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A FURTHER PATHOLOGY OF THE COULOMB GAUGE
IN NON - ABELIAN YANG-MILLS THEORIES

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A B S T R A C T

In the first part the vacuum structure of $SU(2)$ Yang-Mills theories in the Coulomb gauge is discussed. It is proved that the only transverse pure gauge field $A_\mu(\vec{x}) = U^{-1} \partial_\mu U$ with $U(\vec{x}) \xrightarrow{r \rightarrow \infty} \text{const.}$ is the trivial one $A_\mu(\vec{x}) \equiv 0$; the features of other possible vacua with $U(\vec{x}) \xrightarrow{r \rightarrow \infty} U(\theta, \varphi)$ are studied. In the second part, regular Euclidean configurations that connect a vacuum state at $x_4 = -\infty$ to another at $x_4 = +\infty$ are discussed. It is proved, always working in the Coulomb gauge, that the perturbative vacuum $A_\mu(\vec{x}) \equiv 0$ cannot tunnel into any other one and that regular configurations with non-vanishing Pontryagin number q cannot affect such a vacuum. Moreover strong arguments are given to show that many-instanton configurations ($|q| \geq 2$) cannot be written at all in the Coulomb gauge, that is by a regular field A_μ satisfying the transversality condition $\partial_i A_i(\vec{x}, x_4) = 0$.

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SECTION 1

In these last years many different mechanisms have been proposed in order to explain quark confinement in QCD. Till now none of these attempts has been completely successful in solving this challenging problem.

The last proposal of a quark confining mechanism is due to Gribov ¹⁾. His starting point is the observation that, for non-Abelian gauge theories, the transversality condition

$$\partial_i A_i = 0 \quad (i=1,2,3) \quad (1.1)$$

does not actually fix the gauge completely. The effect of this gauge fixing degeneracy is evaluated in Refs. 1) and 2) : it is shown that the Coulomb propagator becomes very singular for vanishing momentum and hence that the potential energy between two charges increases at large distances.

However, it is not quite clear if these important physical results are really gauge independent.

In this work we will cast some doubts on the gauge independence of the results obtained in the "Coulomb gauge" (1.1) ; in fact we will show that, for Yang-Mills theories, the transversality condition (1.1) is not actually an (ambiguous) gauge fixing condition, but that it is much more restrictive. To be more precise, we will study the SU(2) case and we will exhibit some physical (Euclidean) configurations, with regular $F_{\mu\nu}(\vec{x}, x_4)$, which seem not to admit any regular potential $A_\mu(\vec{x}, x_4)$ satisfying Eq. (1.1). Then the choice of the "Coulomb gauge" implies that some contributions to the Green functions (calculated by means of the path integral method) are arbitrarily neglected.

The work is organized in the following way. In Section 2 we discuss the vacuum structure in the "Coulomb gauge" (1.1). In Section 3 we study the possible Euclidean configurations that connect a vacuum state at $x_4 = -\infty$ to another at $x_4 = +\infty$; in particular it is shown that there exist multi-instanton configurations ³⁾ that cannot be written in the "Coulomb gauge". Section 4 contains some final remarks ; in the Appendix one finds the proof of a statement of Section 2.

SECTION 2

Our first aim is to discuss the structure of the vacuum states in the "Coulomb gauge" ; in other words, we want to see which potentials $A_\mu(\vec{x})$, satisfying the transversality condition (1.1), generate a vanishing field $F_{\mu\nu}(\vec{x}) = 0$. We explicitly remark that all over this section we will work at fixed time ; the temporal evolution of the vacuum states will be the object of Section 3.

In order to classify the possible vacua it is very useful to introduce the quantity ³⁾

$$\varphi = -\frac{1}{24\pi^2} \int d^3x \epsilon_{ijk} \text{Tr}(A_i A_j A_k). \quad (2.1)$$

In the definition (2.1) the usual notation for $SU(2)$ gauge potentials has been used :

$$A_\mu = e \frac{A_\mu^i \sigma^i}{2i}. \quad (2.2)$$

The geometrical meaning of the quantity φ becomes clear if one writes the pure gauge potential A_μ in the form :

$$A_\mu = U^{-1} \partial_\mu U \quad (2.3)$$

where the $SU(2)$ group element $U(\vec{x})$ is parametrized by a unit four-vector $N_a(\vec{x})$ ($a=1,2,3,4$) :

$$U = N_4 + i\sigma_i N_i. \quad (2.4)$$

By substituting (2.3) and (2.4) in (2.1), one gets :

$$\varphi = -\frac{1}{2\pi^2} \int_{I(R_3)} d\alpha d\beta d\gamma \sin^2\alpha \sin\beta, \quad (2.5)$$

where α, β, γ are the polar co-ordinates on the S_3 sphere spanned by the unit vector N_a :

$$N_a = \begin{pmatrix} \sin \alpha \sin \beta \cos \gamma \\ \sin \alpha \sin \beta \sin \gamma \\ \sin \alpha \cos \beta \\ \cos \alpha \end{pmatrix}. \quad (2.6)$$

The integrand in the right-hand side of (2.5) is just the measure of the surface elements on the S_3 sphere ; the integral is extended to the image $I(R_3)$ of the Euclidean space R_3 on the S_3 sphere, according to the mapping :

$$N_a(\vec{x}) : \vec{x} \rightarrow N_a(\vec{x}). \quad (2.7)$$

All over the work we will assume single-valuedness, continuity and differentiability of the mapping $N_a(\vec{x})$.

If and only if the space R_3 , spanned by the spatial co-ordinates, is compactified, that is if the condition ^{*)}

$$\lim_{r \rightarrow \infty} N_a(\vec{x}) = \text{const.}, \quad r = \sqrt{\vec{x}^2} \quad (2.8)$$

holds, the mapping (2.7) is a $S_3 \rightarrow S_3$ mapping ³⁾ ; only in this case the quantity φ has a topological meaning ; then it classifies the homotopy classes of the mapping $S_3 \rightarrow S_3$ and it must be an integer number.

However, in general, the condition (2.8) is not satisfied and φ can, a priori, assume any real value according to which part of the sphere S_3 (the group manifold) is covered by the image $I(R_3)$ of the co-ordinate space R_3 .

Of course, the simplest pure gauge potential satisfying the transversality condition (1.1) is $A_\mu(\vec{x}) \equiv 0$; in terms of the mapping (2.7) this trivial vacuum is characterized by $N_a(\vec{x}) \equiv \text{const.}$; then the image $I(R_3)$ is just a point on the S_3 sphere and φ vanishes.

*) It can be useful to remark that the condition (2.8) can be rephrased in terms of the potential $A_i(\vec{x})$. If Eq. (2.8) holds, $A_i(\vec{x})$ falls down more quickly than $1/r$, for $r \rightarrow \infty$; instead if the limit on $N_a(\vec{x})$ depends on the direction $\vec{x}/r$, a non-vanishing component of $\partial_i U$ appears in some direction orthogonal to $\vec{x}$ and $|A_i(\vec{x})|$ behaves just like $1/r$.

The most general transverse pure gauge field must be a solution of the equation obtained by substituting Eq. (2.3) in the transversality condition (1.1) ; by using the parametrization (2.4) one gets the three independent equations (remember that $N_a N_a = 1$) :

$$\Delta N_a = -(\partial_i N_b \partial_i N_b) N_a, \quad a=1,2,3,4; \quad (2.9)$$

the explicit expression of Eqs. (2.9) in terms of the polar co-ordinates α, β, γ [defined in (2.6)] is :

$$\partial_i \partial_i \alpha = \frac{\sin 2\alpha}{2} (\partial_i \beta \partial_i \beta + \sin^2 \beta \partial_i \gamma \partial_i \gamma) \quad (2.10a)$$

$$\partial_i (\sin^2 \alpha \partial_i \beta) = \frac{\sin 2\beta}{2} \sin^2 \alpha \partial_i \gamma \partial_i \gamma \quad (2.10b)$$

$$\partial_i (\sin^2 \alpha \sin^2 \beta \partial_i \gamma) = 0. \quad (2.10c)$$

Spherically symmetric solutions of Eqs. (2.10) can be obtained by considering the ansatz ¹⁾

$$\begin{pmatrix} \sin \beta \cos \gamma \\ \sin \beta \sin \gamma \\ \cos \beta \end{pmatrix} = \frac{\vec{x}}{r}$$

$$\alpha(\vec{x}) = \alpha(r) \quad [\text{with } \alpha(0) = 0 \pmod{\pi}], \quad (2.11)$$

which means

$$\begin{aligned} \beta(\vec{x}) &= \theta \\ \gamma(\vec{x}) &= \varphi \end{aligned} \quad ; \quad (2.12)$$

in (2.11) and (2.12) the variables r, θ, φ are polar co-ordinates, with respect to an arbitrary pole, on the space R_3 spanned by the spatial co-ordinates $\vec{x}$.

If the ansatz (2.11), (2.12) is used, Eqs. (2.10b) and (2.10c) are identically satisfied and Eq. (2.10a) reduces to the ordinary differential equation :

$$\frac{d^2\alpha}{dz^2} + \frac{2}{z} \frac{d\alpha}{dz} - \frac{\sin 2\alpha}{z^2} = 0 \quad (2.13)$$

which becomes the damped pendulum equation ¹⁾, once the variable $t = \log r$ is introduced. As Gribov has noted, beyond the trivial solution $\alpha(r) \equiv 0$, which corresponds to $A_\mu \equiv 0$, Eq. (2.13) has two other classes of solutions, characterized by the asymptotic behaviour :

$$\lim_{z \rightarrow \infty} \alpha(z) - \alpha(0) = \pm \frac{\pi}{2} . \quad (2.14)$$

These latter solutions do not satisfy Eq. (2.8) ; for very large r the vector $N_a (r = \infty, \theta, \varphi)$ is not a constant, but, on the contrary, it spans a great S_2 sphere on the S_3 sphere ^{*}). Then the space R_3 is mapped on one-half of the sphere S_3 (see Fig. 1) ; in fact, from (2.5), one gets, respectively,

$$\varphi = \mp \frac{1}{2} . \quad (2.14')$$

We were not able to find the general solution of the coupled non-linear partial derivative differential equations (2.10), without using any restrictive ansatz. However, we managed to establish a certain number of results which strongly limit the classes of possible solutions of Eqs. (2.9), (2.10) and which then allow us to clarify the vacuum structure in the "Coulomb gauge" (1.1). At first we will prove the following :

Statement A : The only solution of Eqs. (2.9) with the asymptotic behaviour (2.8) is the trivial one : $N_a(\vec{x}) \equiv \text{const.}$ In other words, we are stating that it is impossible to find a non-trivial mapping $N_a(\vec{x})$, subjected to Eqs. (2.9), whose image $I(R_3)$ covers the sphere S_3 an integer (or vanishing) number of times.

^{*}) We have borrowed our language from the usual one for one less dimension : a great circle on the sphere.

Proof

Equations (2.9) can be obtained ¹⁾ as "equations of motion" relevant to the "action" ^{*)}

$$W = \int d^3x \partial_i N_a \partial_i N_a \quad (2.15)$$

with the constraint

$$N_a N_a = 1. \quad (2.15')$$

One can immediately realize ⁴⁾ that the solution $N_a(\vec{x}) \equiv \text{const.}$ is the only one which gives a finite "action" W . In fact if $N_a(\vec{x})$ is a solution and $W[N_a(\vec{x})]$ is finite, the action $W[N_a(\vec{x}, \lambda)]$, with $N_a(\vec{x}, \lambda) = N_a(\vec{x}/\lambda)$, must have an extremum for $\lambda = 1$. From (2.15), one gets :

$$W[N_a(\vec{x}, \lambda)] = \lambda W[N_a(\vec{x})] ;$$

then $(\partial W / \partial \lambda)|_{\lambda=1} = 0$ implies $W[N_a(\vec{x})] = 0$, which is obtained only if $N_a(\vec{x}) = \text{const.}$ The next, and final, step to prove Statement A is to show that any solution of Eqs. (2.9) which satisfies condition (2.8) gives a finite "action" W . In fact, if condition (2.8) holds, one can factor out the leading $r \rightarrow \infty$ behaviour :

$$N_a(r, \theta, \varphi) \underset{r \rightarrow \infty}{\Rightarrow} M_a + \varepsilon(r) K_a(\theta, \varphi), \quad (2.16)$$

where M_a is a constant unit vector and $\varepsilon(r)$ is a function which vanishes for $r \rightarrow \infty$. By substitution in Eqs. (2.9) one gets the asymptotic equation :

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\varepsilon}{dr} \right) K_a(\theta, \varphi) - \frac{\varepsilon(r)}{r^2} L^2 K_a(\theta, \varphi) = 0 \quad (2.17)$$

where L^2 is the angular momentum operator :

$$L^2 = - \left[\frac{1}{r \sin \theta} \partial_\theta (r \sin \theta \partial_\theta) + \frac{1}{r \sin^2 \theta} \partial_\varphi^2 \right]. \quad (2.18)$$

*) Of course, W is a fictitious action, whose equations of motion are $\partial_i A_i = 0$; it has nothing to do with the Yang-Mills action.

By multiplying Eq. (2.17) by $K_a(\theta, \varphi)$ and integrating on the spherical surface one gets

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\varepsilon}{dr} \right) - \frac{c}{r^2} \varepsilon(r) = 0, \quad (2.19)$$

where c is a positive (or vanishing) constant :

$$c = \frac{\int d\Omega K_a(\theta, \varphi) L^2 K_a(\theta, \varphi)}{\int d\Omega K_a(\theta, \varphi) K_a(\theta, \varphi)} \geq 0.$$

The asymptotic behaviour of $\varepsilon(r)$ is immediately obtained by Eq. (2.19) by means of standard methods ; under the condition that $\varepsilon(r)$ vanishes one gets

$$\varepsilon(r) = O(r^\nu) \quad \text{with} \quad \nu \leq -1. \quad (2.20)$$

The substitution of (2.20) in (2.16) and (2.15) shows that condition (2.8) implies a finite action W and then, for the first part of the proof, $N_a(\vec{x}) \equiv \text{const.}$ Q.e.d.

Forgetting now Eq. (2.8), we are led to consider solutions $N_a(\vec{x})$ of Eqs. (2.9) whose image $I(r_3)$ covers the sphere S_3 a non-integer number of times. About these possible solutions we can formulate the following :

Statement B : There are no solutions of Eqs. (2.9) such that the image $I(R_3)$ of the corresponding mapping $N_a(\vec{x})$ covers the sphere S_3 more than once.

Our proof of this statement is a bit cumbersome and can be found in the Appendix.

Here we only say that our proof generalizes to Eqs. (2.9) the considerations on Eq. (2.13) suggested by the physical analogy with a damped pendulum.

In such a simple case one considers the quantity

$$E(r) = \frac{1}{2} r^2 \left(\frac{d\alpha}{dr} \right)^2 - \sin^2 \alpha \quad (2.21)$$

which plays the rôle of the total energy of the pendulum ; by showing that, due to the friction, $E(r)$ is a decreasing function of r , one realizes that starting from $\alpha(r=0)=0$ [and $(d\alpha/dr)|_{r=0} = \gamma$, finite] the function $\alpha(r)$ can neither reach or overcome the value π . As it is shown in the Appendix, also in the general case of Eqs. (2.9), similar considerations can be done starting from the quantity

$$E(r, \theta, \varphi) = \frac{1}{2} r^2 \left(\partial_r N_a \partial_r N_a - \partial_T N_a \partial_T N_a \right) \quad (2.22)$$

where $\partial_r = (\partial/\partial r)$ is the radial derivative and ∂_T is the transverse part of the gradient :

$$\partial_T = \frac{1}{r} \left(\frac{\partial}{\partial \theta} + \frac{1}{\sin \theta} \frac{\partial}{\partial \varphi} \right) . \quad (2.23)$$

One could be tempted to summarize the statements A and B by saying that Eqs. (2.9) imply that the absolute value of the quantity φ , defined in (2.1), must be less than one. However, this statement is too strong ; in fact it is possible to imagine mappings $N_a(\vec{x})$ with $\varphi \geq 1$ which are not forbidden by statements A and B.

For instance, the value $\varphi = 1$ can be obtained at least in two different ways :

- i) the most obvious way, which is forbidden by statement A, is the one-to-one mapping $S_3 \rightarrow S_3$; in terms of the polar co-ordinates α, β, γ the image $I(R_3)$ is defined by :

$$\begin{aligned} 0 &\leq \alpha \leq \pi \\ 0 &\leq \beta \leq \pi \\ 0 &\leq \gamma \leq 2\pi . \end{aligned} \quad (2.24)$$

ii) an alternative way is a two-to-one mapping of R_3 on one half of the sphere S_3 ; that is the image $I(R_3)$ would be defined by

$$\begin{aligned} 0 \leq \alpha \leq \frac{\pi}{2} \\ 0 \leq \beta \leq \pi \\ 0 \leq \gamma \leq 4\pi. \end{aligned} \quad (2.25)$$

Even if several attempts of finding such a kind of solutions of Eqs. (2.9), by means of different ansatz, were unsuccessful, we are not able to prove that Eqs. (2.9) definitely exclude the possibility ii) or analogous ones (with possible decreasing oscillations of α around $\pi/2$).

A useful way to classify these other possible solutions, if they exist, stems from the consideration that, due to statement A, the image $I(R_3)$ corresponding to any non-trivial solution of Eqs. (2.9) must have a non-vanishing boundary ; the continuity of the mapping $N_a(\vec{x})$ requires that such a boundary on the spherical surface S_3 is homeomorphic to S_2 . Then the mapping $N_a(\vec{x})$, which sends R_3 into S_3 , induces an $S_2 \rightarrow S_2$ mapping $N_a(\omega, \theta, \varphi)$ between the very large sphere in the co-ordinate space and the boundary of $I(R_3)$ in the group manifold :

$$N_a(\infty, \theta, \varphi): \quad \frac{\vec{x}}{2} \rightarrow \lim_{r \rightarrow \infty} N_a(r, \frac{\vec{x}}{r}). \quad (2.26)$$

The homotopy classes of such an $S_2 \rightarrow S_2$ mapping can be classified by an integer number m . Of course, the Gribov solutions discussed in (2.11)-(2.14) have $m=1$. The arguments discussed in the last part of this section can be summarized in the following :

Statement C : If they exist, solutions $N_a(\vec{x})$ of Eqs. (2.9) characterized by $|\varphi| \geq 1$ [φ is defined in (2.1), (2.5)] must induce a mapping $N_a(\omega, \theta, \varphi)$ [defined in (2.26)] belonging to a homotopy class with topological number $|m| \geq 2$.

SECTION 3

In Section 2 we have discussed the structure of the vacuum states, working at fixed time ; instead in this section we are going to discuss classical Euclidean configurations $F_{\mu\nu}(\vec{x}, x_4)$ that interpolate between a vacuum $|a\rangle$ in the remote past and another one $|b\rangle$ in the far future.

From the path integral formulation of quantum theory we know that if such configurations exist, and have a finite Euclidean action

$$S_E = -\frac{1}{2e^2} \int d^4x \text{Tr}(F_{\mu\nu} F_{\mu\nu}), \quad (3.1)$$

then a quantum tunnelling among the vacua $|a\rangle$ and $|b\rangle$ can occur.

Let us explicitly remark that these configurations must not necessarily be solutions of the Yang-Mills equations of motion, even if these ones are believed to give the most important contribution to the path integral.

In order to have a finite Euclidean action S_E , we limit our analysis to the configurations $F_{\mu\nu}(\vec{x}, x_4)$ which are regular ^{*)} everywhere and which vanish quickly enough at infinity ; that is we impose that for very large $R = \sqrt{\vec{x}^2 + x_4^2}$ the potential A_μ becomes a pure gauge one :

$$A_\mu \underset{R \rightarrow \infty}{=} U^{-1} \partial_\mu U. \quad (3.2)$$

Such configurations can be classified by the topological quantum number ³⁾

$$q = -\frac{1}{32\pi^2} \int d^4x \epsilon_{\mu\nu\alpha\beta} \text{Tr}(F_{\mu\nu} F_{\alpha\beta}); \quad (3.3)$$

as we have assumed that $F_{\mu\nu}$ (and A_μ) is regular everywhere, the Pontryagin number q can be written as a flux across a large surface S'_3 (in the R_4 space x_μ) homeomorphic to the sphere S_3 :

^{*)} One could admit gauge removable singularities, of the kind of those which appear in the 't Hooft-Jackiw-Nohl-Rebbi form of the multi-instanton solutions ³⁾ ; however, in this work we suppose that such singularities have been in fact already removed ³⁾.

$$q = \frac{1}{24\pi^2} \int_{S_3'} dS_\mu \varepsilon_{\mu\nu\alpha\beta} \text{Tr}(A_\nu A_\alpha A_\beta) \quad (3.4)$$

with A_μ given by (3.2).

By using for U the parametrization (2.4) (but now with N_a depending not only on $\vec{x}$ but also on x_4), the quantity q can be written in a form that makes clearer its topological meaning :

$$q = -\frac{1}{2\pi^2} \int_{I(S_3')} d\alpha d\beta dy \sin^2\alpha \sin\beta, \quad (3.5)$$

where $I(S_3')$ is the image on the group manifold S_3 of some large surface S_3' homeomorphic to S_3 .

As the mapping

$$N_a(x_\mu): \quad x_\mu \rightarrow N_a(x_\mu) \quad ; \quad x_\mu \in S_3' \quad (3.6)$$

is supposed to be continuous, q is an integer number indicating how many times the group manifold is covered when x_μ spans the surface S_3' .

The relationship between the topological number q and the quantity φ defined in (2.1) and (2.5) can be seen by choosing S_3' to be the cylindrical surface depicted in Fig. 2. In such a case one gets :

$$q = \varphi_+ - \varphi_- + \varphi_L, \quad (3.7)$$

where

$$\varphi_\pm = -\frac{1}{24\pi^2} \int d^3x \varepsilon_{ijk} \text{Tr}(A_i A_j A_k) \Big|_{x_4 = \pm\infty} \quad (3.8)$$

is just the quantity defined in (2.1), (2.5) calculated at time $x_4 = \pm\infty$,
and ^{*})

$$\varphi_L = \frac{1}{8\pi^2} \int_{-\infty}^{+\infty} dx_4 \lim_{r \rightarrow \infty} \int d^2 S_i \varepsilon_{ijk} \text{Tr}(A_j A_k A_4) \quad (3.9)$$

is the flux across the lateral surface of the cylinder of Fig. 2.

By using the parametrization (2.4), both $\varphi_{\pm}$ and φ_L can be written in a form analogous to Eqs. (3.5) and (2.5) ; now the integration surfaces are, respectively, the images $I(R_3, \pm)$ and $I(L)$ of the top, bottom and lateral surfaces of the cylinder of Fig. 2, according to the mappings :

$$N_a(\vec{x}, \pm) : \vec{x} \rightarrow N_a(\vec{x}, x_4 = \Lambda_{\pm}) \quad \text{for } \Lambda_{\pm} \rightarrow \pm\infty \quad (3.10a)$$

$$N_a(\infty, \theta, \varphi, x_4) : \frac{\vec{x}}{r}, x_4 \rightarrow N_a(r \rightarrow \infty; \frac{\vec{x}}{r}, x_4). \quad (3.10b)$$

As we are interested to discuss the "Coulomb gauge" properties, we will impose, at all times, the transversality condition on the spatial components :

$$\partial_i A_i(\vec{x}, x_4) = 0 \quad (3.11)$$

and a good large distance behaviour for the time component :

$$\lim_{r \rightarrow \infty} A_4(\vec{x}, x_4) = 0. \quad (3.12)$$

We want at once to remark that all the results we will obtain in this section are not affected if condition (3.12) is dropped out.

However, our discussion will be simpler assuming for the moment that condition (3.12) is fulfilled ; the proof that Eq. (3.12) is not actually important will be given at the end of this section.

^{*}) Both in (3.8) and (3.9) the limit $x_4 \rightarrow \pm\infty$ must be taken after having performed the space integral ⁵⁾.

Equation (3.12) implies a drastic simplification because it forces the lateral flux φ_L defined in (3.9) to vanish ; in fact from Eq. (3.12) and Eq. (3.2) it follows that the vector $N_a(r=\infty; \vec{x}/r, x_4)$ is actually time-independent ; hence the image $I(L)$ of the lateral surface of the cylinder of Fig. 2 simply shrinks to the boundary of $I(R_3, \pm)$.

Then the boundaries of $I(R_3, -)$ and $I(R_3, +)$ must coincide and the covering of the group manifold S_3 must be got by the union of $I(R_3, -)$ and $I(R_3, +)$, without any contribution by $I(L)$ ^{*)}. In particular :

Statement D : If the boundaries of $I(R_3, -)$ and $I(R_3, +)$ are not simply a point, they must have the same $S_2 \rightarrow S_2$ topological number m , defined at the end of Section 2 ; of course, if $\partial I(R_3, -)$ is simply a point, also $\partial I(R_3, +)$ must be a point.

Now we can easily state a further result :

Statement E : In the Coulomb gauge (3.11), (3.12), the trivial vacuum $A_i(\vec{x}) \equiv 0$ cannot tunnel into any other vacuum. Moreover, any regular Euclidean configuration $A_\mu(\vec{x}, x_4)$ that connects at times $\pm\infty$ the vacuum $A_i(\vec{x}) \equiv 0$ with itself (with finite action) is topologically trivial, that is it has Pontryagin number $q=0$.

Proof

The first part of Statement E is a direct consequence of Statement A of Section 2, as, according to Statement D, the trivial vacuum $A_i(\vec{x}) \equiv 0$ could only tunnel to a vacuum whose image covered an integer (or vanishing) number of times the S_3 sphere.

Also the second part is obvious, once that Eq. (3.12) is supposed to hold, and then $\varphi_L = 0$. Q.e.d.

We want to remark that Statement E is quite strong. In fact, it shows that the "Coulomb gauge" is a very peculiar one and behaves quite differently from the other gauges. In the $A_0 = 0$ gauge, for instance,

^{*)} This statement is much stronger than the one that $\varphi_L = 0$; in fact, even if $\varphi = 1$ could be written as $\varphi_+ - \varphi_-$ with $\varphi_+ = 1$ and $\varphi_- = 0$, such a situation is not allowed if $\varphi_+ = 1$ is obtained in the way ii) described at the end of Section 2, that is with $m = 2$.

there is an infinite number of classes of different vacua ³⁾, classified by $\varphi = n$ (n integer); a (multi) instanton with topological charge $q \neq 0$ sends the vacuum $A_i(\vec{x}) \equiv 0$ ($\varphi = 0$) into a vacuum with $\varphi = q$. In the axial gauge ($A_3 = 0$) the description is quite different as, by definition, the vacuum must have $\varphi = 0$; however, in this gauge it is possible to choose the boundary conditions (for large r) in such a way that the vacuum $\varphi = 0$ at $x_4 = -\infty$ can be sent into itself at $x_4 = +\infty$ by means of configurations with any q .

On the contrary, Statement E shows that in the "Coulomb gauge" it is not possible to write configurations with $q \neq 0$ by means of a continuous potential $A_\mu(\vec{x}, x_4)$ which, at a certain very large time (either positive or negative), reproduces the trivial vacuum $A_\mu(\vec{x}) \equiv 0$.

From Statements B and C of Section 2, one easily deduces also the following result, which was already proved in Ref. 5), but only in the case of the spherical symmetry ansatz (2.11) :

Statement F : In the "Coulomb gauge" (3.11), (3.12), any regular Euclidean configuration $A_\mu(\vec{x}, x_4)$ with $q = 1$ necessarily connects the Gribov vacuum with $\varphi_- = -\frac{1}{2}$ (and $m = 1$) at $x_4 = -\Lambda$ with the analogous one with $\varphi_+ = +\frac{1}{2}$ at $x_4 = +\Lambda$ (Λ very large and positive). An analogous statement holds for configurations with $q = -1$.

Proof

According to Statement E, a configuration with $q = 1$ cannot start at $x_4 = -\Lambda$ from the vacuum $A_i(\vec{x}) \equiv 0$; neither such a configuration can start at $x_4 = -\Lambda$ from a configuration with $\varphi_- = +\frac{1}{2}$ and $m = 1$, because, as stated in B and C, configurations with $\varphi_+ = \frac{3}{2}$ and $m = 1$ do not exist (remember that Statement D forbids tunnelling among vacua with different m).

Moreover, even if we are not able to exclude the possibility of vacuum states with $|m| = 2, 3, \dots$, Statement D implies that regular configurations with $q = 1$ cannot have anything to do with such hypothetical vacua; in fact, as the topological number m is conserved, a regular Euclidean configuration $F_{\mu\nu}(\vec{x}, x_4)$ with a finite action which starts, at $x_4 = -\infty$, from a vacuum with a given m , must have $|q| = |m|$ or $q = 0$ (see Fig. 3). Hence the only possibility left is the one stated in Statement F. Q.e.d.

We can finally state the main result of this work :

Statement G : There exist Euclidean configurations $F_{\mu\nu}(\vec{x}, x_4)$ that do not admit any regular potential $A_\mu(\vec{x}, x_4)$ which satisfies the Coulomb conditions (3.11) and (3.12).

Proof

Let us consider for instance a double instanton ($q=2$), built up as a sum of two non-overlapping single instantons, whose sizes λ_1 and λ_2 are very small with respect to the difference T between the time co-ordinates of their centres ^{*)} (see Fig. 4, where it is also assumed that the spatial co-ordinates of the centres of the two instantons coincide). It follows immediately from Statement F that such a configuration cannot be written in the "Coulomb gauge" ; in fact one can arrange only one of the two subsequent instantons, for instance the first, making it to tunnel from a vacuum with $\varphi = -\frac{1}{2}$ to another with $\varphi = +\frac{1}{2}$; the other instanton should tunnel from a vacuum with $\varphi = +\frac{1}{2}$ to another with $\varphi = \frac{3}{2}$ (and $m=1$) but the latter does not exist. Q.e.d.

The proofs we have till now given of the statements of this Section heavily use condition (3.12) on the large distance asymptotic behaviour of the time-component A_4 . Then one could wonder if the strange features of the "Coulomb gauge" outlined in the statements E, F and G are just a consequence of condition (3.12).

In such a case one could think that the pathologies of the "Coulomb gauge" would disappear by dropping out condition (3.12), that seems less important than the transversality condition (3.11).

^{*)} The limit $(\lambda_1/T, \lambda_2/T) \rightarrow 0$ is harmless ; anyway one can avoid any limit by considering a double instanton configuration with $\lambda_1 = \lambda_2 = \epsilon T$; then one can modify $F_{\mu\nu}$ a bit in such a way to get exactly

$$F_{\mu\nu}(\vec{x}, 0) = F_{\mu\nu}(\vec{x}, -T) = F_{\mu\nu}(\vec{x}, T) \equiv 0.$$

The configuration obtained in this way is no longer an exact solution of the equations of motion (even if it can be made as close to a solution as one wants, by choosing a suitable ϵ) ; however, it has the same topological properties of a double instanton and must be taken into account in the path integral formulation of quantum theory.

For instance, such is the case for the dyon solution discussed in Appendix A of Ref. 6) : such a configuration satisfies Eq. (3.11) and not Eq. (3.12) and it seems very difficult to find a potential A_μ that satisfies both conditions (3.11) and (3.12). On the contrary, we are going to show that our statements do not change even if one neglects condition (3.12).

In fact, a potential

$$A_4(\vec{x} \rightarrow \infty, x_4) = U^{-1} \partial_4 U \quad (3.13)$$

different from zero can be obtained in essentially two ways.

i) The gauge group element U depends only on the time x_4 and does not depend on $\vec{x}$. A potential A_4 of this kind can be gauge transformed to zero by means of the space independent transformation $U^{-1}(x_4)$; it is easy to see that such a gauge transformation affects neither the transversality equation (3.11) nor our topological considerations, in particular Statement D. In fact, $U^{-1}(x_4)$ is homotopically equivalent to $U = \text{const.}$ and then it does not modify the topological number q ; moreover, such a transformation affects the mappings $N_a(\vec{x}, \pm)$ defined in (3.10) and the boundaries of their images $I(R_3, \pm)$ only by time-dependent rigid rotations. Thus the results of this section do not change if condition (3.12) is substituted by (3.13) with U only depending on the time.

ii) The time-dependence of $\lim_{r \rightarrow \infty} A_4(\vec{x}, x_4)$ cannot be factorized out ; in this case $U(\vec{x}, x_4)$ is determined from the transversality equation (3.11), modulo trivial gauge transformations of the kind i).

For a general $F_{\mu\nu}(\vec{x}, x_4) \neq 0$, such an equation is, of course, very complicated ; however, if we consider configurations with the asymptotic behaviour (3.2), then for very large $r = \sqrt{\vec{x}^2}$ the field $F_{\mu\nu}(\vec{x}, x_4)$ vanishes at any time and Eq. (3.10) reduces to the asymptotic behaviour for $r \rightarrow \infty$ of Eq. (2.9) discussed in Section 2.

In such a limit, Eq. (2.9) becomes :

$$-L^2 N_a(\theta, \varphi, x_4) = -(2\partial_T N_b \ 2\partial_T N_b) N_a(\theta, \varphi, x_4), \quad (3.14)$$

where

$$N_a(\theta, \varphi, x_4) = \lim_{z \rightarrow \infty} N_a(\vec{x}, x_4); \quad (3.15)$$

the transverse derivatives ∂_T and the angular momentum operator L^2 are defined in Eqs. (2.23) and (2.18), respectively.

As $\partial_T N_b \partial_T N_b$ is a positive definite quantity, it is easy to prove the

Statement H : any vector $N_a(\theta, \varphi, x_4)$ which satisfies Eq. (3.14) must belong to one of the two following classes :

a)

$$N_a(\theta, \varphi, x_4) = M_a(x_4), \quad (3.16)$$

where $M_a(x_4)$ does not depend on θ, φ ;

b) the angular dependence of $N_a(\theta, \varphi, x_4)$ is such that, at fixed x_4 , it is impossible to find a constant vector M_a such that

$$M_a N_a(\theta, \varphi, x_4) > 0 \quad (3.17)$$

for any value of θ and φ .

In other words, the angular dependence of $N_a(\theta, \varphi, x_4)$ is such that it is impossible to find a "great sphere S_2 " that divides S_3 into two halves such that the vector $N_a(\theta, \varphi, x_4)$ at fixed x_4 always lies in one of them.

Proof

Let us integrate both sides of Eq. (3.14) on the large S_2 sphere in the Euclidean space $\vec{x}$:

$$-\int d\Omega L^2 N_a = -\int d\Omega (2\partial_T N_b \partial_T N_b) N_a. \quad (3.18)$$

By definition of L^2 , the left-hand side of Eq. (3.18) vanishes ; the right-hand side must also vanish and then either Eq. (3.16) follows, or Eq. (3.17) is excluded. Q.e.d.

It is important to remark that :

Statement I : The vector $N_a(\theta, \varphi, x_4)$ cannot continuously pass from the class a) to the class b) or vice versa. In fact, if such a continuous transition were allowed, at a certain time x_4 the vector $N_a(\theta, \varphi, x_4)$ should span a small S_2 sphere around a vector $M_a(x_4)$; but such a θ, φ dependence of $N_a(\theta, \varphi, x_4)$ belongs neither to the class a) nor to the class b) and then it is forbidden by Statement H.

Finally, Statement I applied to the temporal evolution of $N_a(\theta, \varphi, x_4)$ from $x_4 = -\Lambda$ to $x_4 = +\Lambda$ (Λ very large and positive) reproduces Statement D. Then the Statements E, F, G follow in the same way as before, without any use of the asymptotic condition (3.12) on the time-component A_4 .

SECTION 4

The main result of our work is that some physical configurations (the multi-instanton solutions of pure Yang-Mills theory) cannot be described in the "Coulomb gauge" (Statement G of Section 3) ; that is, for such configurations, it is impossible to find a regular potential $A_\mu(\vec{x}, x_4)$ which satisfies the transversality condition $\partial_i A_i(\vec{x}, x_4) = 0$.

We remark, as stated at the end of Section 3, that such a restriction to the possible physical situations reachable in the "Coulomb gauge" is due only to the transversality condition, without any need of imposing a particular, large r behaviour to the time-component $A_4(\vec{x}, x_4)$.

At the end of the Appendix, we have observed that, to be strictly rigorous, our proof of Statement B of Section 2 (and then of Statement G of Section 3) is not quite complete.

However, even if a very peculiar loop-hole to our proof of Statement B could be found, the behaviour of the Coulomb gauge would remain very pathological all the same.

In fact we have proved (Statement E), independently of Statement B, that in the Coulomb gauge any configuration with non-vanishing Pontryagin number q (instantons) cannot affect the perturbative vacuum $A_\mu(\vec{x}) \equiv 0$. As we have also proved that the perturbative vacuum cannot tunnel into any other one, in the Coulomb gauge we should deal with a Hilbert space, not communicating with any other one, in which instantons are not present. As instantons are not pure gauge effects and bring to observable consequences (think of the arbitrary phase θ which appears in the $A_0 = 0$ gauge) one should understand where they are hidden in the Coulomb gauge. As a final remark we wish, however, to precise that all our observations would lose their validity if one admits singular potentials and singular gauge transformations.

P.S. : After this work was completed we have seen a preprint by L.F. Abbott and T. Eguchi : "Structure of the Yang-Mills Vacuum in Coulomb Gauge", whose results seem to contradict those of our work and of Ref. 5). However, it seems to us that in the Abbott-Eguchi paper, some limits are arbitrarily interchanged : the ordering of the limit for $r \rightarrow \infty$ and for $t \rightarrow \infty$ in their Eq. (35) agrees neither with Eqs. (28), (29), (30), nor with Eq. (36) [for a brief discussion of the ordering of these limits, see Ref. 5].

P.P.S. : We have just seen a paper by R. Jackiw, I. Muzinich and C. Rebbi which partially overlaps our work and Ref. 5). In that work, where only spherically symmetric configurations are discussed, it is proved that configurations with non-vanishing Pontryagin number can be written in the Coulomb gauge if and only if discontinuous potentials $A_\mu(\vec{x}, x_4)$ are admitted.

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APPENDIX - PROOF OF THE STATEMENT B OF SECTION 2

Let us suppose that the Statement B is not true ; then, in the Euclidean space R_3 , it must exist at least a simply connected surface σ which is mapped in a single point of the sphere S_3 ; that is the vector $N_a(\vec{x})$ must be constant when $\vec{x}$ varies on the surface σ .

We choose as origin of the polar co-ordinates a point interior to this surface and we write its equation in the form ^{*)} :

$$r = \bar{r}(\theta, \varphi). \quad (A.1)$$

Then we can consider the identity :

$$\int d\Omega \, \varepsilon(\bar{r}(\theta, \varphi), \theta, \varphi) = \int d\Omega \int_0^{\bar{r}(\theta, \varphi)} dr \, \frac{\partial \varepsilon}{\partial r}, \quad (A.2)$$

where

$$\int d\Omega = \int_0^\pi d\theta \, 2\sin\theta \int_0^{2\pi} d\varphi \quad (A.3)$$

and $\varepsilon(r, \theta, \varphi)$ is the quantity defined in (2.2) [in writing Eq. (A.2) we have used the fact that $\varepsilon(0, \theta, \varphi) = 0$].

By means of Eqs. (2.9), one gets for the quantity $\partial \varepsilon / \partial r$ the following expression :

$$\frac{\partial \varepsilon}{\partial r} = -r \partial_r N_a \partial_r N_a - r^2 (\partial_r N_a \partial_T N_a)_T \quad (A.4)$$

where

$$\begin{aligned} (\partial_r N_a \partial_T N_a)_T &= \\ &= \frac{1}{r^2 \sin\theta} \left[\partial_\theta (\partial_r N_a \, 2\sin\theta \, \partial_\theta N_a) + \partial_\varphi (\partial_r N_a \, \frac{\partial_\varphi N_a}{\sin\theta}) \right] \end{aligned} \quad (A.5)$$

*) We suppose that it is possible to find an origin such that $\bar{r}(\theta, \varphi)$ is a single valued function.

is the transverse contribution to the divergence of $(\partial_r N_a \partial_i N_a)$.

We can immediately observe that $\partial \varepsilon / \partial r$ is negative definite, apart from a total derivative ; then we can hope to reproduce the results obtained by Gribov for the spherical ansatz [see the discussion of Eq. (2.21)].

The calculations can be done more easily if one chooses a suitable set of co-ordinates on the sphere S_3 ; we will fix the south pole of the sphere S_3 just in the point $N_a(\vec{x})$ for $\vec{x} \in \sigma$.

Then in terms of the co-ordinates α, β, γ defined in (2.6) the south pole is defined by $\alpha = \pi$ and we get for ε and $\partial \varepsilon / \partial r$ the following expressions :

$$\varepsilon(\bar{r}(\theta, \varphi), \theta, \varphi) = \frac{1}{2} \bar{r}^2 (\partial_r \alpha \partial_r \alpha - \partial_T \alpha \partial_T \alpha) \Big|_{r=\bar{r}(\theta, \varphi)} \quad (A.6)$$

where the equation

$$\alpha(\bar{r}(\theta, \varphi), \theta, \varphi) = \pi \quad (A.7)$$

has been used ;

$$\begin{aligned} \frac{\partial \varepsilon}{\partial r} = & -r (\partial_r \alpha \partial_r \alpha + r \sin^2 \alpha \partial_r \beta \partial_r \beta + r \sin^2 \alpha r \sin^2 \beta \partial_r \gamma \partial_r \gamma) - \\ & - r^2 (\partial_r \alpha \partial_T \alpha + r \sin^2 \alpha \partial_r \beta \partial_T \beta + r \sin^2 \alpha r \sin^2 \beta \partial_r \gamma \partial_T \gamma)_T \end{aligned} \quad (A.8)$$

where notations analogous to (A.5) have been used.

By substituting the expressions (A.6) and (A.8) into Eq. (A.2) and by taking into account the identity :

$$\begin{aligned}
 & \int d\Omega \int_0^{\bar{r}(\theta, \varphi)} dz \, z^2 \left(\partial_z \alpha \partial_T \alpha + 2 \sin^2 \alpha \partial_z \beta \partial_T \beta + 2 \sin^2 \alpha \sin^2 \beta \partial_z \gamma \partial_T \gamma \right) \Big|_T = \\
 & = \int d\Omega \left(\partial_T \alpha \partial_T \alpha \right) \Big|_{z=\bar{r}(\theta, \varphi)} \quad , \quad (A.9)
 \end{aligned}$$

one finally gets

$$\begin{aligned}
 & \frac{1}{2} \int d\Omega \, \bar{r}^2 \left(\partial_z \alpha \partial_z \alpha + \partial_T \alpha \partial_T \alpha \right) \Big|_{z=\bar{r}(\theta, \varphi)} = \\
 & = - \int d\Omega \int_0^{\bar{r}(\theta, \varphi)} dz \, z \left(\partial_z \alpha \partial_z \alpha + 2 \sin^2 \alpha \partial_z \beta \partial_z \beta + 2 \sin^2 \alpha \sin^2 \beta \partial_z \gamma \partial_z \gamma \right) \quad (A.10)
 \end{aligned}$$

or, in the four-vector notation,

$$\begin{aligned}
 & \frac{1}{2} \int d\Omega \, \bar{r}^2 \left(\partial_i N_a \partial_i N_a \right) \Big|_{z=\bar{r}(\theta, \varphi)} = \\
 & = - \int d\Omega \int_0^{\bar{r}(\theta, \varphi)} dz \, z \left(\partial_z N_a \partial_z N_a \right) . \quad (A.10')
 \end{aligned}$$

As the right-hand side of (A.10) is certainly negative [but the trivial case $N_a(\vec{x}) \equiv \text{const.}$] and the left-hand side is positive definite, Eq. (A.10) is absurd. Thus it is impossible to have a surface σ such that $N_a(\vec{x}) = \text{const.}$ for $\vec{x} \in \sigma$ and the statement B of Section 2 is proved to be true.

To be rigorous, we have, however, to remark that we were not able to exclude the possibility of a surface σ whose equation could not be expressed in the form (A.1) with respect to any origin.

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FIGURE CAPTIONS

- Figure 1 Figures 1a and 1b represent the images $I(R_3)$ of the Euclidean space R_3 on the sphere S_3 , according to the mapping induced by the two classes of solutions (2.14).
The figure is oversimplified, not only because the sphere S_3 is drawn like a sphere S_2 , but also because the oscillations of α before reaching the asymptotic value $\pi/2$ are not represented.
- Figure 2 A particular form of the surface S_3^1 , homeomorphic to a S_3 sphere; is represented ; a single axis $\vec{x}$ is drawn to represent the three axes x, y, z . The surface is supposed to lie completely in the region where $F_{\mu\nu} = 0$.
- Figure 3 Both in Figs. 3a and 3b, the image $I(R_3, -)$ (continuous line) covers m times the Northern hemisphere. $I(R_3, +)$ (dashed lines) covers m times the Southern hemisphere in Fig. 3a ; instead in Fig. 3b it covers again, with opposite orientation, the Northern hemisphere.
- Figure 4 A particular two-instanton configuration ; the field $F_{\mu\nu}$ is appreciably different from zero only inside the dashed spheres.

$$\alpha(0) = 0 ; \quad \alpha(\infty) = \pi/2$$

$$\varphi = -1/2$$

$$\alpha(0) = \pi ; \quad \alpha(\infty) = \pi/2$$

$$\varphi = +1/2$$

FIG. 1

FIG. 2

FIG. 3 a

FIG. 3 b

FIG. 4