

2. FORMULATION OF THE PROBLEM

Before starting any calculations, we briefly define the quantities that we are going to investigate and give an outline of our calculational program. Details of our calculations will be presented in the following sections, and in Section 6

we give a summary of the results. We begin with the Lagrangian of our RFT; since we are working in zero transverse dimensions, we drop the transverse derivative terms (for a graphical representation of the interaction part see Fig. 2):

$$\mathcal{L}(p, q) = \frac{1}{2} (p^\circ \overleftrightarrow{\partial}_y q^\circ + p_i^I \overleftrightarrow{\partial}_y q_i^I + p_\alpha^{II} \overleftrightarrow{\partial}_y q_\alpha^{II}) - H(p, q) \quad (2.1)$$

$$H(p, q) = -\mu_0 p^\circ q^\circ - \mu_I p_i^I q_i^I - \mu_{II} p_\alpha^{II} q_\alpha^{II} \quad (2.2)$$

$$\begin{aligned} & + \lambda_{00}^\circ (p^\circ + q^\circ) p^\circ q^\circ \\ & + \lambda_{II, II}^\circ (p^\circ q_\alpha^{II} q_\alpha^{II} + q^\circ p_\alpha^{II} p_\alpha^{II}) \\ & + \lambda_{II, II}^{II} (p_\alpha^{II} q_\beta^{II} q_\gamma^{II} d_{\alpha\beta\gamma} + q_\alpha^{II} p_\beta^{II} p_\gamma^{II} d_{\alpha\beta\gamma}) \\ & + \lambda_{0, II}^{II} (p_\alpha^{II} q^\circ q_\alpha^{II} + q_\alpha^{II} p^\circ p_\alpha^{II}) \\ & + \lambda_{0, I}^I (p_i^I q^\circ q_i^I + q_i^I p^\circ p_i^I) \\ & + \lambda_{II, I}^I (p_i^I q_j^I q_\alpha^{II} \lambda_{ij}^\alpha + q_i^I p_j^I p_\alpha^{II} \lambda_{ij}^\alpha) \\ & + i \lambda_{I, I}^\circ (p^\circ q_i^I q_i^I + q^\circ p_i^I p_i^I) \\ & + i \lambda_{I, I}^{II} (p_\alpha^{II} q_i^I q_j^I \lambda_{ij}^\alpha + q_\alpha^{II} p_i^I p_j^I \lambda_{ij}^\alpha) . \end{aligned}$$

Here q and p stand for the field variables and their Hermitian conjugate, respectively, and they are functions of rapidity y only (as usual, rapidity plays the role of time). The upper labels 0, I, II of p and q refer to isospin $T = 0, 1, 2$, respectively, and the lower indices label the components: for the triplet p_i^I and q_i^I we use Latin subscripts ($i = 1, 2, 3$), and for the isospin-two quintet

Greek ones ($\alpha = 1, 3, 4, 6, 8$). Without any labels p and q are vectors with the components: $p = (p^0, p_1^I, p_2^I, p_3^I, p_1^{II}, p_3^{II}, p_4^{II}, p_6^{II}, p_8^{II})$, etc. The μ 's are defined as the negative Reggeon masses:

$$\begin{aligned}\mu_0 &= -\Delta_0 = \alpha_0(0) - 1 < 0 \\ \mu_I &= -\Delta_I = \alpha_I(0) - 1 < 0 \\ \mu_{II} &= -\Delta_{II} = \alpha_{II}(0) - 1 > 0\end{aligned}\quad (2.3)$$

and $\mu_0 < 0$, $\mu_I < 0$, $\mu_{II} > 0$ [cf. Eqs. (1.1)-(1.3)]. A convenient way to write the SU(2)-invariant interaction between the $T = 0, 1, 2$ fields is the use of SU(3) structure constants $d_{\alpha\beta\gamma}$ and λ^α -matrices: restricting the indices α, β, γ to the values 1, 3, 4, 6, 8 one finds a five-dimensional representation of SU(2). For completeness, we list the structure constants as well as the λ -matrices in Tables 1 and 2. Finally, the way we have written the Lagrangian in (2.1) is somewhat different from the expression in Ref. 16; the connection between the two versions is made through the substitutions: $\mu_0 = -\Delta_0$, $\mu_I = -\Delta_I$, $\mu_{II} = -\Delta_{II}$, $i\phi = q^0$, $i\phi^+ = p^0$, $i\phi_i = q_i^I$, $i\phi_i^+ = q_i^I$, $i\phi_{ij} = \sum_\alpha (\lambda^\alpha)_{ij} q_\alpha^{II}$, $i\phi_{ij}^+ = \sum_\alpha (\lambda^\alpha)_{ij} p_\alpha^{II}$.

The quantity we want to compute is the scattering matrix. For its definition we follow Refs. 1 and 2 and introduce

$$Z(\mathcal{F}, G) = \frac{\mathcal{U}(\infty, y_2) \mathcal{F}(y_2) \mathcal{U}(y_2, y_1) G(y_1) \mathcal{U}(y_1, -\infty)}{\mathcal{U}(\infty, -\infty)} \quad (2.4)$$

$\mathcal{U}(y_2, y_1)$ is the time evolution operator:

$$\mathcal{U}(y_2, y_1) = \exp \left\{ - (y_2 - y_1) H \right\}, \quad (2.5)$$

and G and F , being functionals of $q(y)$ and $p(y)$, respectively, denote the coupling of the Reggeons to the external particles. The interpretation of (2.4) is the following (Fig. 3a): between $y = -\infty$ and y_1 , the system is in its ground state. Then, at time y_1 , the source $G[q(y), y_1]$ acts instantaneously and creates Reggeon fields. $\mathcal{U}(y_2, y_1)$ describes the propagation of the system until y_2 , where the source $F[p(y), y_2]$ annihilates the state, and the system then remains in the ground state until $+\infty$. In order to obtain the S-matrix element we need a proper normalization:

$$S(\mathcal{F}, G) = \frac{Z(\mathcal{F}, G)}{Z(\mathcal{F}, 0) Z(0, G)} \quad (2.6)$$

If H is the Hamiltonian of the system, and if the eigenfunctions $|\psi_n\rangle$ with energy E_n ,

$$H |\psi_n\rangle = E_n |\psi_n\rangle \quad (2.7)$$

form a complete set, we have an explicit expression for the evolution operator.

$$U(y_2, y_1) = \sum_n |\psi_n\rangle \langle \psi_n| e^{-E_n(y_2 - y_1)}. \quad (2.8)$$

For the S-matrix element (2.6) we find

$$S(F, G) = \frac{\sum \langle \psi_0 | F | \psi_n \rangle \langle \psi_n | G | \psi_0 \rangle e^{-(E_n - E_0)(y_2 - y_1)}}{\langle \psi_0 | F | \psi_0 \rangle \langle \psi_0 | G | \psi_0 \rangle} \quad (2.9)$$

The contribution of the ground state $|\psi_0\rangle$ to the sum in (2.9) gives unity, and, for large $y_2 - y_1$, the leading contribution to the scattering amplitude comes from the next-lowest eigenvalue E_1 . In particular, if the ground state is degenerate, and a second state with the same energy $E_1 = E_0$ exists, the scattering amplitude is a constant and the theory has a Regge singularity with intercept one.

Since our theory has an internal $SU(2)$ symmetry (= isospin), we wish to consider configurations with definite isospin quantum numbers. Our one-dimensional field theory has become a quantum mechanical problem, and the $SU(2)$ invariance of our Hamiltonian is treated in the same fashion as rotational invariance [O(3)-symmetry] in ordinary non-relativistic quantum mechanics: H and the generators of $SU(2)$ rotations commute with each other, and the eigenstates $|\psi_n\rangle$ of H can be chosen to carry definite isospin quantum number T (like angular momentum in ordinary quantum mechanics). We then define P_T as the projection operator on the subspace of states with isospin T , and U_T as the evolution operator in that subspace:

$$\begin{aligned} U_T(y_2, y_1) &= P_T U(y_2, y_1) \\ &= \sum_n' |\psi_n(T)\rangle \langle \psi_n(T)| e^{-E_n^T(y_2 - y_1)}. \end{aligned} \quad (2.10)$$

The sum in (2.10) goes only over states with isospin T. This then allows us to define the scattering matrix with definite isospin quantum number in the t-channel (Fig. 3b)

$$Z_T(F, G) = \frac{U_0(\infty, y_2) F(y_2) U_T(y_2, y_1) G(y_1) U_0(y_1, -\infty)}{U(\infty, -\infty)} \quad (2.11)$$

and

$$S_T = \frac{Z_T(F, G)}{Z(0, G) Z(F, 0)} \quad (2.12)$$

The difference between (2.11) and (2.9) is that between time y_1 and time y_2 the space of states is restricted to field configurations with isospin T. [In $(-\infty, y_1)$ and (y_2, ∞) the system is in its ground state which, as we will show later, has vacuum quantum number $T = 0$.] With the form (2.10) of U_T we obtain the analogue of (2.9):

$$S_T(F, G) = \frac{\sum' \langle \psi_0 | F | \psi_n(T) \rangle \langle \psi_n(T) | G | \psi_0 \rangle e^{-(E_n^T - E_0)(y_2 - y_1)}}{\langle \psi_0 | F | \psi_0 \rangle \langle \psi_0 | G | \psi_0 \rangle} \quad (2.13)$$

For large $y_2 - y_1$, the leading contribution to (2.13) comes from the lowest state with isospin T:

$$S_T(F, G) \sim e^{-(E_n^T - E_0)(y_2 - y_1)} \quad (2.14)$$

Our requirement that the vacuum singularity is higher than all others and that the $T = 1$ exchange is higher than the $T = 2$ exchange is satisfied if

$$E_0^0 < E_0^I < E_0^{II} \quad (2.15)$$

(E_0^T means lowest energy level in the sector with isospin T). In particular, we want the ground state of our theory to belong to isospin zero. In the analogous case in ordinary quantum mechanics, where angular momentum plays the same role as isospin in our case, the positive sign of the centrifugal barrier guarantees that the lowest energy state has angular momentum zero, and that the lowest $\ell = 1$ state lies below the lowest $\ell = 2$ state. Because of the different structure of RFT (the kinetic term of the Lagrangian is of the type $\vec{p} \vec{\partial}_t q$, and the potential depends on both p and q), the form in which a "centrifugal" term emerges is

completely different. However, one of our results will be that the ordering (2.15) holds and, hence, a physically acceptable situation is obtained.

So far we have given a definition of the objects that we are interested in and demonstrated the relevance of the energy spectrum of our RFT. We now want to describe how actual computations will be done. In supercritical Pomeron theory it was possible²⁾ to find the RFT energy spectrum by solving the Schrödinger equation (2.7). In our case, H depends on many more variables (p and q are nine-component vectors), and the differential equation (2.7) is too complicated. Instead of this we shall make use of path integral representation for the S -matrix elements and study the classical solutions which minimize the action.

In order to be able to write down such a path integral representation for (2.4) and (2.11), we observe that the behaviour of (2.4) and (2.11) for large $y_2 - y_1$, depends only on the energy spectrum of our RFT but not on the specific form of the source terms F and G (the Reggeon particle couplings). This allows us to use a particularly simple $SU(2)$ invariant expression for F and G :

$$G(y_1) = \exp \left\{ \int dy \, \delta(y-y_1) \, G \cdot q(y) \right\} \quad (2.16)$$

$$F(y_2) = \exp \left\{ \int dy \, \delta(y-y_2) \, F \cdot p(y) \right\}, \quad (2.17)$$

where

$$F \cdot p(y) = F^0 p^0 + F_i^I p_i^I + F_\alpha^{II} p_\alpha^{II}, \quad (2.18)$$

and F and G are constant (i.e. time-independent) vectors. The results on the high-energy behaviour that we obtain with (2.16) and (2.17) will be unchanged, even if the true physical Reggeon particle couplings are not as simple as (2.16) and (2.17). The path integral for (2.4) has the form

$$Z(F, G) = \frac{1}{N} \int [dp dq] \exp \left\{ - \int_{-\infty}^{\infty} dy \left[\frac{1}{2} p^T \partial_y^2 q - H(p, q) - \delta(y-y_1) G \cdot q - \delta(y-y_2) F \cdot p \right] \right\}. \quad (2.19)$$

Here p and q stand for all components $p^0, p_i^I, p_\alpha^{II}$, etc, H is taken from (2.1), dot products of two vectors are understood in the sense of (2.18), and the factor N stands for the path integral without the source terms F and G . In order to find the analogous expression for (2.11), we need to know how the projection operator P_I enters the path integral. In the time interval (y_1, y_2) the sum over

intermediate states in (2.11) [or (2.13)] extends only over states with isospin T ; the Hamiltonian which governs the time evolution of this subspace is H^T , i.e. the form of H in which the Hamiltonian is diagonal in the isospin quantum number. For illustration, we consider the familiar example of an ordinary quantum mechanical Hamiltonian with rotation $[O(2)]$ symmetry:

$$H(p, q) = \frac{\vec{p}^2}{2} + V(|\vec{q}|) \quad (2.20)$$

[p and q are two-dimensional vectors, $\vec{p} = (p_x, p_y)$, etc.]. The Hamiltonian for angular momentum ℓ is:

$$H_\ell = \frac{p_r^2}{2} + \frac{\ell^2 - 1/4}{2r^2} + V(r) \quad r = |\vec{q}|, \quad (2.21)$$

and the usual way to find this form is by separation of variables. H^T is the analogue of (2.21) for our case (we shall derive it in Section 5), and the path integral for (2.11) is

$$\begin{aligned} Z_T(T, G) = \frac{1}{N} \int [dp dq] \exp \left\{ - \int_{-\infty}^{\infty} dy \left(\frac{1}{2} p^2 \dot{q}^2 - \theta(y, y_1) H^0(p, q) \right. \right. \\ \left. \left. - \theta(y, y_2) H^0(p, q) - \theta(y_2 - y) \theta(y - y_1) H^T(p, q) \right. \right. \\ \left. \left. - \delta(y - y_1) G \cdot q - \delta(y - y_2) T \cdot p \right) \right\}. \end{aligned} \quad (2.22)$$

In this path integral, only those variables appear which are invariant under $SU(2)$ transformations -- the analogue of r and p_r in (2.20). The angles are already integrated out, and their conjugate variables [the angular momentum ℓ in (2.20)] define the quantum number T . The decomposition of H in (2.22) takes care of the fact that for the time interval (y_1, y_2) the system has isospin T , whereas for $y < y_1$ and $y > y_2$ the system is in its ground state with $T = 0$. We mention that for the simple case (2.20) the path integral (2.22) can explicitly be derived by using a representation of the projection operator P_ℓ and carrying out the angular part of the functional integration¹⁸⁾. For a more complicated case, such as ours, the methods of collective coordinates can be used in order to derive (2.22)¹⁷⁾.

In our attempt to evaluate the path integral (2.22) we follow the usual pattern and first find the classical solutions which minimize the action. At a later stage one then computes the effect of small oscillations in (2.22) around the classical solutions (WKB approximation). In the present paper we restrict ourselves to the classical approximation to (2.22). Furthermore, in the absence of quantum corrections, the definition of the S-matrix element (2.12) simplifies: $S_T = Z_T$. In this approximation, therefore, the S-matrix element reads:

$$S_T(\overline{F}, G) = \frac{\exp[-A_{cl}^T(\overline{F}, G)]}{\exp[-A_{cl}^0(0,0)]} \quad (2.23)$$

with

$$A_{cl}^T(\overline{F}, G) = \int_{-\infty}^{\infty} dy \left[\frac{1}{2} p_{\alpha} \dot{q}_{\alpha} - (\theta(y-y_1) + \theta(y-y_2)) H^0(p_{\alpha}, q_{\alpha}) - \theta(y_2-y) \theta(y-y_1) H^T(p_{\alpha}, q_{\alpha}) - \int (y-y_1) G \cdot q_{\alpha} - \int (y-y_2) \overline{F} \cdot p_{\alpha} \right]. \quad (2.24)$$

3. FIXED POINTS AND CLASSICAL SOLUTIONS IN THE ZERO QUANTUM NUMBER CHANNEL

After this general outline of our program, we now are ready to begin with a more detailed discussion of our calculations. The quantity we wish to calculate is the classical approximation to the S-matrix element (2.23). In this section we take the zero quantum number case ($T = 0$), and in the following sections we shall deal with the general case $T \neq 0$. In order to find the classical approximation (2.23) to the path integral (2.22) we have to solve the equations of motion in the presence of the two source terms. Because of the special form of the sources -- they act only at times $y = y_1$ and $y = y_2$ and are zero otherwise -- our task is mainly to solve the classical equations of motion:

$$\dot{q} = \frac{\partial H^0}{\partial p} \quad -\dot{p} = \frac{\partial H^0}{\partial q} \quad (3.1)$$

As we have explained at the end of the previous section, the index "0" of H^0 is to remind us that we should consider only those solutions which carry isospin zero. In ordinary classical mechanics with $O(2)$ symmetry, solutions with zero angular momentum are those which have zero angular velocity and hence do not change direction in space. Similarly, solutions with zero isospin have a fixed direction in group space (a formal derivation of this will be given in the following section), and solving the equations (3.1) is the same as solving the full equations:

$$\dot{q} = \frac{\partial H}{\partial p} \quad -\dot{p} = \frac{\partial H}{\partial q} \quad (3.2)$$

[with H from (2.2)] in a fixed direction in group space.

Equation (3.2) represents a system of first-order differential equations. In order to find its solutions it is most important to know the fixed points (or singular points) where all time derivatives vanish:

$$0 = \frac{\partial H}{\partial p} \quad 0 = \frac{\partial H}{\partial q} \quad . \quad (3.3)$$

We, therefore, will spend most of our time in finding values of p and q that satisfy (3.3). Throughout our discussion, we shall make use of the results that have been obtained in the supercritical Pomeron theory. We, therefore, start with a brief review of these results (Section 3.1). Then we shall search for solutions to Eq. (3.3) (Section 3.2), and, out of that, we construct the classical solutions (Section 3.3).

3.1 Review of supercritical Pomeron theory

In our whole discussion of fixed points and classical solutions to (3.3) we shall make use of the experience that has been gained by solving supercritical Pomeron theory^{1,2)}. We, therefore, consider it to be useful to briefly review those parts of results which will be relevant for our subsequent discussion. In the supercritical Pomeron theory, one has only the two variables p and q , and the Hamiltonian is

$$H(p, q) = -\mu pq + \lambda pq(p+q) \quad . \quad (3.4)$$

$(\mu, \lambda > 0)$

The solutions to the equations of motion are shown in Fig. 4a: the arrows indicate the direction of motion when time y increases, and the dots at the corner of the triangle as well as in its centre are the fixed points where $\partial H/\partial p = \partial H/\partial q = 0$.

When evaluating the scattering matrix element [it is of the form (2.19), with only one-component fields p and q], one has to include the effect of the sources G and F which act instantaneously at time y_1 and y_2 , respectively. In the equations of motion they lead to a jump in p and q at times y_1 and y_2 by the amount G and F , respectively. As a result of this, the solution to the equations of motion, which leads to an extremum of the action integral (2.19), can have the forms shown in Figs. 4b-c. For the first case (Fig. 4b), which leads to the non-falling cross-section of Refs. 1-8, the solution starts at $y = -\infty$ at the right-hand fixed point, goes along the q -axis until time y_1 , jumps upwards by the amount G (we consider only the case $F, G < \mu/\lambda$), follows the classical orbit until time y_2 , jumps by the amount F onto the p -axis, and reaches the upper corner of the triangle at time $+\infty$. In the limit $y_1 \rightarrow -\infty, y_2 \rightarrow \infty$ the orbit is pressed against

the sides of the triangle. In the second case (Fig. 4c), which leads to the falling cross-section of Ref. 9, one chooses the hypotenuse of the triangle as the classical solution to jump off from and back onto. In both cases, the energy of the classical solutions goes to zero when $y_1 \rightarrow -\infty$, $y_2 \rightarrow \infty$, and the action is equal to the area under the orbit. The first case corresponds to the solution with minimal action; arguments for choosing the second case as the relevant physical one are given in Ref. 9.

3.2 Existence of fixed points

When discussing solutions to the system of differential equations (3.3), one starts by searching for fixed points (singular points). These are those points in the p - and q -space where all time derivatives vanish, and, hence, the solution $[p(y), q(y)]$ "spends an infinitely long time". In order to simplify our discussion we shall first only look for fixed points with $p^I = q^I = 0$. Furthermore, we shall search for fixed points and solutions to (3.3) for which the $T = 2$ vectors p^{II} and q^{II} point into a certain direction in group space: $p^{II} = (0, 0, 0, 0, p_8^{II})$, etc. (this is the analogue of distinguishing the z -axis in ordinary mechanics). It is then easy to see that $\dot{p}_i^I = 0$, $\dot{q}_i^I = 0$ ($i = 1, 2, 3$), $\dot{p}_1^{II} = \dots = \dot{p}_8^{II} = 0$, $\dot{q}_1^{II} = \dots = \dot{q}_8^{II} = 0$, and the only non-vanishing equations of motion are:

$$\begin{aligned} \dot{q}^0 &= -\mu_0 q^0 + \lambda_{00}^0 q^0 (q^0 + 2\rho^0) + \lambda_{II,II}^0 (q_8^{II})^2 + \lambda_{0,II}^{\Pi} p_8^{\Pi} q_8^{\Pi} \\ \dot{q}_8^{\Pi} &= -\mu_{II} q_8^{\Pi} + \lambda_{II,II}^{\Pi} d_{sss} q_8^{\Pi} (q_8^{\Pi} + 2\rho_8^{\Pi}) + 2\lambda_{II,II}^0 q^0 p_8^{\Pi} + \lambda_{0,II}^{\Pi} q_8^{\Pi} (q^0 + \rho^0) \\ -\dot{\rho}^0 &= -\mu_0 \rho^0 + \lambda_{00}^0 \rho^0 (\rho^0 + 2q^0) + \lambda_{II,II}^0 (\rho_8^{\Pi})^2 + \lambda_{0,II}^{\Pi} q_8^{\Pi} \rho_8^{\Pi} \\ -\dot{\rho}_8^{\Pi} &= -\mu_{II} \rho_8^{\Pi} + \lambda_{II,II}^{\Pi} d_{sss} \rho_8^{\Pi} (\rho_8^{\Pi} + 2q_8^{\Pi}) + 2\lambda_{II,II}^0 \rho^0 q_8^{\Pi} + \lambda_{0,II}^{\Pi} \rho_8^{\Pi} (q^0 + \rho^0). \end{aligned} \quad (3.5)$$

Trying to find values of P and q that make all equations (3.5) vanish, one immediately sees that with $p^0 = p_8^{II} = 0$ the last two equations are zero. Vanishing of the first two equations is obtained if:

$$\left(q^0 - \frac{\mu_0}{2\lambda_{00}^0}\right)^2 + \frac{\lambda_{II,II}^0}{\lambda_{00}^0} (q_8^{\Pi})^2 = \left(\frac{\mu_0}{2\lambda_{00}^0}\right)^2 \quad (3.6)$$

$$0 = q_8^{\Pi} \left(-\mu_{II} + \lambda_{II,II}^{\Pi} d_{sss} q_8^{\Pi} + \lambda_{0,II}^{\Pi} q^0\right). \quad (3.7)$$

Solutions to these equations are the analogue of the two fixed points on the q -axis in Fig. 4a: in our case this q -axis, in fact, represents a plane with q^0 and q_8^{II} as coordinates (Fig. 5), and the zeros of the first two equations (3.5) are the intersection points of the two curves (3.6) and (3.7). In Figs. 5a and b we have illustrated the two cases $\lambda_{II,II}^0/\lambda_{0,0}^0 < 0$ and $\lambda_{II,II}^0/\lambda_{0,0}^0 > 0$: Eq. (3.6) then describes a hyperbola or ellipse, respectively. For the straight line given by the vanishing of the bracket expression in (3.7) there are several possibilities. Depending on the intercept $\mu_{II}/\lambda_{0,II}^{II}$ and slope $2\sqrt{3} \lambda_{II,II}^{II}/\lambda_{0,II}^{II}$ ($d_{888} = -1/\sqrt{3}$ from Table 2), the line intersects with the hyperbola (or ellipse) twice, once, or not at all. Figures 5a and b show those two situations which, as we shall discuss in the following, lead to acceptable situations. With $\lambda_{0,0}^0 > 0$, Fig. 5a corresponds to $\lambda_{II,II}^{II} > 0$ and $\lambda_{0,II}^0 < 0$, whereas in Fig. 5b, $\lambda_{II,II}^{II} > 0$ and $\lambda_{0,II}^0 < 0$ [$\mu_0 < 0$ and $\mu_{II} > 0$ from (2.3)].

When searching for classical solutions that minimize the action in the path integral (2.19) we can accept only those which when time goes to $-\infty$ or $+\infty$ start or end at stable fixed points. In order to illustrate the notation "stable" we show in Figs. 6a-d a few different characteristics of a fixed point (the arrows denote the direction of time evolution)¹⁹⁾: in Figs. 6a and b the fixed point is a stable departure point (being sufficiently close, all classical solutions will fall into the fixed point when time goes to $-\infty$; in Fig. 6b the solutions spiral). In Fig. 6c the fixed point is unstable (depending on the direction, the solutions either approach or move away from the fixed point), and the fixed point in Fig. 6d ("vortex point") is never reached by any classical solution. When in Figs. 6a and b the arrows go in the opposite direction, the point will be called "arrival point". A mathematical criterion for distinguishing between these different cases is obtained by linearizing the equations of motion around the fixed point: if, for example, the equations of motion are of the form $\dot{q} = \partial H / \partial p|_{p=0}$ (q being the two-dimensional vector with the components q^0 and q_8^{II}), then $\dot{q} = 0$ at the fixed point, and the linearized equations of motion near the fixed point $q = \bar{q}$ are:

$$\dot{q} \approx \left. \frac{\partial^2 H}{\partial p \partial q} \right|_{q=\bar{q}, p=0} (q - \bar{q}) . \quad (3.8)$$

The characteristic values of the matrix $\partial^2 H / \partial p \partial q$ then tell whether the fixed point at $\bar{q}$ is stable or not: if both eigenvalues are real and positive (negative), we have a stable departure (arrival) point (Fig. 6a). In the case of Fig. 6b, they are complex conjugate to each other with positive real parts; for Fig. 6c, they are real with opposite signs; for Fig. 6d, they are purely imaginary.

Let us now analyse the stability properties of the fixed points in Fig. 5. Rather than analytically calculating the eigenvalues of $\partial^2 H / \partial p \partial q$ for each of these points we use a simple graphical method. Along the hyperbola in Fig. 5a (ellipse in Fig. 5b) Eq. (3.6) $\dot{q}^0 = 0$; above the upper and below the lower branch of the hyperbola (outside the ellipse) $\dot{q}^0$ is positive and changes sign when going across the hyperbola (ellipse). Similarly for $\dot{q}_8^{II}$: it vanishes on the two straight lines given by Eq. (3.7) and changes sign when crossing the line (see Fig. 5). Since $dq^0/dq_8^{II} = \dot{q}^0/\dot{q}_8^{II}$, we thus have a simple way to find the sign of the slope: furthermore, dq^0/dq_8^{II} is zero when $\dot{q}^0 = 0$ and becomes infinite when $\dot{q}_8^{II} = 0$. From this we immediately find that the origin is an unstable fixed point; the point B on the hyperbola in Fig. 5a with $\dot{q}_8^{II} = 0$ is a stable arrival point, and the lower fixed point in Fig. 5b is unstable, too. The remaining points in Figs. 5a and b can be spiralling arrival, spiralling departure points or of the type Fig. 6d. In order to find out which situation is realized, we observe that at any rate for each of these points the two eigenvalues of $\partial^2 H / \partial p \partial q$ are pairwise complex conjugate; furthermore, $\text{tr} (\partial^2 H / \partial p \partial q) = \text{sum of eigenvalues} = 2 \times \text{real part of the eigenvalues}$. The trace of $\partial^2 H / \partial p \partial q$ is found to be

$$\text{tr} \left(\frac{\partial^2 H}{\partial p \partial q} \Big|_{p=0} \right) = -\mu_0 + 2\lambda_{00}^0 q^0 - \lambda_{II,II}^{\Pi} q^{\Pi} / \sqrt{3}, \quad (3.9)$$

and the zeros of (3.9) are the dotted lines in Figs. 5a and b. If the parameters μ and λ are such that the two spiralling points are above this line (as we have drawn in Figs. 5a and b), the points A in both cases are stable, spiralling departure points.

In order to complete our discussion of stability within the $p = 0$ subspace, we must consider the remaining directions q_1^{II} , q_3^{II} , q_4^{II} , q_6^{II} , and q_i^I which we have put equal to zero. For the arrival point B in Fig. 5a it is easy to see that the matrix $\partial^2 H / \partial p \partial q$ is diagonal in the variables q_1^{II} , q_3^{II} , q_4^{II} , q_6^{II} , and the diagonal elements = eigenvalues are

$$\frac{\partial^2 H}{\partial p_1^{II} \partial q_1^{II}} = \frac{\partial^2 H}{\partial p_3^{II} \partial q_3^{II}} = \frac{\partial^2 H}{\partial p_4^{II} \partial q_4^{II}} = \frac{\partial^2 H}{\partial p_6^{II} \partial q_6^{II}} = -\mu_{II} + \lambda_{0,II}^{\Pi} q^0 < 0 \quad (3.10)$$

Thus this point is a stable arrival point for all these directions. For the two spiral points A in Figs. 5a and b the matrix $\partial^2 H / \partial p \partial q$ is also diagonal for these variables, and the eigenvalues are

$$\frac{\partial^2 H}{\partial p_1^{II} \partial q_1^{II}} = \frac{\partial^2 H}{\partial p_3^{II} \partial q_3^{II}} = \sqrt{3} \lambda_{II,II}^{\Pi} q^{\Pi} > 0, \quad (3.11)$$

$$\frac{\partial^2 H}{\partial p_4^{\text{II}} \partial q_4^{\text{II}}} = \frac{\partial^2 H}{\partial p_6^{\text{II}} \partial q_6^{\text{II}}} = 0. \quad (3.12)$$

The first condition is satisfied for the right-hand side spiralling point in both Figs. 5a and b; hence they are stable departure points for the directions q_1^{II} and q_3^{II} . The appearance of two zero eigenvalues in (3.12) reflects the invariance against SU(2) rotations. Finally, we consider the part of $\partial^2 H / \partial p \partial q$ which refers to the q^{I} fields:

$$\frac{\partial^2 H}{\partial p^{\text{I}} \partial q^{\text{I}}} = (-\mu_{\text{I}} + \lambda_{0,\text{I}}^{\text{I}} q^0) + \lambda_{\text{I},\text{II}}^{\text{I}} q_8^{\text{II}} \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & -\frac{2}{\sqrt{3}} \end{pmatrix} \quad (3.13)$$

In order that the arrival point B of Fig. 5a remains stable with respect to the q^{I} components we need

$$-\mu_{\text{I}} + \lambda_{0,\text{I}}^{\text{I}} q^0 = -\mu_{\text{I}} + \mu_0 \frac{\lambda_{0,\text{I}}^{\text{I}}}{\lambda_{0,0}^0} < 0 \quad (3.14)$$

or

$$\frac{\lambda_{0,\text{I}}^{\text{I}}}{\lambda_{0,0}^0} > \frac{|\mu_0|}{|\mu_{\text{I}}|}.$$

$[\mu_0$ and μ_{I} are negative, see (2.3)] which, in particular, requires $\lambda_{0,\text{I}}^{\text{I}} < 0$. For the departure point A in both Figs. 5a and 5b the stability condition is

$$-[-\mu_{\text{I}} + \lambda_{0,\text{I}}^{\text{I}} q^0] < q_8^{\text{II}} \lambda_{\text{II},\text{I}}^{\text{I}} \frac{1}{\sqrt{3}} < \frac{1}{2} [-\mu_{\text{I}} + \lambda_{0,\text{I}}^{\text{I}} q^0]. \quad (3.15)$$

In the case of Fig. 5a, the q^0 and q_8^{II} coordinates of A are positive and both (3.14) and (3.15) can be satisfied by requiring that $\lambda_{\text{II},\text{I}}^{\text{I}}$ is not too large. In the case of Fig. 5b we have only the condition (3.15), and this can easily be fulfilled by putting appropriate restrictions on $\lambda_{0,\text{I}}^{\text{I}}$ and $\lambda_{\text{II},\text{I}}^{\text{I}}$.

The result of this discussion can be summarized as follows. We have determined two regions in the parameter space (μ, λ) for which a stable departure point exists in the $p = 0$ subspace. The first region corresponds to Fig. 5a, and the restrictions on the λ 's are

$$\lambda_{0,0}^0, \lambda_{\text{II},\text{II}}^{\text{II}}, \lambda_{0,\text{II}}^{\text{II}}, \lambda_{0,\text{I}}^{\text{I}} > 0 \quad (3.16)$$

$$\lambda_{\text{II},\text{I}}^{\text{I}} < 0.$$

In addition to that, the departure point A must be above the dotted line in Fig. 5a [given by putting (3.9) equal to zero], and $\lambda_{0,I}^I$ and $\lambda_{II,I}^I$ have to satisfy (3.14) and (3.15). The second case is that of Fig. 5b and requires

$$\lambda_{0,0}^0, \lambda_{II,II}^{\Pi}, \lambda_{II,II}^0 > 0$$

$$\lambda_{0,II}^{\Pi} < 0. \quad (3.17)$$

Again, the departure point A must be above the dotted line in Fig. 5b, and $\lambda_{0,I}^I$ and $\lambda_{II,I}^I$ have to satisfy (3.15).

Since the Hamiltonian (2.2) is invariant under

$$\text{all } \lambda \rightarrow -\lambda, p \rightarrow -p, q \rightarrow -q, \quad (3.18)$$

and

$$\lambda_{II,II}^{\Pi} \rightarrow -\lambda_{II,II}^{\Pi}, \lambda_{II,I}^I \rightarrow -\lambda_{II,I}^I, \lambda_{I,I}^{\Pi} \rightarrow -\lambda_{I,I}^{\Pi}$$

$$p^{\Pi} \rightarrow -p^{\Pi}, q^{\Pi} \rightarrow -q^{\Pi}, \quad (3.19)$$

further regions in λ -space can be obtained from (3.16) and (3.17) by applying the symmetry transformations (3.18) and (3.19). With the same methods that we have used to discuss (3.16) and (3.17) one can show that for all other regions in λ -space no stable departure point exists. The departure point that we have found is the analogue of the left corner of the triangle (Fig. 4) in supercritical Pomeron theory. As we shall show later on, it plays exactly the same role. From now on we shall assume that the λ -parameters are either in the region given by (3.16) and Fig. 5a or in the region (3.17) and Fig. 5b.

In order to complete our discussion on the existence of fixed points we have to leave the $p = 0$ subspace. Because of the p - q symmetry of the Hamiltonian (2.2), the $q = 0$ subspace (p -axis in Fig. 4) has the same fixed point structure as the $p = 0$ subspace with only the change of time direction [owing to the minus sign in the second equation of motion in (3.31)]. What was the stable departure point in the $p = 0$ subspace becomes, therefore, a stable arrival point in the $q = 0$ subspace, the analogue of the upper corner of the triangle Fig. 4. When leaving the $p = 0$ subspace, we also have to extend our investigation of stability properties. What we have shown so far is that the fixed point is a stable departure point for solutions in the $p = 0$ subspace. When p becomes different from zero the linearized equations of motion near the fixed point ($q = \bar{q}, p = 0$) are

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} H_{pq} & H_{pp} \\ -H_{qq} & -H_{qp} \end{pmatrix} \begin{pmatrix} q - \bar{q} \\ p \end{pmatrix}, \quad (3.20)$$

where $H_{qp} = H_{pq} = \partial^2 H / \partial p \partial q$, etc. are 9×9 dimensional block matrices. As before, stability properties of the fixed point $(\bar{q}, 0)$ are determined by the eigenvalues of the 18×18 matrix in (3.20). It is easy to see that for $p = 0$ the block matrix H_{qq} vanishes. Therefore, the vectors $(Q, 0)$ with Q being an eigenvector of H_{pq} are eigenvectors of the matrix in (3.20), and their eigenvalues are those of the block matrix H_{pp} . These, however, we know already from our analysis of motions in the $p = 0$ subspace [see Eq. (3.8)], and we have found that they are positive semidefinite (two of them are complex with positive real parts). The second set of eigenvectors of the matrix in (3.20) is given by the vectors (P, Q) , where Q again is an eigenvector of H_{pq} and P satisfies:

$$H_{pq} P + H_{pp} Q = -(\text{eigenvalue of } H_{pq}) \cdot Q. \quad (3.21)$$

The eigenvalues of these vectors (P, Q) are equal to minus the eigenvalues of H_{pq} , i.e. they are negative or complex with negative real parts.

This means that, in the total p - q space, there is a subspace, formed by the vectors (P, Q) satisfying (3.21), in which the fixed point has the characteristics of a stable arrival point. The situation is shown in Fig. 7: the horizontal line represents the $p = 0$ subspace in which orbits depart from the fixed point, the intersecting line the other subspace in which the fixed point acts as an arrival point. Points slightly outside the $p = 0$ subspace have their main projection along the departing subspace and a small component along the arrival subspace: the sum of these two components then leads to a motion away from the fixed point towards the $p = 0$ subspace (Fig. 7). This is completely analogous to supercritical Pomeron theory (Fig. 4). For the other arrival or departure points in the $p = 0$ or $q = 0$ subspace the same consideration applies (Fig. 7).

For the origin we have found that it is unstable within the $p = 0$ subspace and the $q = 0$ subspace. In order to determine the orbits of classical solutions near this point we look at the equations of motion (3.5) in the four-dimensional $p^0, q^0, p_s^{II}, q_s^{II}$ space. Let us consider the region where all these variables are small compared to μ/λ : $p^0 \ll \mu/\lambda$, etc., where μ and λ can be any of the μ 's and λ 's in (3.5). Furthermore, we confine ourselves to the case where $p_s^{II}, q_s^{II} \ll p^0, q^0$, i.e. we look for the orbits near the p^0 - q^0 plane. The equations of motion can be approximated:

$$\begin{aligned}\dot{q}^0 &= -\mu_0 q^0, & -\dot{p}^0 &= -\mu_0 p^0 \\ \dot{q}_s^\Pi &= (-\mu_\Pi + \lambda_{0,\Pi}^\Pi (p^0 + q^0)) q_s^\Pi + 2 \lambda_{\Pi,\Pi}^0 q^0 p_s^\Pi \\ -\dot{p}_s^\Pi &= (-\mu_\Pi + \lambda_{0,\Pi}^\Pi (p^0 + q^0)) p_s^\Pi + 2 \lambda_{\Pi,\Pi}^0 p^0 q_s^\Pi.\end{aligned}\quad (3.22)$$

From this we see that if $p_s^{\text{II}} = q_s^{\text{II}} = 0$ at some time, they will remain zero and the orbits stay in the $p^0 - q^0$ plane (Fig. 8): coming from the p^0 -direction, we are directed into the q^0 -direction. Next we look for motions near the $p_s^{\text{II}} - q_s^{\text{II}}$ plane. If $p_s^{\text{II}}, q_s^{\text{II}} \gg p_0, q_0$ the equations of motion are:

$$\begin{aligned}\dot{q}_s^\Pi &= -\mu_\Pi q_s^\Pi, & -\dot{p}_s^\Pi &= -\mu_\Pi p_s^\Pi \\ \dot{q}^0 &= -\mu_0 q^0 + q_s^\Pi (\lambda_{\Pi,\Pi}^0 q_s^\Pi + \lambda_{0,\Pi}^\Pi p_s^\Pi) \\ -\dot{p}^0 &= -\mu_0 p^0 + p_s^\Pi (\lambda_{\Pi,\Pi}^0 p_s^\Pi + \lambda_{0,\Pi}^\Pi q_s^\Pi).\end{aligned}\quad (3.23)$$

They have the solutions:

$$\begin{aligned}q_s^\Pi &= Q_s e^{-\mu_\Pi y}, & p_s^\Pi &= P_s e^{\mu_\Pi y} \\ q^0 &= \frac{Q_s^2 \lambda_{\Pi,\Pi}^0}{\mu_0 - 2\mu_\Pi} e^{-2\mu_\Pi y} + \frac{\lambda_{0,\Pi}^\Pi}{\mu_0} P_s Q_s \\ &\quad + e^{-\mu_0 y} \left\{ Q_0 - \frac{Q_s^2 \lambda_{\Pi,\Pi}^0}{\mu_0 - 2\mu_\Pi} - \frac{P_s Q_s \lambda_{0,\Pi}^\Pi}{\mu_0} \right\}\end{aligned}\quad (3.24)$$

$$\begin{aligned}p^0 &= \frac{P_s^2 \lambda_{\Pi,\Pi}^0}{\mu_0 - 2\mu_\Pi} e^{2\mu_\Pi y} + \frac{\lambda_{0,\Pi}^\Pi}{\mu_0} P_s Q_s \\ &\quad + e^{\mu_0 y} \left\{ P_0 - \frac{P_s^2 \lambda_{\Pi,\Pi}^0}{\mu_0 - 2\mu_\Pi} - \frac{P_s Q_s \lambda_{0,\Pi}^\Pi}{\mu_0} \right\}\end{aligned}\quad (3.25)$$

(P_0, Q_0, P_s, Q_s are the values of $p_0, q_0, p_s^{\text{II}}, q_s^{\text{II}}$ at time $y = 0$). Remembering that $\mu_{\text{II}} > 0, \mu_0 < 0$, we see that, for positive y , p_s^{II} and p^0 grow, whereas q_s^{II} and q^0 stay small as long as the curly brackets in (3.25) vanish. For negative y the q 's grow and the p 's stay small as long as the bracketed expression of Eq. (3.26) vanishes. Therefore, if we are at time $y = 0$ in the two-dimensional

plane determined by the requirement that the bracketed terms of Eq. (3.25) and (3.26) vanish, the motion for positive y approaches the $q = 0$ subspace, and for negative y the $p = 0$ subspace.

These two calculations show that in the four-dimensional space around the origin there are two two-dimensional planes such that within these planes there are orbits which connect motion in the $p = 0$ subspace with motion in the $q = 0$ subspace. Outside these planes solutions arrive at or leave from the origin somewhere between the $p = 0$ and $q = 0$ subspaces. All this can be generalized to include the remaining $p^{\text{II}}, q^{\text{II}}$ and $p^{\text{I}}, q^{\text{I}}$ components.

As to the question whether other fixed points exist outside the $p = 0$ or $q = 0$ subspace, we only mention that the analogue of the fixed point in the centre of the triangle in Fig. 4a exists. It has the same stability properties as in supercritical Pomeron theory, i.e. it is a vortex point surrounded by closed orbits and cannot be reached by classical solutions. In the following we shall assume that this information about the fixed point structure is sufficient to explain the orbits of classical solutions. In particular, we make the assumption that classical solutions exist which connect the fixed points in the $p = 0$ subspace with those of the $q = 0$ subspace.

We then summarize the fixed point structure that we have discussed in this section in Figs. 7 and 9. In Fig. 7 the q -axis represents the ensemble of all q -coordinates; for the fixed points A and B, however, only q^0 and q_8^{II} are different from zero. The q^0 - q_8^{II} plane is shown in Fig. 9. Figure 9a corresponds to the situation of Fig. 5a, Fig. 9b to that of Fig. 5b (for the latter the fixed points B and B' in Fig. 7 are absent). For the orbits of classical solutions near the fixed points we have found that A is a stable departure point in the $p = 0$ subspace and B a stable arrival point. Outside the $p = 0$ subspace there are certain directions in which A becomes an arrival point and B a departure point. The origin is not stable within the $p = 0$ or the $q = 0$ subspace. Orbits exist which approach the origin from the $p = 0$ subspace and turn into the $q = 0$ subspace, and others which come from the $q = 0$ direction and leave along the $p = 0$ direction.

This completes our discussion of fixed points (and the classical solutions around them), for which only $p^0, p_8^{\text{II}}, q^0$, and q_8^{II} are different from zero. Because of the SU(2) invariance of our Hamiltonian H we could have chosen any other direction for the p^{II} and q^{II} vectors. Applying all possible SU(2) rotations to the fixed points that we have discussed we obtain a closed surface. In Appendix B we show that, within the $p = 0$ subspace or the $q = 0$ subspace, there are no other stable fixed points outside this surface. This implies that we know all stable fixed points in the $p = 0$ and $q = 0$ subspaces for which q^0 and q_α^{II} or p^0 and q_α^{II}

($\alpha = 1, 3, 4, 6, 8$) are different from zero. As to the question whether fixed points exist with $p^I \neq 0$ and $q^I = 0$, we only can say that, if they exist, the vectors p^I and q^I must be complex valued because of the imaginary couplings $\lambda_{I,I}^0$ and $\lambda_{I,I}^{II}$ in (2.2). If we restrict ourselves to real values of p and q , only fixed points with $p^I = 0$ and $q^I = 0$ are possible.

3.3 Classical solutions

Let us now use our knowledge of the fixed point structure to construct classical solutions. We are interested in orbits which, when time is large and negative, start near a departure point and, for large positive time, end at an arrival point. Furthermore, the solutions should preserve the p - q symmetry of the Hamiltonian, and in view of quantum oscillations we demand that departure and arrival points be stable. These requirements leave us with the following possibilities (Fig. 10): solutions might

- i) leave point A in the $p = 0$ subspace, reach the $q = 0$ subspace via the origin and end in A';
- ii) leave A' within the p - q subspace, where A' is a stable departure point, and reach A within the analogous subspace;
- iii) leave B within its departure direction and end at B';
- iv) leave B' within the $q = 0$ subspace, reach the $p = 0$ subspace via the origin and end at B;
- v) leave point A within the $p = 0$ subspace, reach B and turn into the departure direction of B; at B' enter the $q = 0$ subspace and end at A.

One can, of course, find more possibilities by adding these orbits together; but then the action integral (2.24) receives additional contributions and is no longer minimal. If there are more fixed points outside $p = 0$ and/or $q = 0$, more orbits might exist, but nothing can be said about that. Our list of solutions is complete for orbits that leave and end at $p = 0$ or $q = 0$. We shall now discuss these solutions, using our knowledge about the local neighbourhood of the fixed points. Since we do not know the global structure of the solutions we have to make the assumption that the solutions do connect the fixed points.

Starting with solution (i) we assume that the orbits in the $p = 0$ and the $q = 0$ subspace have the structure given in Fig. 9. A solution that starts in the $p = 0$ subspace near A then can end either at B, at the origin (one particular orbit only), or at infinity. Starting near the fixed point A slightly outside the $p = 0$ subspace, a solution still will approximately have the same motion as in the $p = 0$ subspace: sufficiently close to $p = 0$ all $\dot{p}$ are arbitrarily small, whereas the q -velocities are finite except at the fixed points. From what we

said before about solutions near the origin, it follows that a two-dimensional subset of orbits (approximately in the p_8 - q_8 plane) approaches the origin close to the $p = 0$ subspace, turns into a motion very close to the $q = 0$ subspace and then moves towards the fixed point A' . These are the solutions of type (i).

For solutions (ii) we only note that all orbits near A' , even outside the departure subspace leave A' while approaching the departure subspace. We then have to assume that at least some of these orbits connect the neighbourhood of A' with that of A . For type (iii) exactly the same discussion applies.

For type (iv) we need those solutions which start near A , a little outside the $p = 0$ subspace, and then move towards B . Near B the orbits turn around and come close to the departure direction. Having reached B' , another turn into the $q = 0$ direction is made, and the orbits move towards A' .

Finally, type (v) belongs to those orbits which approach the origin from the p^0 -direction and leave in the q^0 -direction. Since B' and B are in the p^0 - and q^0 -directions, respectively, these orbits stay all the time in the p^0 - q^0 plane. This completes our discussion on classical solutions. Let us now turn to the path integral (2.22) and its classical approximation (2.23).

4. THE TOTAL CROSS-SECTION

We now want to calculate the action integral (2.24). In our present discussion, we have put $T = 0$, and (2.24) reads

$$A_{\alpha}^0(F, G) = \int_{-\infty}^{\infty} dy \left[\frac{1}{2} p_{\alpha} \overleftrightarrow{\partial}_y q_{\alpha} - H^0(p_{\alpha}, q_{\alpha}) - \delta(y-y_1) G \cdot q_{\alpha} - \delta(y-y_2) F p_{\alpha} \right]. \quad (4.1)$$

In order to compute this integral we have to solve the equations of motion in the presence of the source terms. Since these source terms act instantaneously [cf. (2.22)], our knowledge of solutions to (3.1) is completely sufficient: in the three time intervals $(-\infty, y_1)$, (y_1, y_2) , and (y_2, ∞) the sources are zero, and at times y_1 and y_2 they cause a jump of p and q by the amounts G and $-F$, respectively. Time integration goes from $-\infty$ to ∞ , and in order to obtain a finite action integral, solutions must start and end at fixed points. Corresponding to the different possibilities of connecting fixed points by classical solutions [types (i) to (v) in the previous section], different values for the action integral are obtained. In this section we shall deal with (i) and (v), which are analogous to the non-decreasing solution of supercritical Pomeron theory. Our type (ii) parallels the other solution in supercritical Pomeron theory, and we

shall discuss this case in a separate publication¹⁷⁾. In the two remaining cases (iii) and (iv), the motion takes place in the p^0-q^0 plane, and all the other components are zero (the fixed points B and B', where the motion starts and ends, lie in this plane, and in order to pass the origin when going from B' to B, the orbits must lie in the p^0-q^0 plane). These solutions, therefore, represent the special case where all the $T = 1$ and $T = 2$ fields are switched off, and for the remaining $T = 0$ field we are back to undercritical Pomeron theory with falling cross-section, and no further discussion is needed.

Let us now discuss how the solutions (i) and (v) contribute to the action integral (4.1). We take y_1 negative and large, y_2 positive and large, and the size of the sources F and G small compared with μ/λ , the scale of the fixed point coordinates. The classical solution under the integral (2.24) is determined by the requirements:

$$p_\alpha(y_1) = G \quad (4.2)$$

$$q_\alpha(y_2) = F. \quad (4.3)$$

In order to construct the solution, we start near A (Fig. 11): given the condition (4.2) for the p-components, we find q-components such that we are close to the arrival subspace [the arrival subspace is determined by a linear relation between p and q components, Eq. (3.17)]. When time increases, we know from our discussion of the preceding section that the orbit of our solution passes near A and then moves close to the $p = 0$ subspace either towards the origin, fixed point B or towards infinity. Among the solutions approaching the origin there are those which, having passed the origin, are close to the $q = 0$ subspace and move towards A' [type (i)]. Near A' they turn around and, for certain values of F, they intersect with the subspace $q = +F$. Our freedom in choosing q at the starting time y_1 can then be used to satisfy (4.3). Other orbits coming from A move close to B, turn around into the direction towards B', again change direction and, finally, approach A' [type (v)]. For certain values of F, (4.3) can be satisfied. In the limit $y_1 \rightarrow -\infty, y_2 \rightarrow \infty$ all these solutions come closer and closer to A and A' and approach the $p = 0$ and $q = 0$ subspaces. It is now easy to compute (4.1). In the time interval $(-\infty, y_1)$ we have $p \equiv 0$. This implies $H(p_{cl}, q_{cl}) = 0$ [from (2.2)], and, hence, the integrand of (4.1) vanishes. The same is true for the range (y_2, ∞) . Between y_1 and y_2 , $H(p_{cl}, q_{cl})$ is a constant of motion (energy), and for the solutions passing through A and A' we have $H(p_{cl}, q_{cl}) = 0$. We are left with:

$$A^0_\alpha(\overline{F}, G) = \frac{1}{2} \int_{y_1}^{y_2} p_\alpha \overleftrightarrow{\partial}_y q_\alpha dy \quad (4.4)$$

$$= \int_{y_1}^{y_2} (p^0 dq^0 + p_i^I dq_i^I + p_\alpha^{\overline{II}} dq_\alpha^{\overline{II}})$$

(repeated indices are summed over). Each part of this integral has a simple geometrical interpretation: it is the area under the curve in the p-q space. In the limit $y_1 \rightarrow -\infty$, $y_2 \rightarrow \infty$ the integral (4.4) approaches a finite constant, whose value depends, of course, on the orbit [type (i) gives a value other than type (v)]. The S-matrix element, within our approximation (2.23), therefore becomes

$$S_0(\overline{F}, G) = \text{constant}. \quad (4.5)$$

The constant depends on F and G and is different from one. This leads to the result

$$G_{\text{tot}} = \text{constant}. \quad (4.6)$$

It is clear from our discussion that the question about which classical orbit [either type (i) or type (v)] has to be taken when evaluating (4.1) strongly depends on the sources F and G. For some values F and G, solutions of type (i) exist, for others type (v), and there will also be values for which neither of them exists. Such a dependence of the scattering amplitude on the sources can most easily be understood from the quantum mechanical definition (2.13). The matrix element $\langle \psi_n(T) | G | \psi_0 \rangle$ denotes the projection of the state $G | \psi_0 \rangle$ onto the energy and isospin eigenstate $|\psi_n(T)\rangle$. Depending upon the choice of G (directions in the group space), this projection takes different values.

5. NON-ZERO QUANTUM NUMBERS

We now want to evaluate the action integral (2.24) for non-zero isospin T. The way we attack this problem is the same as before: we first study the equations of motion

$$\dot{q} = \frac{\partial H^T}{\partial p}, \quad -\dot{p} = \frac{\partial H^T}{\partial q}, \quad (5.1)$$

where H^I is the Hamiltonian for definite isospin T . In order to find the form of H^I , we introduce a different parametrization of the components of the $T = 2$ field and the $T = 1$ field: this is the analogue of transforming to polar coordinates in classical mechanics with rotational $[= O(3)]$ symmetry.

Let us start with the $T = 2$ fields p^{II} and q^{II} only. If we denote by L_x , L_y and L_z the three generators of the group and by $D_x(\theta) = \exp(i\theta L_x)$ the corresponding finite rotations, then any vector q^{II} can be written as (we show this in Appendix B):

$$q^{II} = R^{II}(\varphi, \vartheta, \psi) \hat{q}^{II}, \quad (5.2)$$

where $R^{II}(\phi, \theta, \psi)$ is a $SU(2)$ rotation depending on three angles

$$R^{II}(\varphi, \vartheta, \psi) = D_z^{II}(\varphi) D_y^{II}(\vartheta) D_z^{II}(\psi). \quad (5.3)$$

D_z^{II} , etc. are the $T = 2$ representation matrices, given in Appendix A. The vector $\hat{q}^{II}$ in (5.2) is a standard vector lying in a segment of the $q_8^{II} - q_3^{II}$ plane (Fig. 12)

$$\hat{q}^{II} = (0, \hat{q}_3^{II}, 0, 0, \hat{q}_8^{II}) \quad , \quad 0 \leq \hat{q}_3^{II} \leq \sqrt{3} \hat{q}_8^{II}. \quad (5.4)$$

Next we define -- for each vector q^{II} -- three normal vectors n_x , n_y , and n_z , which point into the direction in which the three $SU(2)$ generators act on q^{II}

$$\begin{aligned} n_x &= i R^{II}(\varphi, \vartheta, \psi) L_x^{II} \hat{q}^{II} \\ n_y &= i R^{II}(\varphi, \vartheta, \psi) L_y^{II} \hat{q}^{II} \\ n_z &= i R^{II}(\varphi, \vartheta, \psi) L_z^{II} \hat{q}^{II} \end{aligned} \quad (5.5)$$

They are related to the angular derivatives of q^{II}

$$\begin{aligned} \frac{\partial q^{II}}{\partial \varphi} &= \cos \vartheta n_z + \sin \vartheta \cos \psi n_x - \sin \vartheta \sin \psi n_y \\ \frac{\partial q^{II}}{\partial \psi} &= n_z \\ \frac{\partial q^{II}}{\partial \vartheta} &= \cos \psi n_y - \sin \psi n_x \end{aligned} \quad (5.6)$$

Using the explicit form of L^{II} (Appendix A), one finds that these vectors are orthogonal to each other and perpendicular to q^{II} .

$$0 = q^{II} \cdot n_x = q^{II} \cdot n_y = q^{II} \cdot n_z \quad (5.7)$$

$$0 = n_x \cdot n_y = n_x \cdot n_z = n_y \cdot n_z. \quad (5.8)$$

For the vector p^{II} we make use of the fact that it is the conjugate momentum of q^{II} . We now abandon the symmetric treatment of p and q that we have kept so far, and consider p^{II} as the momentum associated with each space point q^{II} . For each vector q^{II} , we decompose p^{II} into components along n_x , n_y , and n_z and a remainder perpendicular to them

$$p^{II} = p_x^{II} + p_y^{II} + p_z^{II} + p_{||}^{II}. \quad (5.9)$$

With this parametrization we rewrite the kinetic part of the Lagrangian (2.1), using the relations (5.6)

$$p^{II} \cdot \dot{q}^{II} = \hat{p}^{II} \cdot \hat{\dot{q}}^{II} + \dot{\vartheta} \ell_{\vartheta} + \dot{\psi} \ell_{\psi} + \dot{\varphi} \ell_{\varphi}, \quad (5.10)$$

where

$$\ell_{\vartheta} = \hat{p}_z |\hat{n}_z| \cos \vartheta + \hat{p}_x |\hat{n}_x| \sin \vartheta \cos \psi - \hat{p}_y |\hat{n}_y| \sin \vartheta \sin \psi \quad (5.11)$$

$$\ell_{\psi} = \hat{p}_z |\hat{n}_z|$$

$$\ell_{\varphi} = \hat{p}_y |\hat{n}_y| \cos \psi - \hat{p}_x |\hat{n}_x| \sin \psi$$

and

$$\hat{p}^{II} = R^{II}(\varphi, \vartheta, \psi)^{-1} p_{||}^{II} \quad (5.12)$$

$$\hat{p}_x^{II} = R^{II}(\varphi, \vartheta, \psi)^{-1} p_x, \text{ etc.}$$

$$\hat{n}_x = R^{II}(\varphi, \vartheta, \psi)^{-1} n_x = i L_x^{II} \hat{q}^{II}, \text{ etc.}$$

The ℓ 's defined in (5.11) play a similar role to the polar and azimuthal angular momentum components in classical mechanics (that is why we use the letter ℓ). Only one of them is a constant of motion. For each generator of SU(2) rotations which leaves H invariant, there is a conserved quantity

$$\begin{aligned} \ell_x^{\Pi} &= i (L_x^{\Pi})_{\alpha\beta} p_{\alpha}^{\Pi} q_{\beta}^{\Pi} \\ \ell_y^{\Pi} &= i (L_y^{\Pi})_{\alpha\beta} p_{\alpha}^{\Pi} q_{\beta}^{\Pi} \\ \ell_z^{\Pi} &= i (L_z^{\Pi})_{\alpha\beta} p_{\alpha}^{\Pi} q_{\beta}^{\Pi} . \end{aligned} \quad (5.13)$$

Isospin is defined by

$$T^2 = (\ell_x^{\Pi})^2 + (\ell_y^{\Pi})^2 + (\ell_z^{\Pi})^2. \quad (5.14)$$

The relation between the ℓ 's in (5.11) and the Cartesian ℓ 's of (5.13) is rather complicated. However, we may choose the vector $\vec{\ell}^{\Pi}$ in a particular direction

$$\vec{\ell}^{\Pi} = (0, 0, \ell_z^{\Pi}). \quad (5.15)$$

Then one finds

$$\begin{aligned} \hat{p}_x |\hat{n}_x| &= -\ell_z^{\Pi} \sin\vartheta \cos\varphi \\ \hat{p}_y |\hat{n}_y| &= \ell_z^{\Pi} \sin\vartheta \sin\varphi \\ \hat{p}_z |\hat{n}_z| &= -\ell_z^{\Pi} \cos\vartheta \end{aligned} \quad (5.16)$$

and

$$\begin{aligned} \ell_y &= -\ell_z^{\Pi} \\ \ell_y &= -\ell_z^{\Pi} \cos\vartheta \\ \ell_y &= \ell_z^{\Pi} \sin\vartheta \sin 2\varphi . \end{aligned} \quad (5.17)$$

Next we rewrite the Hamiltonian (2.2) (leaving out, for the moment, the terms with p^I, q^I)

$$\begin{aligned} H^{\Pi}(p, q) &= H(\hat{p}, \hat{q}) + \lambda_{\Pi, \Pi}^0 q^0 (\hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2) + \frac{\lambda_{\Pi, \Pi}^{\Pi}}{2\sqrt{3}} \cdot \\ &\cdot [2 \hat{p}_z^2 \hat{q}_z^{\Pi} + \hat{p}_y^2 (\sqrt{3} \hat{q}_3^{\Pi} - \hat{q}_8^{\Pi}) - \hat{p}_x^2 (\sqrt{3} \hat{q}_3^{\Pi} + \hat{q}_8^{\Pi})] . \end{aligned} \quad (5.18)$$

The first part $H(\hat{p}, \hat{q})$ is independent of $\vec{\ell}$ and is obtained from (2.2) by simply replacing p^{II} and q^{II} by $\hat{p}^{II}$ and $\hat{q}^{II}$, respectively. [Both $\hat{p}^{II}$ and $\hat{q}^{II}$ have only "3" and "8" components. Therefore, $H(\hat{p}, \hat{q})$ depends on p^0 , q^0 , $\hat{p}_3^{II}$, $\hat{q}_3^{II}$, $\hat{p}_8^{II}$, $\hat{q}_8^{II}$.] The remainder of (5.18) depends through $\hat{p}_x$, $\hat{p}_y$, and $\hat{p}_z$ upon the ℓ 's [cf. (5.11)] and is the analogue of the centrifugal barrier in classical mechanics. If we choose $\vec{\ell}$ as in (5.15), this "centrifugal term" can be written as

$$\begin{aligned} \ell_z^{II^2} \left\{ \frac{\cos^2 \vartheta}{4(\hat{q}_3^{II})^2} \left[\lambda_{II, II}^0 q^0 + \frac{\lambda_{II, II}^{II}}{\sqrt{3}} \hat{q}_8^{II} \right] + \frac{\sin^2 \vartheta}{((\hat{q}_3^{II})^2 - 3(\hat{q}_8^{II})^2)^2} \cdot \right. \\ \cdot \left[\lambda_{II, II}^0 q^0 (2\sqrt{3} \hat{q}_3^{II} \hat{q}_8^{II} \cos 2\psi + (\hat{q}_3^{II})^2 + (\hat{q}_8^{II})^2) - \lambda_{II, II}^{II} \frac{1}{2\sqrt{3}} \cdot \right. \\ \cdot \left. \left(\cos 2\psi \sqrt{3} \hat{q}_3^{II} ((\hat{q}_3^{II})^2 + 5(\hat{q}_8^{II})^2) + \hat{q}_8^{II} (7(\hat{q}_3^{II})^2 + 3(\hat{q}_8^{II})^2) \right) \right] \left. \right\}. \end{aligned} \quad (5.19)$$

Similar to the centrifugal barrier in classical mechanics, it becomes singular when the coordinates $\hat{q}_3$ and $\hat{q}_8$ are zero.

The equations of motion are

$$\dot{\psi} = \frac{\partial H^T}{\partial \ell_\psi}, \quad \dot{\psi} = \frac{\partial H^T}{\partial \ell_\psi}, \quad \dot{\vartheta} = \frac{\partial H^T}{\partial \ell_\vartheta}, \quad (5.20)$$

$$-\dot{\ell}_\psi = \frac{\partial H^T}{\partial \psi}, \quad -\dot{\ell}_\psi = \frac{\partial H^T}{\partial \psi}, \quad -\dot{\ell}_\vartheta = \frac{\partial H^T}{\partial \vartheta}, \quad (5.21)$$

and, for the remaining variables q^0 , $\hat{q}^{II}$, p^0 , $\hat{p}^{II}$

$$\dot{q} = \frac{\partial H^T}{\partial p}, \quad -\dot{p} = \frac{\partial H^T}{\partial q}. \quad (5.22)$$

The partial derivatives of H are taken from (5.18), where $\hat{p}_x^2$, etc. are expressed in terms of the ℓ 's and the angles through (5.11). Since H does not depend on ϕ , ℓ_ϕ is always conserved. Rather than writing down the explicit form of the equations of motion, we again choose $\vec{\ell}^{II}$ as in (5.15) and restrict ourselves to motions with

$$\psi = 0, \quad \vartheta = \frac{\pi}{2}. \quad (5.23)$$

This has a simple analogue in classical mechanics: if we identify our standard vector $\hat{q}^{II}$ with the z-axis in ordinary mechanics, then $D_y(\pi/2)$ takes us into the x-y plane, and (5.23) describes motions in this plane, rotating around the z-axis. [There are, of course, more complicated motions than (5.23); but as the collective coordinate treatment in Ref. 17 shows, (5.23) is the correct generalization of the

$T = 0$ case.] From (5.17) it follows that $\ell_\theta = \ell_\psi = 0$, and from the equations of motion (5.20) and (5.21): $\dot{\psi} = \dot{\theta} = \dot{\ell}_\theta = \dot{\ell}_\psi = 0$. So our problem is reduced to the dynamical variables $q^0, p^0, \hat{q}_3^{II}, \hat{p}_3^{II}, \hat{q}_8^{II}, \hat{p}_8^{II}$, and ϕ (ℓ_ϕ is constant of motion). The Hamiltonian (5.18) simplifies

$$H^T = H(\hat{p}, \hat{q}) + \frac{\ell_z^2}{(\hat{q}_3^{II} - \sqrt{3}\hat{q}_8^{II})^2} \left\{ \lambda_{II,II}^0 q^0 - \lambda_{II,II}^{II} \frac{1}{2\sqrt{3}} (\hat{q}_8^{II} + \sqrt{3}\hat{q}_3^{II}) \right\}. \quad (5.24)$$

Let us now study the classical solutions described by this Hamiltonian (5.24). It will be useful to do this as a function of $T^2 = \ell_z^2$: first take $T = 0$, then we turn on ℓ_z but keep it small (or, alternatively, we keep ℓ_z finite and take μ/λ large). For $\ell_z = 0$, we find from (5.24) that $\dot{\phi} = \partial H^T / \partial \ell_\phi = 0$, i.e. solutions with $T = 0$ do not change their direction in group space. For the remaining dynamical variables -- $q^0, p^0, \hat{q}_3^{II}, \hat{p}_3^{II}, \hat{q}_8^{II}, \hat{p}_8^{II}$ -- the equations of motion have been studied in the previous section, and we have found the fixed points in the $p = 0$ and $q = 0$ subspaces (Fig. 7). Now we take $\ell_z \neq 0$ but small: ϕ is no longer zero, and the solutions rotate. Fixed points are defined by

$$0 = \frac{\partial H^T}{\partial p}, \quad 0 = \frac{\partial H^T}{\partial q}, \quad (5.25)$$

where p and q stand for $q^0, p^0, \hat{q}_3^{II}, \hat{p}_3^{II}, \hat{q}_8^{II},$ and $\hat{p}_8^{II}$, and H^T is taken from (5.24). For small ℓ_z , the solutions to (5.25) will be close to the fixed points of the $\ell_z = 0$ case; let us first look how the fixed point A is displaced. We simply expand (5.25) around A

$$0 = \frac{\partial H^T}{\partial p} \approx \frac{\partial^2 H^T}{\partial p \partial p} \Delta p + \frac{\partial^2 H^T}{\partial p \partial q} \Delta q \quad (5.26)$$

$$0 = \frac{\partial H^T}{\partial q} \approx \frac{\partial^2 H^T}{\partial q \partial p} \Delta p + \frac{\partial^2 H^T}{\partial q \partial q} \Delta q. \quad (5.27)$$

In these vector equations, $\Delta p = (\Delta p^0, \Delta \hat{p}_3^{II}, \Delta \hat{p}_8^{II})$, and the second-order derivative matrices of H^T are evaluated at the fixed point A. Using the form H^T from (5.24) and observing that the ℓ -dependent part of H^T contains no p 's, Eq. (5.26) becomes

$$0 = \frac{\partial^2 H}{\partial p \partial p} \Delta p + \frac{\partial^2 H}{\partial p \partial q} \Delta q \quad (5.28)$$

and gives a linear relation between the vectors Δp and Δq . Equation (2.27) has the form

$$0 = \frac{\partial^2 H}{\partial q \partial p} \Delta p + \frac{\partial^2 H}{\partial q \partial q} \Delta q + \ell_z^2 \cdot \text{const} \quad (5.29)$$

and leads to the result that

$$\Delta p \sim \Delta q \sim O(\ell_z^2). \quad (5.30)$$

Hence the displacement of the fixed point A is of the order of $O(T^2)$. For the energy of the shifted fixed point we find -- again by Taylor expanding --

$$H^T(\Delta p, \bar{q} + \Delta q) = \frac{\ell_z^2}{3(\hat{q}_s^{\text{II}})^2} \left\{ \lambda_{\text{II}, \text{II}}^0 q^0 - \lambda_{\text{II}, \text{II}}^{\text{II}} \hat{q}_s^{\frac{\text{II}}{2\sqrt{3}}} \right\} + O(\ell_z^4), \quad (5.31)$$

where $\hat{q}_s^{\text{II}}$, and q^0 are the coordinates of A. Using their sign structure (Fig. 5), together with the signs of $\lambda_{\text{II}, \text{II}}^0 < 0$ and $\lambda_{\text{II}, \text{II}}^{\text{II}} > 0$, we find the important result ^{*}

$$E^T = -H^T = T^2 \cdot \text{positive number} \quad (5.32)$$

In particular, for $T = 0$ the fixed point A has the lowest energy value, and the energy grows like T^2 .

As to the other fixed point, we show in Appendix C that both the origin and the fixed point B move into complex p - q space when ℓ_z becomes different from zero, and also their energy becomes complex. The fixed point A' stays real and has the same energy (5.30) as A (this follows from the symmetry between A and A'). The orbits which connect A and A' are, by continuity, close to the $T = 0$ orbits. In the limit $y_1 \rightarrow -\infty$, $y_2 \rightarrow \infty$, however, they no longer pass through the origin or the fixed points B and B'. The stability properties of A and A' are approximately the same as in the $T = 0$ case. An illustration of the situation for small T is given in Fig. 13.

Before we come to evaluate the action integral we have to include the $T = 1$ fields. Exactly in the same way as we did before for the $T = 2$ field, we introduce

^{*}) The minus sign in front of the Hamiltonian is due to the way in which H enters the path integral (2.19) (Ref. 2).

a standard vector $\hat{q}^I$ pointing in the q_3^I -direction. Then any vector q^I can be written as

$$q^I = R^I(\alpha, \beta) \hat{q}^I, \quad (5.33)$$

with

$$R^I(\alpha, \beta) = \mathcal{D}_z^I(\alpha) \mathcal{D}_y^I(\beta). \quad (5.34)$$

In analogy with (5.5), we define two vectors:

$$\begin{aligned} n_x &= i R^I(\alpha, \beta) L_x^I \hat{q}^I \\ n_y &= i R^I(\alpha, \beta) L_y^I \hat{q}^I \end{aligned} \quad (5.35)$$

(n_z is zero in this case), which have the same properties as (5.7) and (5.8). Decomposing p^I as in (5.9):

$$p^I = p_x^I + p_y^I + p_{||}^I, \quad (5.36)$$

we find for the kinetic part of the $T = 1$ field:

$$p^I \cdot \dot{q}^I = \hat{p}^I \cdot \dot{\hat{q}}^I + \dot{\alpha} \ell_\alpha + \dot{\beta} \ell_\beta \quad (5.37)$$

with

$$\begin{aligned} \ell_\alpha &= \hat{p}_x^I |\hat{n}_x| \sin \beta \\ \ell_\beta &= \hat{p}_y^I |\hat{n}_y| \\ \hat{p}^I &= R^I(\alpha, \beta)^{-1} p_{||}^I \\ \hat{p}_x^I &= R^I(\alpha, \beta)^{-1} p_x^I \\ \hat{n}_x &= R^I(\alpha, \beta)^{-1} n_x = i L_x^I \hat{q}^I. \end{aligned} \quad (5.38)$$

In analogy with (5.13), we define a vector $\vec{\ell}^I$ with the components:

$$\ell_x^I = i (L_x^I)_{ke} p_k^I q_e^I \text{ etc.} \quad (5.39)$$

The vectors $\vec{\ell}^I$ and $\vec{\ell}^{II}$ are not conserved separately. The constants of motion that belong to the SU(2) invariance of our model are the components of

$$\vec{\ell} = \vec{\ell}^I + \vec{\ell}^{II}. \quad (5.40)$$

The isospin is defined [replacing (5.14)]:

$$T^2 = \ell^2 = (\vec{\ell}^I + \vec{\ell}^{II})^2. \quad (5.41)$$

With the parametrization (5.33) and (5.36) one rewrites the part of the Hamiltonian (2.2) that contains the $T = 1$ field. The result is quite similar to (5.18): there is a part independent of $\vec{\ell}^I$ which contains $p^0, q^0, \hat{p}^I, \hat{q}^I, \hat{p}^{II}, \hat{q}^{II}$, and another term containing $\vec{\ell}^I$. Rather than writing down the explicit form of the ℓ -dependent term, we mention only two important properties: it vanishes if $\vec{\ell}^I = 0$ [in the same manner as the ℓ -dependent part of (5.18) does when $\vec{\ell}^{II} = 0$], and it is complex valued because of the "i" in front of $\lambda_{I,I}^0$ and $\lambda_{I,I}^{II}$. The full H^T thus becomes

$$H^T = H(p^0, q^0, \hat{p}^I, \hat{q}^I, \hat{p}^{II}, \hat{q}^{II}) + \ell^{\frac{II}{-}} \text{-dependent piece of} \quad (5.42) \\ \text{eq. (5.18) + } \ell^{\frac{I}{-}} \text{-dependent piece.}$$

We now look at the motion of fixed points as a function of $T^2 = \vec{\ell}^2$. For $\vec{\ell}^I = \vec{\ell}^{II} = 0$ we have the $T = 0$ situation of Section 3, and the q^I and p^I coordinates of our fixed points were zero. If now $\vec{\ell}$ is turned on (small against μ/λ), we have some freedom as regards how to distribute $\vec{\ell}$ among $\vec{\ell}^I$ and $\vec{\ell}^{II}$. If we choose $\vec{\ell}^I = 0$ and $\vec{\ell} = \vec{\ell}^{II}$ the fixed points start to move in the way we have discussed before, but q^I and p^I stay zero. For $\vec{\ell}^I \neq 0$, the p^I and q^I components of the fixed point coordinates move away from zero, but due to the complex valuedness of the interaction term, they become complex. So if we restrict ourselves to real values p and q we have to choose $\vec{\ell}^I = 0$, and we are back to the results (5.32) or Fig. 13.

Finally, we come to the evaluation of the action integral (2.24). Again we require that the classical solution under the integral satisfies the boundary conditions (4.2) and (4.3)

$$p_{ce}(y_1) = G \quad (4.2)$$

$$q_{ce}(y_2) = F. \quad (4.3)$$

Between times y_1 and y_2 , p and q satisfy the equations of motion (5.1), outside this time interval (3.1). In the limit $y_1 \rightarrow -\infty$, $y_2 \rightarrow \infty$ the solution approaches the shifted fixed points A and A' (Fig. 13), whose energy we have evaluated in (5.31) and (5.32). The action integral is

$$\begin{aligned} A_{ce}^T(F, G) &= \int_{y_1}^{y_2} dy \left(p_{ce}^{0,0} \dot{q}_{ce}^{0,0} + \hat{p}_{ce}^I \dot{\hat{q}}_{ce}^I + \hat{p}_{ce}^{II} \dot{\hat{q}}_{ce}^{II} + E^T \right) \\ &= \text{const} + (y_2 - y_1) \cdot E^T \end{aligned} \quad (5.43)$$

For the S-matrix element we find

$$S^T(F, G) \sim e^{-(y_2 - y_1) \cdot E^T}, \quad (5.44)$$

and since $E^T > 0$, S^T goes to zero. This most important result of this section shows that our RFT spectrum has the correct features: the vacuum channel ($T = 0$) has lowest energy and, hence, the total cross-section dominates over quantum number exchange channels. For $T \neq 0$, the energy is a monotonic function of the isospin size.

6. SUMMARY AND CONCLUSIONS

In this paper we have studied a Reggeon field theory model with an internal $SU(2)$ (= isospin) symmetry, where one of the quantum-number-carrying fields (isospin = two) has a negative Reggeon mass (intercept above one). The model that we have chosen has been suggested by an analysis of the high-energy behaviour of spontaneously broken non-Abelian gauge theories, and it contains three isomultiplets: Reggeon fields with isospin zero, one, and two. These fields interact via triple couplings. As a first step, we have been working in zero transverse dimensions and the classical approximation.

It has been our main interest to ask whether such a model can, without requiring very specific values for its parameters (intercepts and coupling constants), account for some of the gross features of high-energy scattering: an (almost) constant total cross-section, and decreasing cross-sections for quantum number exchange channels. Obviously, the input trajectories are in disagreement with

this: one Reggeon has intercept above one, and the vacuum singularity is not the leading one. The question then is whether the interaction changes this energy spectrum in an appropriate way. In a relativistic theory one would expect Goldstone particles; Reggeon field theory, however, has a completely different structure, and, correspondingly, the particle spectrum will be different.

Our answers to these questions can be summarized in the following way.

- i) For a certain range of values for the intercepts and coupling constants, fixed points and stable classical solutions do exist. Their structure (Fig. 7, Fig. 13) is very similar to that found in supercritical Pomeron theory. In particular, the ambiguity of having two solutions exists in our theory, too. In this paper, we have restricted ourselves to that solution which is the analogue of Amati's solution in supercritical Pomeron theory. The analogue of White's solution which yields a falling cross-section will be described in a separate paper.
- ii) The spectrum of our Reggeon field theory satisfies the physical requirements: the lowest state with zero energy has vacuum quantum number (isospin zero) and leads to a constant total cross-section

$$\sigma_{tot} = \frac{1}{s} \text{Im } T^{(0)} = \text{const.} \quad (6.1)$$

For non-zero isospin the energy grows proportional to T^2

$$E^T = T^2 \cdot \text{positive number,} \quad (6.2)$$

and gives rise to decreasing cross-sections for amplitudes with quantum number exchanges

$$\frac{1}{s} T^{(T)} \sim s^{-E^T} \quad (6.3)$$

We conclude from this that our model satisfies all the requirements that we have mentioned. As to the question whether we find evidence for spontaneous symmetry breaking and Goldstone particles, there are no constant (= time independent) stable classical solutions in our theory which would be necessary for the appearance of Goldstone modes.

The next question to be asked is to what extent quantum corrections and transverse dimensions will affect our results. Guided by the similarity with supercritical Pomeron theory we expect that quantum corrections to our classical

solution will lead to a splitting of the lowest energy levels. The main reason for this is that a more accurate calculation of the path integral has to take into account not only the classical solution that runs from A to A' but also those which return to A and complete more and more loops. For supercritical Pomeron theory it has been shown⁹⁾ that the effect of adding these solutions is the splitting of the ground state level which had been found before by solving the Schrödinger equation directly. For large μ/λ , this splitting is exponentially small, and, hence, does not alter our result (6.2) [cf. (5.31)]. The result of turning on transverse dimensions will be the existence of a phase transition (again from similarity with supercritical Pomeron theory), such that a finite negative Reggeon mass belongs to the ordered phase. The total cross-section then may be modified by a power of $\ln s$: in supercritical Pomeron theory one finds for $D = 1$ just a factor $\ln s$ [and $(\ln s)^2$ for $D = 2$]. In our case, the negative slope of the $T = 2$ trajectory may very well lead to a different answer. All this remains a subject of future work.

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APPENDIX A

In this first Appendix we present a few formulae which are useful in connection with transformation properties of the $T = 1$ and $T = 2$ fields as well as $H(p, q)$.

For the $T = 1$ field, which transforms as a triplet under $SU(2)$ rotations, we have the usual infinitesimal rotation matrices:

$$(L_k^I)_{ij} = -i \varepsilon_{kij} \quad , \quad (A.1)$$

which satisfy

$$[L_k^I, L_m^I] = i \varepsilon_{kmn} L_n^I \quad (A.2)$$

(ε_{ijk} is the antisymmetric tensor). In the following we shall use L_x^I, L_y^I, L_z^I instead of L_1^I, L_2^I, L_3^I , respectively.

For the $T = 2$ field we make use of the $SU(3)$ structure constants and λ -matrices; they are listed in Table 1. From the relations among these structure constants it follows that

$$\begin{aligned} (L_x^\pi)_{\alpha\beta} &= -2if_{7\alpha\beta} \\ (L_y^\pi)_{\alpha\beta} &= 2if_{5\alpha\beta} \\ (L_z^\pi)_{\alpha\beta} &= -2if_{2\alpha\beta} \end{aligned} \quad (A.3)$$

satisfy the $SU(2)$ commutation relations (A.2) (according to our convention of Section 2, Greek indices take the values 1, 3, 4, 6, 8). Finite rotations are obtained by exponentiating the matrices (A.3)

$$D_z^\pi(\theta) = e^{i\theta L_z^\pi} = \begin{pmatrix} \cos 2\theta & -\sin 2\theta & 0 & 0 & 0 \\ +\sin 2\theta & \cos 2\theta & 0 & 0 & 0 \\ 0 & 0 & \cos \theta & +\sin \theta & 0 \\ 0 & 0 & -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (A.4)$$

$$D_x^{\text{II}}(\theta) = e^{i\theta L_x^{\text{II}}} = \begin{pmatrix} \cos \theta & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} \cos 2\theta + \frac{3}{4} & 0 & 0 & -\frac{\sqrt{3}}{4}(\cos 2\theta - 1) \\ 0 & 0 & \cos \theta & 0 & 0 \\ 0 & 0 & 0 & \cos 2\theta & 0 \\ 0 & \frac{\sqrt{3}}{4}(\cos 2\theta - 1) & 0 & 0 & \frac{3}{4} \cos 2\theta + \frac{1}{4} \end{pmatrix}$$

$$- \begin{pmatrix} 0 & 0 & -\sin \theta & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \sin \theta & 0 \\ \sin \theta & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} \sin \theta & 0 & 0 & \frac{\sqrt{3}}{2} \sin 2\theta \\ 0 & 0 & 0 & -\frac{\sqrt{3}}{2} \sin 2\theta & 0 \end{pmatrix} \quad (\text{A.5})$$

$$D_y^{\text{II}}(\theta) = e^{i\theta L_y^{\text{II}}} = \begin{pmatrix} \cos \theta & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} \cos 2\theta + \frac{3}{4} & 0 & 0 & \frac{\sqrt{3}}{4}(\cos 2\theta - 1) \\ 0 & 0 & \cos 2\theta & 0 & 0 \\ 0 & 0 & 0 & \cos \theta & 0 \\ 0 & \frac{\sqrt{3}}{4}(\cos 2\theta - 1) & 0 & 0 & \frac{3}{4} \cos 2\theta + \frac{1}{4} \end{pmatrix} \quad (\text{A.6})$$

$$- \begin{pmatrix} 0 & 0 & 0 & -\sin \theta & 0 \\ 0 & 0 & -\frac{1}{2} \sin 2\theta & 0 & 0 \\ 0 & \frac{1}{2} \sin \theta & 0 & 0 & \frac{\sqrt{3}}{2} \sin 2\theta \\ \sin \theta & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\sqrt{3}}{2} \sin 2\theta & 0 & 0 \end{pmatrix}.$$

APPENDIX B

In this Appendix we first show that for any vector q^{II} there exist a uniquely defined $SU(2)$ rotation R^{II} and a standard vector $\hat{q}^{II}$ such that

$$q^{II} = R^{II}(\varphi, \vartheta, \psi) \hat{q}^{II}. \quad (B.1)$$

The $SU(2)$ rotation R^{II} can be written as

$$R^{II}(\varphi, \vartheta, \psi) = \mathcal{D}_z^{II}(\varphi) \mathcal{D}_y^{II}(\vartheta) \mathcal{D}_z^{II}(\psi), \quad (B.2)$$

and the standard vector has the form

$$\hat{q}^{II} = (0, \hat{q}_3^{II}, 0, 0, \hat{q}_8^{II}), \quad (B.3)$$

where $\hat{q}_3^{II}, \hat{q}_8^{II} > 0$ and $\hat{q}_3^{II} \leq \sqrt{3} \hat{q}_8^{II}$ (Fig. 12). In order to prove (B.1) we first note that the symmetric and traceless tensor

$$T_{ij} = q_\alpha^{II} \lambda_{ij}^\alpha \quad (B.4)$$

transforms under $SU(2)$ transformations like

$$T_{ij} \rightarrow (R^I)_{im} (R^I)_{jn} T_{mn}, \quad (B.5)$$

where R^I is the $T = 1$ representation of an $SU(2)$ rotation. For the symmetric matrix T there exists an orthogonal matrix Σ which diagonalizes T

$$\Sigma^T T \Sigma = \hat{T}. \quad (B.6)$$

The matrix Σ is uniquely defined if we require that the diagonal elements of $\hat{T}$ are ordered

$$\hat{T}_{11} \geq \hat{T}_{22} \geq \hat{T}_{33}. \quad (B.7)$$

Parametrizing $\hat{T}$ as in (B.4)

$$\hat{T} = \hat{q}_\alpha^{II} \lambda^\alpha, \quad (B.8)$$

we know that only $\hat{q}_3^{II}$ and $\hat{q}_8^{II}$ can be different from zero, and

$$\begin{aligned}\hat{T}_{11} &= \hat{q}_3^{II} + \frac{1}{\sqrt{3}} \hat{q}_8^{II} \\ \hat{T}_{22} &= -\hat{q}_3^{II} + \frac{1}{\sqrt{3}} \hat{q}_8^{II} \\ \hat{T}_{33} &= -\frac{2}{\sqrt{3}} \hat{q}_8^{II}\end{aligned}\quad (B.9)$$

From (B.7) it then follows that $\hat{q}_8^{II}$, and $\hat{q}_3^{II}$ are positive and $\hat{q}_3^{II} \leq \sqrt{3} \hat{q}_8^{II}$. The diagonalization matrix Σ can be written as

$$\Sigma = R_z^I(\varphi) R_Y^I(\vartheta) R_z^I(\psi), \quad (B.10)$$

and R^{II} in Eq. (B.1) is the $T = 2$ representation of this $SU(2)$ rotation.

Using this result it is now easy to see that all stable fixed points in the $p = 0$ subspace are connected with those that we have already found in the $q^0 - q_8^{II}$ subspace by $SU(2)$ rotations. Because of (B.1) it is sufficient to restrict ourselves to the $q^0 - q_8^{II} - q_3^{II}$ subspace with $0 \leq q_3^{II} \leq \sqrt{3} q_8^{II}$ and to show that our fixed points with $q_3^{II} = 0$ are the only stable ones. From the three fixed point conditions

$$\begin{aligned}0 = \dot{q}^0 &= -\mu_0 q^0 + \lambda_{00}^0 (q^0)^2 + \lambda_{II,II}^0 (q_3^{II})^2 + \lambda_{II,II}^0 (q_8^{II})^2 \\ 0 = \dot{q}_3^{II} &= -\mu_{II} q_3^{II} + 2 d_{338} \lambda_{II,II}^{II} q_3^{II} q_8^{II} + \lambda_{0,II}^{II} q^0 q_3^{II} \\ 0 = \dot{q}_8^{II} &= -\mu_{II} q_8^{II} + d_{888} \lambda_{II,II}^{II} (q_8^{II})^2 + d_{833} \lambda_{II,II}^{II} (q_3^{II})^2 + \lambda_{0,II}^{II} q^0 q_8^{II}.\end{aligned}\quad (B.11)$$

one finds -- by dividing the second equation by q_3^{II} , multiplying by q_8^{II} and subtracting from the third equation -- that any solution satisfies

$$q_3^{II} = 0 \quad \text{or} \quad (q_3^{II})^2 = (\sqrt{3} q_8^{II})^2. \quad (B.12)$$

The first case is the one we have discussed before; in the second case, we re-write the first and third equations of (B.11)

$$\left(q^0 - \frac{\mu_0}{2\lambda_{00}^0}\right)^2 + \frac{\lambda_{II,II}^0}{\lambda_{0,0}^0} (2q_8^{II})^2 = \left(\frac{\mu_0}{2\lambda_{00}^0}\right)^2 \quad (B.13)$$

$$0 = q_8^{II} \left(-\mu_{II} + \lambda_{II,II}^{II} \frac{2}{\sqrt{3}} q_8^{II} + \lambda_{0,II}^{II} q^0\right). \quad (B.14)$$

They are the same as Eqs. (3.6) and (3.7) if q_8^{II} in (3.6) and (3.7) is replaced by $-2q_8^{II}$. The fixed-point structure (B.11), therefore, is the same as in Figs. 5a,b except for $q_8^{II} \rightarrow -2q_8^{II}$. In particular, the analogue of the fixed point A has a negative q_8^{II} coordinate and, hence, lies outside the region $0 \leq q_3^{II} \leq \sqrt{3} q_8^{II}$. [Using the form of D_y^{II} , given in Appendix A, one can see that this fixed point is obtained by applying $D_y^{II}(\pm\pi/2)$ to A in Figs. 5a and b.]

APPENDIX C

In this Appendix we study the displacement of the fixed point at the origin as a function of the isospin T . Our starting point is the Hamiltonian H^T in (5.24), and the equations we want to solve are those of Eq. (5.25). For $T^2 = \ell_z^2 = 0$, H^T reduces to the Hamiltonian we have studied in Section 3, and we have found that the origin is a (unstable) fixed point. When ℓ_z^2 is small but different from zero, this fixed point moves, and we expect this shift to be small. Owing to the singularity of the ℓ_z -dependent term in (5.24) at $q = 0$, we cannot simply Taylor-expand around the origin but have to take a closer look at the equations (5.25). In order to simplify the algebra we transform from $\hat{q}_3^{II}$, $\hat{q}_8^{II}$, $\hat{p}_3^{II}$, $\hat{p}_8^{II}$ to new variables

$$\hat{q}_3^{II} = -\frac{Q_3 + \sqrt{3}Q_8}{2} \quad \hat{q}_8^{II} = -\frac{Q_8 - \sqrt{3}Q_3}{2} \quad (C.1)$$

$$\hat{p}_3^{II} = -\frac{P_3 + \sqrt{3}P_8}{2} \quad \hat{p}_8^{II} = -\frac{P_8 - \sqrt{3}P_3}{2} \quad (C.2)$$

Equation (5.24) then becomes

$$\begin{aligned} H^T = & -\mu_0 p^0 q^0 - \mu_{II} (P_8 Q_8 + P_3 Q_3) + \lambda_{00}^0 p^0 q^0 (p^0 + q^0) \\ & + \lambda_{0,II}^{\pi} (p^0 + q^0) (P_8 Q_8 + P_3 Q_3) + \lambda_{II,II}^0 \left[p^0 (Q_8^2 + Q_3^2) + q^0 (P_8^2 + P_3^2) \right] \\ & + \frac{\lambda_{II,II}^{\pi}}{\sqrt{3}} \left[2(P_8 + Q_8)P_3 Q_3 + P_8 Q_3^2 + Q_8 P_3^2 - P_8 Q_8 (P_8 + Q_8) \right] \\ & + \frac{\ell_z^2}{Q_3^2} \left(\lambda_{II,II}^0 q^0 + \frac{\lambda_{II,II}^{\pi}}{\sqrt{3}} Q_8 \right), \end{aligned} \quad (C.3)$$

and the derivatives in (5.25) have to be taken with respect to the variables p^0 , q^0 , P_3 , Q_3 , P_8 , Q_8 . In order to find a solution to (5.25) which for small ℓ_z^2 is close to the origin, we make the ansatz

$$q^0, Q_3, Q_8 \sim O(\ell_z) \quad (C.4)$$

$$P_3 \sim O(1), \quad p_0, P_8 \sim O(\ell_z)$$

and keep only the leading terms in (5.25):

$$\begin{aligned}
 0 &= \frac{\partial H^T}{\partial p_0} = -\mu_0 q^0 + \lambda_{0,\Pi}^{\Pi} P_3 Q_3 \\
 0 &= \frac{\partial H^T}{\partial P_3} = -\mu_{\Pi} Q_3 + 2 \lambda_{\Pi,\Pi}^0 P_3 q^0 + 2 \frac{\lambda_{\Pi,\Pi}^{\Pi}}{\sqrt{3}} Q_8 P_3 \\
 0 &= \frac{\partial H^T}{\partial P_8} = -\mu_{\Pi} Q_8 + 2 \frac{\lambda_{\Pi,\Pi}^{\Pi}}{\sqrt{3}} P_3 Q_3 \\
 0 &= \frac{\partial H^T}{\partial q_0} = \lambda_{\Pi,\Pi}^0 P_3^2 + \frac{\ell_z^2}{Q_3^2} \lambda_{\Pi,\Pi}^0 \\
 0 &= \frac{\partial H^T}{\partial Q_3} = -\mu_{\Pi} P_3 - 2 \frac{\ell_z^2}{Q_3^2} (\lambda_{\Pi,\Pi}^0 q^0 + \frac{\lambda_{\Pi,\Pi}^{\Pi}}{\sqrt{3}} Q_8) \\
 0 &= \frac{\partial H^T}{\partial Q_8} = \frac{\lambda_{\Pi,\Pi}^{\Pi}}{\sqrt{3}} P_3^2 + \frac{\ell_z^2}{Q_3^2} \frac{\lambda_{\Pi,\Pi}^{\Pi}}{\sqrt{3}}
 \end{aligned} \tag{C.5}$$

These equations have the solutions

$$\begin{aligned}
 q^0 &= \frac{\lambda_{0,\Pi}^{\Pi}}{\mu_0} \sqrt{-\ell_z^2} \\
 Q_8 &= 2 \frac{\lambda_{\Pi,\Pi}^{\Pi}}{\sqrt{3} \mu_{\Pi}} \sqrt{-\ell_z^2} \\
 Q_3 &= \sqrt{-\ell_z^2} \left(2 \frac{\lambda_{0,\Pi}^{\Pi} \lambda_{\Pi,\Pi}^0}{\mu_0 \mu_{\Pi}} + 4 \frac{(\lambda_{\Pi,\Pi}^{\Pi})^2}{3 \mu_{\Pi}^2} \right)^{1/2} \\
 P_3 &= -\frac{\ell_z^2}{Q_3^2} = \left(2 \frac{\lambda_{0,\Pi}^{\Pi} \lambda_{\Pi,\Pi}^0}{\mu_0 \mu_{\Pi}} + 4 \frac{(\lambda_{\Pi,\Pi}^{\Pi})^2}{3 \mu_{\Pi}^2} \right)^{-1}
 \end{aligned} \tag{C.6}$$

For real $\ell_z^2 = T^2$, all q-coordinates become complex, and so does the energy of the fixed point

$$H^T \approx -\mu_{\Pi} \sqrt{-\ell_z^2} \tag{C.7}$$

We conclude from this that the fixed point, which for $T^2 = 0$ sits at the origin, moves into complex p-q space when T^2 is turned on. Conservation of energy then implies that classical solutions that start at the shifted fixed point A with real energy cannot reach this fixed point.

A similar analysis holds for the fixed point B: when T^2 is small but different from zero, its coordinates become complex valued. We only quote the energy (to leading order ℓ_2^2):

$$H^T \approx -\frac{1}{2} \left(-\mu_{II} + \lambda_{0,II}^{\frac{I}{\mu_I}} \frac{\mu_0}{\mu_I} \right) \sqrt{-\ell_2^2} . \quad (C.8)$$

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Table 1

The SU(3) λ -matrices, taken from, P. Carruthers,
Introduction to unitary symmetry
(Interscience-Wiley, NY, 1966)

$$\begin{aligned}\lambda_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \lambda_2 &= \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \lambda_3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \lambda_4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \\ \lambda_5 &= \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} & \lambda_6 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\ \lambda_7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} & \lambda_8 &= \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & -\frac{2}{\sqrt{3}} \end{pmatrix}\end{aligned}$$

Table 2

The f and d structure constants of SU(3),
taken from the same reference as Table 1

ijk	f_{ijk}	ijk	d_{ijk}
123	1	118	$1/\sqrt{3}$
147	$1/2$	146	$1/2$
156	$-1/2$	157	$1/2$
246	$1/2$	228	$1/\sqrt{3}$
257	$1/2$	247	$-1/2$
345	$1/2$	256	$1/2$
367	$-1/2$	338	$1/\sqrt{3}$
458	$\sqrt{3}/2$	344	$1/2$
678	$\sqrt{3}/2$	355	$1/2$
		366	$-1/2$
		377	$-1/2$
		448	$-1/(2\sqrt{3})$
		558	$-1/(2\sqrt{3})$
		668	$-1/(2\sqrt{3})$
		778	$-1/(2\sqrt{3})$
		888	$-1/\sqrt{3}$

Figure captions

- Fig. 1 : The three input trajectories.
- Fig. 2 : The interaction vertices.
- Fig. 3 : Graphical illustration of a) Eq. (2.4), b) Eq. (2.11).
- Fig. 4 : Classical solutions in supercritical Pomeron theory: a) in the absence of sources; b) and c) in the presence of sources. (In Fig. 4c we take $F = G$.)
- Fig. 5 : The $q^0 - q_8^{II}$ plane. On the hyperbola (ellipse) is $\dot{q}^0 = 0$, and on the straight lines $\dot{q}_8^{II} = 0$. Signs on both sides of these curves denote the sign of $\dot{q}^0$ and $\dot{q}_8^{II}$. Fixed points are the intersections of $\dot{q}^0 = 0$ and $\dot{q}_8^{II} = 0$. For the dotted lines see discussion after (3.9). In Fig. 5a the signs of the λ 's are: $\lambda_{0,0}^0, \lambda_{II,II}^{II}, \lambda_{0,II}^{II} > 0$, and $\lambda_{II,II}^0 < 0$. In Fig. 5b: $\lambda_{0,0}^0, \lambda_{II,II}^{II}, \lambda_{II,II}^0 > 0$, and $\lambda_{0,II}^{II} < 0$.
- Fig. 6 : Characteristics of fixed points (singular points): a) and b) will be called stable departure points, c) is unstable, and d) is called a vortex point.
- Fig. 7 : Fixed points and their stability in p - q space: the horizontal line denotes the subspace with $p = 0$, the vertical one $q = 0$.
- Fig. 8 : Around the origin: the $p^0 - q^0$ subspace.
- Fig. 9 : Orbits in the $q^0 - q_8^{II}$ plane for the two parameter ranges of Figs. 5a and b.
- Fig. 10 : The various possibilities of connecting two fixed points in a p - q symmetric way. The horizontal line represents the full $p = 0$ subspace, the vertical line the $q = 0$ subspace.
- Fig. 11 : Orbit of solution (i) in the presence of the two sources F and G .
- Fig. 12 : The shaded area shows the region in which $\hat{q}^{II}$ is defined.
- Fig. 13 : Displacement of the fixed points A and A' when T^2 becomes different from zero. Note that the fixed points B and B' (as well as the unstable fixed points at the origin) have disappeared into the complex p - q space.

Fig. 1

Fig. 2

Fig. 3

Fig. 4

Fig. 5

Fig. 6

Fig. 7

Fig. 8

Fig. 9

Fig. 10

Fig. 11

Fig. 12

Fig. 13