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SELF-CONSISTENT QUARK BAGS *)

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ABSTRACT

The properties of strongly interacting spin- $\frac{1}{2}$ fields are considered in the frame of self-consistent quark bags. It is assumed here that a hadron is a bound state of a few (very) heavy particles (free quarks), which can be characterized by the properties of light quasi-fermions (bound quarks) bound in an average self-consistent gluon potential. Actual solutions for fermion fields in interaction with meson fields are described. The algorithm used to generate examples of the self-consistent solutions in three space dimensions is discussed. An attempt is made to justify such an approach as an approximation to the q number, strongly interacting field theory. The present investigations deal only with Abelian fields.

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1. INTRODUCTION AND OVERVIEW

We do not understand the forces that keep the quarks (semi-) permanently bound in hadrons. The known experimental properties of quark bound states -- hadrons -- appear self-contradictory, when confronted with our past experience with nuclear and atomic physics. The basic problem is that experimental information indicates that hadronic properties are well described in a model of weakly interacting (quasi) particles. A possible solution to this dilemma, on which we base our investigations, is a model¹⁾ in which the bound state of a few heavy particles²⁾ is viewed as a collection of quasi-particles of relatively small mass, each moving in an average "shell" potential independently of the others. It is only when we forcefully destroy this hadronic "bag" that the true nature of the underlying very massive fields becomes more apparent -- but with no free quarks escaping from the region of the interaction. In this picture there are two possibilities to view the absence of asymptotically free quark states: either their free mass is still higher than presently available energies $m_q \gtrsim 10 \text{ GeV}$, or their decay width is very high²⁾. In the latter case it is necessary to consider quarks as quantities with integral quantum numbers.

Since the spectroscopic information available today is very detailed, it would seem that any profound attempt to deal with the binding of quarks and hadronic structure requires inclusion of known symmetries [flavour SU(4), colour SU(3), etc.]. In our approach we will consider the above symmetries as being exact and will focus on extending the qualitative understanding of strongly bound fermion states. We will introduce the internal symmetries only when needed to construct hadrons from quarks. Our aim is to describe several mechanisms for strong binding inherent in the strongly interacting field theories.

The basic role in our investigations will be played by the (quark) fermion field Ψ and its transition amplitude Ψ_0 . For example, we have

$$\Psi_0(x) = \langle qq | \Psi(x) | q_0 qq \rangle \quad (1.1)$$

We will consider several different interactions of both massive and massless quark fields. In the absence of an exact method to treat strongly interacting quantum fields we will consider the strongly bound states in the self-consistent bag approximation³⁾, discussed in detail further below. In this approximation all quantum fields are replaced by classical fields with some constraints on the fermion fields that arise in view of the anticommutation properties of the quantum fields. This simple prescription must be amended suitably in some cases, in particular when a vector type interaction of the fermion field is considered. This is a particularly interesting case. If the assertion that the interaction

considered here is representative of the interacting Yang-Mills fields is correct, we are led to the conclusion that the properties of strongly bound states generated by the coloured vector gluons will have some unusual properties not at all expected from classical potential models. Also some other subtle properties stemming from the quantum character of the fields must not be forgotten.

The motivation for transition from q-number to c-number fields in the domain of strong binding resides in two intuitive notions:

- i) The c-number theory arises when the expectation value of the Hamiltonian describing a model is considered in the zero-average momentum frame of trial state $|t\rangle$ formed in the direct product space of gluons and quarks using a localized, coherent gluon state and a quasi-fermion state.
- ii) In the cases of interest the interacting c-number fields generate stable, localized solutions which usually are referred to as *solitons*.

The investigations concerning

- a) the region of validity of the bag approximation, and
- b) the relevance of the c-number solutions in describing strongly bound states in three space dimensions

are as yet inconclusive⁴⁾. It is our basic assumption that the bag approximation is valid whenever

$$M \ll Nm_q. \quad (1.2)$$

Here m_q is the lightest "free" field excitation that can be reached from the c-number bound state of mass M (M includes the mass of the gluon field as well). N is the number of constituents. This condition is self-explanatory -- the bound states are "stiff" with respect to vacuum fluctuations if Eq. (1.2) is satisfied.

As the result of the bag approximation we are left with a c-number theory of interacting fields. The properties of soliton solutions^{5,6)} of interacting fermion fields are interesting in themselves, apart from the point of view presented above. We will consider several general properties of such solutions with the help of virial relations in Section 3. In particular, we will derive a relation between the kinetic energy of the fermions and the potential interaction energy of the gluon field⁷⁾.

In Section 4 we present a brief description of the numerical methods and algorithms⁵⁾ used to generate examples of the c-number solutions in three-dimensional space that are of interest to us. We will discuss the properties of

radially symmetric solutions in Section 5. In particular we consider³⁾ the *simplest* possible system consisting of a fermion field $\bar{\Psi}$ (quark field) interacting with a scalar meson (gluon) field Φ [see Eq. (2.1)].

Our investigations differ at this stage in two points from those of Bardeen et al. and Vinciarelli¹⁾ and Creutz⁸⁾:

- i) We obtain actual solutions for the quark bag by extensive numerical studies in the three-dimensional space rather than in the unphysical one-dimensional space;
- ii) choosing (in a special case) the gluon potential $U(\Phi^2) = -\frac{1}{2} u^2 \Phi^2$, we show that in the physical, three-dimensional space the existence of the quark bag does not critically depend on an inherent non-linearity of the gluon field, but can also be viewed as a consequence of the quark-gluon interaction.

In the limit of the vanishing mass of the scalar gluon field we are led to consider in Section 6 the properties of a self-interacting Dirac field. We view the special form of the self-interaction chosen there as representative of many other possible self-interactions as much as the scalar gluon interaction was. Hopefully the results we obtain with these simple Lagrangians remain valid when more complicated interactions are considered.

Finally, in Section 7 we consider the particular case of a vector interaction. In contrast to the scalar interactions considered previously, the vector interaction can bind a c-number field with arbitrary strength, leading to mixtures between particle and antiparticle, single-particle states. The necessary modifications of the theory, as well as particular solutions of a model, are described.

2. SELF-CONSISTENT QUARK BAGS

As indicated in the above discussion, we take the self-consistent solution of the coupled Dirac-meson fields to represent a bound state of strongly interacting fermions. This physical picture corresponds to the idea of a quark trapped in a self-consistent potential well^{1,3)}. If taken seriously, this remark means that the solutions of the interacting fields contain information about experimentally measurable quantities. A preferred model then can emerge that has the best phenomenological properties. I have found that such a pet theory is that of interacting fermion-scalar gluon fields

$$\mathcal{L}_1 = \bar{\Psi} (\gamma \cdot p - m) \Psi + g \bar{\Psi} \Psi \Phi + \frac{1}{2} \partial_\mu \Phi \partial^\mu \Phi - (U(\Phi) - U(0)). \quad (2.1)$$

In the first part of this section we will make a connection between the classical theory and quantum field theory, using as an example the Lagrangian (2.1), where we sometimes also write for the gluon potential U the simplest term

$$U(\Phi^2) \rightarrow \frac{1}{2} \mu^2 \Phi^2. \quad (2.2)$$

In order to treat the strongly interacting ($g^2 \gg 1$) limit of the quantum field theory, induced by the Lagrangian Eq. (2.1), let us now consider quasi-particles. These are entities carrying, typically, the same internal quantum numbers, for example spin, as the particles, but subject to a quite different interaction from the free modes. The advantage that is gained consists in the modification of the interaction; in particular, quasi-particles interact weakly and we may use the well-proved perturbative methods to treat the residual interaction. A quasi-fermion state with quantum numbers $|p\rangle$ can be obtained from the quantum operator $\hat{\Psi}(\vec{x}, 0)$ by projecting with the associated amplitude $\psi_p(\vec{x})$ (a circumflex over a quantity indicates its q-number character)

$$|p\rangle = \int \hat{\Psi}^\dagger(\vec{x}, 0) \psi_p(\vec{x}) |0\rangle d^3x. \quad (2.3)$$

Here $|0\rangle$ denotes the ground state. The equations of motion for the amplitude ψ_p will be determined *a posteriori*. It is advantageous to introduce the quasi-particle (index p) and antiparticle (index n) operators in the usual manner:

$$\begin{aligned} \hat{\beta}_p^\dagger &= \int \hat{\Psi}^\dagger(\vec{x}, 0) \psi_p(\vec{x}) d^3x, \\ \hat{\beta}_n^\dagger &= \int \psi_n^\dagger(\vec{x}) \hat{\Psi}(\vec{x}, 0) d^3x. \end{aligned} \quad (2.4)$$

The completeness of the basis $\{\psi_p, \psi_n\}$ is reflected in

$$\delta^3(\vec{x} - \vec{x}') \delta_{\alpha\beta} = \sum_n \psi_n^\alpha(\vec{x}) \psi_n^{\beta\dagger}(\vec{x}') + \sum_p \psi_p^\alpha(\vec{x}) \psi_p^{\beta\dagger}(\vec{x}'). \quad (2.5)$$

Here (α, β) are the internal structure indices, for example spinor indices, while the sums also stand for possible integrals over continuous parts of the spectra. We now see that the internal structure of the quasi-particles is that of the particles and is a consequence of the (anti-)commutation relations of the fields $\hat{\Psi}$. Any quasi-particle state may be constructed from a product of the quasi-fermion operators acting on the ground state. Thus, for example, the three-quasi-fermion baryon state B is

$$|B\rangle = \sum_{ijk} b_{ijk} \hat{\beta}_i^\dagger \hat{\beta}_j^\dagger \hat{\beta}_k^\dagger e^{-i \int \hat{\pi}_\Phi(\vec{x}, 0) \varphi_c(\vec{x}) d^3x} |0\rangle. \quad (2.6)$$

with suitable coefficients b_{ijk} that generate the symmetry properties of the state. The remaining factor in Eq. (2.6) will be explained in a moment.

The "bag" field for the quarks is generated by the quasi-fermionic source

$$S = \langle B | \hat{\bar{\Psi}} \hat{\Psi} | B \rangle. \quad (2.7)$$

The presence of a strong source, the strength being determined to a large extent by g^2 , generates an average gluon "shell" potential φ_c

$$\varphi_c = \langle B | \hat{\Phi} | B \rangle. \quad (2.8)$$

To achieve this property, we have introduced the exponential in Eq. (2.6), which makes $|B\rangle$ a coherent state for the meson field, where

$$\pi_{\Phi} = \frac{\delta \mathcal{L}}{\delta \Phi} \quad (2.9)$$

and

$$[\hat{\Phi}(\vec{x}, 0), \pi_{\Phi}(\vec{x}', 0)] = i\delta^3(\vec{x} - \vec{x}'). \quad (2.10)$$

Equipped with the trial state $|B\rangle$ (and similarly a meson state $|M\rangle$) we can minimize the invariant mass that is associated with them, as a function of the unknown functions φ_c and ψ_p . The Lorentz invariant momentum P_{μ} of our trial state $|B\rangle$ is given by

$$P_{\mu} = \int \langle B | \hat{T}_{\mu}^{\nu}(x) | B \rangle d^4x_{\nu}, \quad (2.11)$$

where T_{μ}^{ν} is the energy momentum tensor following from the Lagrangian $\mathcal{L}$ under consideration. The hypersurface in the four-dimensional space is denoted by d^4x_{ν} . Note that we have chosen, in the above discussion,

$$d^4x_{\nu} = n_{\nu} d^3x, \quad (2.12)$$

with $n_{\nu} = (1, 0, 0, 0)$ associated with the laboratory frame of reference. We have written the fully covariant form of P_{μ} in Eq. (2.11) to show the Lorentz covariance of the bag approach. Furthermore, within the ansatz for $|B\rangle$ we have chosen, we search for solutions with vanishing momentum $\vec{P}$, i.e.

$$\int \langle B | \hat{T}_0^i | B \rangle d^3x = 0 \quad (2.13)$$

If our solution satisfies condition (2.13), it is called a baryon in zero average momentum frame. By construction of our trial state we obtain the property

$$\langle B | \hat{\pi}_{\Phi}(x) | B \rangle = 0, \quad (2.14)$$

which guarantees the vanishing of the average gluon momentum. Similarly, it is easy to show that the sum of quasi-fermion momenta vanishes. This is in agreement with a picture in which quarks follow closely the motion of the gluon bag and vice versa.

In the zero average momentum frame we therefore have

$$P_0 = M = \int d^3x \langle B | \hat{T}_0^0 | B \rangle = \int d^3x \mathcal{M}_B, \quad (2.15)$$

where for $\mathcal{L}_1$ defined by Eq. (2.1) we find, using the exact symmetry of all quarks in the bag:

$$\begin{aligned} \mathcal{M}_B = N \psi_i^\dagger (\vec{\alpha} \cdot \vec{p} + \beta(m_q - g\varphi_c) \psi_i \\ + \frac{1}{2} |\vec{\nabla} \varphi_c|^2 + U(\varphi_c) - U_0, \end{aligned} \quad (2.16)$$

where $N = 3$ is the number of (valence) quarks.

In principle, the expectation value (2.15) should include the zero-point energy of the fermion field:

$$E_0 = -\frac{1}{2} \left(\sum_p \epsilon_p[\varphi_c] - \sum_n \epsilon_n[\varphi_c] \right). \quad (2.17)$$

Here ϵ_p and ϵ_n are the positive and negative frequency solutions of the Dirac equation: E_0 is a (divergent) implicit function of φ_c . It is possible to include in the functional $U(\varphi_c)$ the necessary counterterms -- the renormalized zero-point energy then is still a function of φ_c and should, in principle, be included in Eq. (2.16). However, such a contribution can be absorbed in the function $U(\varphi_c)$. Since we have, in principle, no idea about the form of U , we need not consider this point in further detail for the moment.

A result similar to Eq. (2.16) is also obtained for the mesons:

$$\begin{aligned} \mathcal{M}_M = \langle M | \mathcal{H} | M \rangle = \psi_i^\dagger [\vec{\alpha} \cdot \vec{p} + \beta(m_q - g\varphi_c)] \psi_i \\ - (i \rightleftharpoons) + \frac{1}{2} |\vec{\nabla} \varphi_c|^2 + U(\varphi_c) - U_0. \end{aligned} \quad (2.18)$$

The usual $(-)$ sign for antiquasi-fermion amplitude ψ_j in Eq. (2.18) is necessary to change the sign of the negative frequency mode associated with the solution of its Dirac equation. For a scalar potential φ_c the spectrum of the Dirac equation is symmetric between particles and antiparticles, that is

$$\epsilon_p = -\epsilon_{ap} \quad (2.19a)$$

and the wave functions transform like

$$\psi_p = \gamma_5 \psi_{ap}^* \quad (2.19b)$$

Therefore the invariant mass for all hadrons can be written as a function of the number of bound quarks and antiquarks in the bag only:

$$M_N = \int d^3x \mathcal{M}_N = \int d^3x \left[N \{ \psi_0^\dagger (\vec{\alpha} \cdot \vec{p} + \beta(m_q - g\varphi_c)) \psi_0 \} + \frac{1}{2} (\vec{\nabla} \varphi_c)^2 + (U(\varphi_c) - U_0) \right] = M_{N,f} + M_g. \quad (2.20)$$

The index "0" on the spinor wave function indicates that we must choose the lowest energy particle solution (which in the case of scalar interaction is the lowest positive energy eigenvalue solution of the Dirac equation) in order to obtain the smallest value of M_N . The equations of motion satisfied by ψ_0 and φ_c follow now from the variational condition:

$$\delta(M_N - \epsilon_0 N \int d^3x \psi_0^\dagger \psi_0) = 0, \quad (2.21)$$

where we have introduced the usual Lagrange multiplier in order to normalize the solutions:

$$\int \psi_0^\dagger \psi_0 d^3x = 1. \quad (2.22)$$

We find from Eq. (2.21) the self-consistent equations of motion

$$[\vec{\alpha} \cdot \vec{p} + \beta(m_q - g\varphi_c)] \psi_0 = \epsilon_0 \psi_0, \quad (2.23)$$

$$-\Delta g\varphi_c + \frac{\partial U}{\partial \varphi_c} = g^2 N \bar{\psi}_0 \psi_0. \quad (2.24)$$

N is equal to three for baryons and two for mesons. When solving the above equations, we consider all dimensional quantities in units of m_q . If we obtain the solutions as a function of g^2 , then it is possible to consider the case $N = 1$ only, and the relevant parameter is

$$g'^2 = g^2 N. \quad (2.25)$$

We note that M_N can be written as

$$M_N = N \left(\epsilon_0 [g'^2] + \frac{1}{g'^2} \int \left(\frac{1}{2} (\vec{\nabla} g\varphi_c)^2 + \frac{1}{2} \mu^2 (g\varphi_c)^2 \right) d^3x \right). \quad (2.26)$$

Thus M_N/N is only a function of g'^2 . We thus have seen that self-consistent solutions of classical interacting fields are closely related to the quark-bag solutions of N quarks. In the quark-bag approach the main approximation has been the following: only the diagonal part of the Hamiltonian matrix of the interacting fields in a trial state has been considered. The off-diagonal parts remain to be diagonalized and the renormalization program carried out.

It is instructive to notice that the choice of φ_c that is self-consistent in the sense of Eqs. (2.23) and (2.24) makes the remaining interaction off-diagonal.

The full Hamiltonian of q-number fields is (up to the necessary symmetrization)

$$\hat{H} = \hat{H}_D + \hat{H}_\phi + \hat{H}_I \quad (2.27)$$

with

$$\hat{H}_D = \int d^3x \hat{\Psi}^\dagger (\vec{\alpha} \cdot \vec{p} + \beta m) \hat{\Psi} \quad (2.28a)$$

$$\hat{H}_\phi = \int d^3x \frac{1}{2} (\hat{\pi}^2 + (\vec{\nabla} \hat{\phi})^2 + (U(\hat{\phi}) - U_0)^2) \quad (2.28b)$$

$$\hat{H}_I = \int d^3x (-g \hat{\phi}) \hat{\Psi} \hat{\Psi} . \quad (2.28c)$$

Now let us add and subtract a term in $\hat{H}$

$$\hat{H} = (\hat{H}_D + \hat{H}_{ex}) + \hat{H}_\phi + (\hat{H}_I - \hat{H}_{ex}), \quad (2.29)$$

where

$$\hat{H}_{ex} = \int d^3x (-g \psi_c) \hat{\Psi} \hat{\Psi} . \quad (2.30)$$

The approach presented above corresponds to such a choice of a trial state that the *diagonal* matrix elements of the new interaction

$$\hat{H}'_I = \hat{H}_I - \hat{H}_{ex} \quad (2.31)$$

vanish, while the Hamiltonian

$$\hat{H}_0 = (\hat{H}_D + \hat{H}_{ex}) + \hat{H}_\phi \quad (2.32)$$

is diagonal.

An argument in favour of the bag approximation, applicable only in the limit of strong coupling, involves the large separation in energy between the mass of the trial state, M_N , and the free fermion mass, m . I believe that the second-order contributions to M from the off-diagonal matrix elements will be of the order of

$$\begin{aligned} \delta M &\lesssim \left[\int \psi_i^\dagger g \psi_c \psi_i d^3x \right]^2 / [Nm_q - M_N] \\ &\approx \left(\frac{M_N}{Nm_q} \right) M_N . \end{aligned} \quad (2.33)$$

If this is the case, then the ratio

$$\delta = M/m_q \quad (2.34)$$

is the small parameter that determines the validity of the bag approximation.

Finally, we consider the adiabaticity of the fermionic motion. When a quasi-fermion is excited inside the bag, we have to know if the bag should be adjusted instantly, that is the excited state considered as the source on the right-hand

side of Eq. (2.23). I will now argue that the opposite should be expected: the bag does not have time to adjust to the internal excitation.

While the transition time Δt_{ij} from the quasiparticle state i to the quasiparticle state j is

$$\Delta t_{ij} \approx \pi / (\epsilon_j - \epsilon_i)^{-1}, \quad (2.35)$$

the simultaneous adjustment of the gluon bag involves the time

$$\Delta t_\varphi \approx \pi / (M_j - M_i)^{-1} > \Delta t_{ij}. \quad (2.36)$$

The latter relation follows from the observation that the adjustment of the bag improves the parameters of the trial state and so requires

$$M_j < N\epsilon_j + (M_i - N\epsilon_i), \quad (2.37)$$

[except in the (undiscovered) case that the lowest particle eigenenergy ϵ is not associated with the lowest mass M]. Thus the bag approximation implies presumably a Born-Oppenheimer type of adiabaticity -- the motion of the fermions is faster than the rate of change of the self-consistent potential.

3. THE VIRIAL APPROACH TO CLASSICAL FIELD EQUATIONS

Before actually discussing the solutions of particular models, let us see how much we can learn about the general properties of self-consistent solutions from the equations of motion⁷⁾. Basic to our considerations will be a (virial) relation describing the kinetic energy of the fermion field ψ in some prescribed potential $V(x)$, where V is a matrix in the spinor space and can, in principle, consist of all possible couplings (S, V, P, A, T). Let H_D be the Dirac operator

$$H_D = \vec{\alpha} \cdot \vec{p} + \beta m + V. \quad (3.1)$$

Then we have

$$[\vec{x} \cdot \vec{p}, H_D] = i\vec{\alpha} \cdot \vec{p} - i\vec{x} \cdot (\vec{\nabla} V). \quad (3.2)$$

Taking the expectation value of Eq. (3.1) between localized eigenfunctions ψ_i of H_D , we find the *virial* equation for the Dirac field:

$$\int d^3x \psi_i^\dagger \vec{\alpha} \cdot \vec{p} \psi_i = \int d^3x \psi_i^\dagger \vec{x} \cdot (\vec{\nabla} V) \psi_i. \quad (3.3)$$

This is a very useful relation which is valid only when ψ_i is an eigenfunction of H_D . We record that all of the ψ_i considered above are localized discrete eigenstates in configuration space. We now apply Eq. (3.3) to the Lagrangian field theory describing a scalar meson field in interaction with the Dirac field: the Lagrangian has been given in Eq. (2.1) and the equations of motion by Eqs. (2.23)

and (2.24) with $N = 1$ and $\varphi_c = 0$. Then Eq. (3.3) for the kinetic energy of the Dirac field becomes

$$\int d^3x \psi_i^\dagger \vec{\alpha} \cdot \vec{p} \psi_i = \int d^3x (-\vec{x} \cdot \vec{\nabla} \varphi_c) g \bar{\psi}_i \psi_i. \quad (3.4)$$

We find after some manipulations that

$$\int d^n x \psi_i^\dagger \vec{\alpha} \cdot \vec{p} \psi_i = \int d^n x \left[\frac{n-2}{2} |\vec{\nabla} \varphi_c|^2 + n(U - U_0) \right] + ST, \quad (3.5)$$

with a vanishing surface term for localized solutions, with

$$n \equiv \vec{\nabla} \cdot \vec{x} \quad (3.6)$$

as the dimensionality of the space. Normally $n = 3$, but to dramatize the uniqueness of the three-dimensional, physical space, let us consider the number of dimensions as a parameter.

The Hamiltonian associated with the Lagrangian (2.1) is, with a time-independent φ_c :

$$H^S = \int d^n x \left[\psi_0^\dagger (\vec{\alpha} \cdot \vec{p} + \beta m) \psi_0 - g \varphi_c \bar{\psi}_0 \psi_0 + \frac{1}{2} |\vec{\nabla} \varphi_c|^2 + U(\varphi_c) - U_0 \right]. \quad (3.7)$$

We may use the above result, Eq. (3.5), together with the equations of motion (2.23) and (2.24) to obtain the following expression for the Hamiltonian:

$$E_{\text{eff}}^S = \int d^n x \left\{ \frac{n-3}{2} |\vec{\nabla} \varphi_c|^2 + (n+1) [U(\varphi_c) - U_0] + \left(\frac{m}{g} - \varphi_c \right) \frac{\partial U}{\partial \varphi_c} \right\}. \quad (3.8)$$

As is well known, the above equation (3.8) cannot be used as a basis of a variational principle. For known φ_c , which minimizes Eq. (3.7), it gives the proper value of H^S . Therefore we may view E_{eff}^S as an expression defining the total energy.

A striking feature of Eq. (3.8) is the fact that for $n < 3$ (n is the number of space dimensions) the effective kinetic energy of the scalar field becomes negative definite. It just vanishes for $n = 3$. Thus the energy content of E_{eff}^S in a normal number of space dimensions is only implicitly dependent on the derivatives of the field φ_c . The negative definite kinetic energy for $n < 3$ signals a possible instability of the Hamiltonian; the energy of the solution could be reduced by a small non-continuous variation of φ_c .

Another notable feature is that any φ_c^4 proportional term in U cancels out in three-dimensional space (in n -dimensional space φ_c^{n+1} cancels). This is an important feature, since φ_c^4 is commonly held responsible for the stability of a theory with spontaneously broken symmetry^{1,4)}. In the case of a one-dimensional

world ($n = 1$), where the φ_c^2 contribution vanishes, this is obviously a necessary term to stabilize the theory. It would therefore seem that a φ^6 plays the role of the ψ^4 term in three space dimensions, as compared with one-dimensional models.

Very often it is the field

$$\chi = m/g - \varphi_c \quad (3.9)$$

that is introduced, instead of φ_c . The advantage is that there is no explicit mass term of the fermion field in the Hamiltonian. Our result may be easily adapted and we find for the Hamiltonian

$$H^{S'} = \int d^n x [\psi_k \vec{\alpha} \cdot \vec{p} \psi_k + g \chi \bar{\psi}_k \psi_k + V(\chi) + \frac{1}{2} (\vec{\nabla} \chi)^2], \quad (3.10)$$

the result

$$E_{\text{eff}}^{S'} = \int d^n x \left[\frac{n-3}{2} (\vec{\nabla} \chi)^2 + (n+1) V(\chi) - \chi \frac{\partial V}{\partial \chi} \right], \quad (3.11)$$

where

$$V(\chi) = U(\varphi_c) - U_0. \quad (3.12)$$

The field χ may be considered the effective mass of the Dirac field ψ . The mass of the *free* Dirac field is now

$$m = g \chi \quad (x \rightarrow \infty), \quad (3.13)$$

which is dependent on the specific form of $V(\chi)$. Further, we note that since the integrability of the Hamiltonian H^S requires $U(\varphi_c = 0) = U_0$, we find $V(m/g) = U_0$.

In the case of the σ model the function V is considered to be

$$V = H (\chi^2 - f^2)^2. \quad (3.14)$$

In three-dimensional space we obtain with this form of V :

$$E_{\text{eff}}^{(3)} = \int d^3 x \ 4 H f^2 (f^2 - \chi^2). \quad (3.15)$$

If χ_s is allowed to vary only between $+f$ and $-f$ then the last expression is explicitly positive definite. Thus the form of the solution is essential for the determination of a lower bound on E_{eff} . However, an upper limit for E_{eff} may be obtained setting $\chi^2 = 0$ in Eq. (3.15) and taking the volume of the solution for the integral:

$$E_{\text{eff}}^{(3)} \lesssim 4\pi/3 R^3 4 H f^4. \quad (3.16)$$

We now consider the case of the self-interacting Dirac field, where H assumes the form, for scalar-type self-interaction⁶⁾

$$H^P = \int d^n x \left[\psi_0^\dagger (\vec{\alpha} \cdot \vec{p} + \beta m_q) \psi_0 - \frac{G}{2} (\bar{\psi}_0 \psi_0)^2 \right]. \quad (3.17)$$

In view of the equations of motion,

$$[\vec{\alpha} \cdot \vec{p} + \beta (m_q - G \bar{\psi}_0 \psi_0)] \psi_0 = \epsilon_0 \psi_0, \quad (3.18)$$

the virial relation reads as

$$T_0 = \int d^n x \psi_0^\dagger \vec{\alpha} \cdot \vec{p} \psi_0 = -\frac{G}{2} \int d^n x [(\vec{x} \cdot \vec{\nabla}) (\bar{\psi}_0 \psi_0)^2]. \quad (3.19)$$

Upon partial integration of the right-hand side, we obtain, up to a vanishing surface term,

$$T_0 = -n V_0, \quad (3.20)$$

where

$$V_0 = -\int d^n x \frac{G}{2} (\bar{\psi}_0 \psi_0)^2. \quad (3.21)$$

The energy for the self-interacting Dirac field can be written now as

$$E_{eff}^P = (n-1) (-V_0) + m S_0. \quad (3.22)$$

S is the scalar integral

$$S_0 = \int d^n x (\bar{\psi}_0 \psi_0). \quad (3.23)$$

This result, Eq. (3.22), corrects a superficial impression that H^P , Eq. (3.17), is unbound, since it has the structure $x^2 - x^4$. This means that the kinetic energy of the Dirac field more than offsets the attractive self-interaction V_0 . Since $-V_0$ is always positive, we find that for all dimensions the positivity of the solution depends on the sign of the scalar integral. Further, we note that the eigenfrequency can be written as

$$\epsilon_0 = (n-2) (-V_0) + m S_0. \quad (3.24)$$

A similar derivation can be carried out in the case of a Lorentz vector field A_μ of mass μ_v in interaction with the Dirac field. For the Lagrangian

$$\begin{aligned} \mathcal{L}_A = & \frac{1}{2} \partial_\mu A^\nu \partial^\mu A_\nu + \frac{1}{2} \mu_v^2 A_\mu A^\mu \\ & - g_v A_\mu \bar{\psi} \gamma^\mu \psi + \psi (\gamma \cdot p - m) \psi, \end{aligned} \quad (3.25)$$

we find

$$E_{eff}^A = -\int d^n x \left[\frac{n-3}{2} (\nabla \cdot A_\mu)^2 + \frac{n-1}{2} \mu_v^2 A^2 - m \bar{\psi}_0 \psi_0 \right], \quad (3.26)$$

which is negative definite, considering the longitudinal component A_0 only. We note that the longitudinal part of the vector-type interaction is repulsive in the particle-particle channel and attractive in the particle-antiparticle channel.

The situation changes when the sign in the part of the Lagrangian corresponding to the free vector field is changed⁹):

$$\mathcal{L}_A = \frac{1}{2} \partial_\mu A^\nu \partial^\mu A_\nu - \frac{1}{2} \mu_\nu^2 A^2 - g_\nu A_\mu \bar{\Psi} \gamma^\mu \Psi + \bar{\Psi} (\gamma \cdot p - m) \Psi. \quad (3.27)$$

We then find

$$E_{\text{eff}}^A = \int d^n x \left[\frac{n-3}{2} (\nabla A_\mu)^2 + \frac{n-1}{2} \mu_\nu^2 A^2 + m \bar{\Psi}_k \Psi_k \right], \quad (3.28)$$

which is positive definite, constrained to the longitudinal part of A_μ , provided that the scalar integral $\int d^n x \bar{\Psi}_0 \Psi_0 > 0$. The above-described change in sign accomplishes at the same time a change in the "polarity" of the vector interaction -- the longitudinal part is now attractive in the particle-particle channel, while the particle-antiparticle channel becomes repulsive. The perturbative quantum field theory, if based on Eq. (3.27) would suffer from the well-known difficulties associated with the possible need for negative metric particles (ghosts) to guarantee a spectrum bounded below. Therefore, such modifications are usually not considered seriously. Such an example considered in the frame of classical field theory can serve as an educational example in order to gain experience with "attractive" vector-type fields encountered in non-Abelian meson theories (quantum chromodynamics). Returning for a moment to Eq. (3.26), it should be mentioned that it is in principle possible to find a solution in which the space-vector part of A_μ dominates -- thus allowing a stable solution, even with the conventional choice for the sign of the vector field action. We will return to the discussion of the vector field further below.

In principle, the solutions of the coupled non-linear equations of motion need not be actual stable minima of the action -- they allow the first variation of the action to vanish, but no information is available about the second variations. Let us consider here again, as an example, the case of an externally prescribed potential V_{ex} . Then the Hamiltonian is

$$H = \int d^3 x \Psi^\dagger (\vec{\alpha} \cdot \vec{p} + \beta m + V_{\text{ex}}) \Psi \quad (3.29)$$

and the equation of motion follows from equating to zero the first variation of $(H - \epsilon N)$ with respect to ψ . Here N is the norm of the field ψ , while ϵ is the Lagrange multiplier that ensures the normalizability of the solution. We find, as usual, the eigenvalue equation

$$(\vec{\alpha} \cdot \vec{p} + \beta m + V_{\text{ex}}) \psi_k = \epsilon_k \psi_k. \quad (3.30)$$

As is well known, the set of eigensolutions $\{\psi_k\}$ is complete. Suppose we take a trial function

$$\psi_t = \sum_k a_k^t \psi_k \quad (3.31)$$

in an attempt to minimize the Hamiltonian, Eq. (3.29). Using the expansion (3.31) with the complete basis set $\{\psi_k\}$ generated by Eq. (3.30), we find

$$(H - \epsilon_t N) = \sum_k |a_k^t|^2 (\epsilon_k - \epsilon_t). \quad (3.32)$$

The spectrum of the Dirac equation is unbounded below as is the expression (3.32).

From this example we recognize the need to constrain the number of allowed degrees of freedom of the Dirac field. We must exclude from our considerations the possibility of a transition that a classical Dirac particle can undergo into a (classically) unoccupied state of arbitrary large negative frequency. In quantum field theory such a state corresponds to an antiparticle state of positive energy. We have seen in Section 2 that this constraint arises naturally in the self-consistent bag approximation.

Thus the solutions under investigation here must be obtained by a method that allows *a priori* rejection of all unwanted modes of the Dirac equation. In a particular application, this means that the Dirac equation must always be solved exactly for some prescribed potential and an eigenstate ψ_k obtained.

In the case of the interacting Dirac-scalar fields described in Section 2, we actually minimize the action constrained by this consideration, provided that we solve Eq. (2.23) exactly for a given φ_c . Then we may consider the Hamiltonian to be given by

$$H^S = \epsilon_0[\varphi_c] + \int d^3x \left[\frac{1}{2}(\nabla\varphi_c)^2 + U(\varphi_c) - U_0 \right], \quad (3.33)$$

where ϵ_0 is given implicitly by the solution of the eigenvalue problem:

$$[\vec{\alpha} \cdot \vec{p} + \beta(m - g\varphi_c)] \psi_0 = \epsilon_0 \psi_0. \quad (3.34)$$

The above equation must always be solved exactly in order to obtain the Dirac eigenvalue ϵ_0 as a functional depending on φ_c . Using the apparent relation

$$\frac{\delta \epsilon_0}{\delta \varphi_c(x)} = -g \bar{\psi}_0(x) \psi_0(x), \quad (3.35)$$

we recover the conventional equation of motion for the field $\varphi_c(x)$ that we now write in the form

$$-\Delta\varphi_c + \frac{\partial U}{\partial \varphi_c} = - \frac{\delta \epsilon_0}{\delta \varphi_c(x)}. \quad (3.36)$$

The question which we consider next is: are the solutions, constrained to the lowest positive frequency (particle) solutions of the Dirac field, stable?

A method employed in the study of classical field theories by Goldstone and Jackiw¹⁰⁾ is the linear transformation

$$x' = \alpha x \quad (3.37)$$

The virial relation (3.4) is then obtained from

$$\left. \frac{\partial}{\partial \alpha} [H(\alpha x) / N(\alpha x)] \right|_{\alpha=1} = 0, \quad (3.38)$$

where N is the norm of the Dirac field, while the stability of the solutions should be tested by computing:

$$T = \left. \frac{\partial^2}{\partial \alpha^2} [H(\alpha x) / N(\alpha x)] \right|_{\alpha=1}. \quad (3.39)$$

Naively, T should be positive definite in order to have a minimum against the elongations, Eq. (3.37). However, should we obtain a negative value for T , it may in our case only indicate that a number of negative frequency modes have been included in the trial wave function. With a complete set ψ_k , cf. Eq. (3.30), we may write

$$\psi_0(\alpha x) = \sum_k b_k(\alpha) \psi_k(x) \quad (3.40)$$

and therefore we obtain for the Hamiltonian H^S in Eq. (3.33):

$$H^S[\alpha x] = \alpha^3 \sum_k |b_k(\alpha)|^2 \epsilon_k[\psi(\alpha x)] + E_\varphi(\alpha x), \quad (3.41)$$

where E_φ is the energy content of the scalar field. We record that the second variation of Eq. (3.41) includes

$$\left. \frac{\partial^2}{\partial \alpha^2} H^S[\alpha x] \right|_{\alpha=1} = \sum_k \left| \left. \frac{\partial b_k}{\partial \alpha} \right|_{\alpha=1} \right|^2 \epsilon_k[\psi(x)] + \dots \quad (3.42)$$

Since the spectrum is not positive definite, there is no way to determine the sign of the expression (3.42). With

$$\left. \frac{\partial b_k}{\partial \alpha} \right|_{\alpha=1} = \int \psi_k^\dagger \vec{x} \cdot \vec{\nabla} \psi_0 d^3x, \quad (3.43)$$

we recognize that the term (3.42) does not vanish.

An explicit example of a stable theory with non-positive T is found considering the self-interacting Dirac field. We find for the Hamiltonian (3.17):

$$\begin{aligned} H^P(\alpha x) / N(\alpha x) = & \int d^3x \{ \psi^\dagger [\frac{1}{2} \vec{\alpha} \cdot \vec{p} + \beta m] \psi \\ & - \frac{1}{2\alpha^3} (\bar{\psi} \psi)^2 \} / \int d^3x \psi^\dagger \psi, \end{aligned} \quad (3.44)$$

wherefrom we find for the second derivative

$$\frac{\partial^2}{\partial a^2} (H(ax)/N(ax)) \Big|_{a=1} = - \frac{6G}{2} \int d^3x (\bar{\psi}\psi)^2, \quad (3.45)$$

a result that is explicitly negative definite. This would make us worry about the stability of the self-interacting field, had we not convinced ourselves that such a result can arise as a consequence of the indefinite character of the Dirac equation spectrum and has nothing to do with the stability of the solutions.

4. NUMERICAL METHODS

Our aim in this section is to illustrate the numerical methods involved in minimization of the mass (or Hamiltonian) M_N in Eq. (2.20) as a function of the fields ψ_0 and φ_c , with the constraint that ψ_0 be the lowest frequency eigenmode of the quasi-fermion field.

From the considerations of the last section, we recognize the need to constrain the number of allowed degrees of freedom of the Dirac field, as arose naturally in the bag approximation. We must exclude from our considerations the possibility of a transition that a classical Dirac particle can undergo into a (classically) empty state of arbitrary large negative frequency. In a particular application this means that the Dirac equation must always be solved exactly for some prescribed potential, and an eigenstate ψ_0 obtained. We must thus consider the Hamiltonian to be given implicitly by Eq. (3.33), where ε_0 is given implicitly by the solution of Eq. (2.23) with the eigenvalue condition

$$\lim_{|\vec{x}| \rightarrow \infty} \psi_0(\vec{x}) = 0. \quad (4.1)$$

Any trial value of φ_c in Eq. (3.33) will give an upper limit for H^S .

But how do we actually obtain a self-consistent solution for φ_c with ψ_0 constrained as described above to the lowest particle solution? It is very inviting to proceed with an iterative algorithm in which, for a given spherical $\varphi_{c,n-1}$ and fixed g and μ/m , a solution ψ_0 of Eq. (2.23) is obtained as described for spherical potentials further below. From Eq. (2.24) we then determine a new function $\varphi_{c,n}$. In practical calculations it turns out that the radius of convergence of such an iteration decreases rapidly as a function of g^2 , making it virtually impossible to obtain a solution for $g^2 \geq 2$, a region in which we are mainly interested.

I have developed⁵⁾ an algorithm that has so far shown an unrestricted radius of convergence when applied to equations of the type (2.23) and (2.24). Let us introduce for the purpose of clarity two additional functions

$$\phi_c = g \psi_c \quad (4.2)$$

$$f = \psi_c / g \quad (4.3)$$

Equations (2.23) and (2.24) can now be written in the generalized and self-explanatory fashion

$$D[\phi] \psi = 0, \quad (4.4)$$

$$K f = g_s[\phi], \quad (4.5)$$

$$\phi_c = g^2 f. \quad (4.6)$$

Here D and K are linear operators, while the source ρ_s is $\bar{\psi}_0 \psi_0$. Next we note that, using an arbitrary linear functional L , we can write Eq. (4.6) in the form

$$g^2 = L \phi / L f. \quad (4.7)$$

We will return to the discussion of the actual form of L further below.

Our algorithm is: given a function ϕ_{n-1} , the scalar density is found solving Eq. (4.4). Then we obtain from Eq. (4.5)

$$f_n = K^{-1} g_s[\phi_{n-1}] \quad (4.8)$$

and Eq. (4.7) gives us in turn

$$g_n^2 = L \phi_{n-1} / L f_n. \quad (4.9)$$

The algorithm loop is closed by the use of Eq. (4.6)

$$\phi_n = g_n^2 f_n. \quad (4.10)$$

Note that we have iterated the coupling constant g^2 . For a given starting value ϕ_0 a convergent point

$$g^2 = \lim_{n \rightarrow \infty} g_n^2 \quad (4.11)$$

may be found that is dependent on the actual form of the initial function ϕ_0 . I have found empirically that

$$\phi_c = \lim_{n \rightarrow \infty} \phi_n \quad (4.12)$$

is uniquely determined by g^2 and that there is only one g^2 for each given ϕ_0 in the case of Eqs. (2.23), (2.24), and other similar theories discussed further below.

Now, as to the choice of the operator L , the difference between ϕ_{n-1} and $g_n^2 f_n = \phi_n$ needs to be small, in the sense that we are searching for the minimum of the integral

$$I_n = \int w(r) [\phi_{n-1} - g_n^2 f_n]^2 dr \quad (4.13)$$

as a function of g_n^2 . Here $w(r)$ is a weighting function that may be dependent on ϕ_{n-1} . This prescription gives us

$$L\phi = \int w(r) \phi dr. \quad (4.14)$$

The choice $w(r) = 1$ is usually sufficient to have a convergent algorithm. However, $w(r) = \rho_g[\phi_{n-1}]$ gives a quicker convergence in most cases. I would like to conjecture that the algorithm described above should be convergent in most similar cases of physical interest, provided that a suitable choice for the operator L has been made.

Although Eqs. (2.23) and (2.24) look very simple, it is extremely difficult to obtain a self-consistent solution. In particular, we have been able only to analyse in detail the spherically symmetric case. With the usual ansatz for the ground-state wave function

$$\psi_0(x) = \frac{1}{r(4\pi)^{1/2}} \begin{pmatrix} u \xi_s \\ i \frac{\vec{\sigma} \cdot \vec{x}}{|\mathbf{x}|} v \xi_s \end{pmatrix}, \quad (4.15)$$

where ξ_s are the usual spin functions, we find the *radial* equations following from Eqs. (2.23) and (2.24)

$$u' - r^{-1}u - (m - g\varphi_c + \epsilon_0)v = 0 \quad (4.16)$$

$$-v' - r^{-1}v + (m - g\varphi_c - \epsilon_0)u = 0 \quad (4.17)$$

$$-r^{-1}(rg\varphi_c)'' + \mu^2 g\varphi_c = Ng^2(u^2 - v^2)r^{-2}. \quad (4.18)$$

Both spin states have the same equations of motion. Thus not only SU(3) (flavour) but also SU(6) (flavour + spin) are exact symmetries at this point. Furthermore, since N enters as a multiplicative factor, the solution is only a functional of (Ng^2) , as noted in Eq. (2.26). A solution of Eqs. (4.16) to (4.18) will be obtained by an iterative procedure.

5. PROPERTIES OF BOUND STATES OF SCALAR SELF-CONSISTENT BAGS

Properties of bound states obtained within several models will be described here. In particular, we are interested in the geometric properties of the quasi-fermion amplitudes ψ_0 and mass densities $\mathcal{M}$. Furthermore, we consider also g^2 and μ/m systematics of the observable matrix elements of ψ_0 , such as the charge radius, the magnetic moment of the proton, and the axial coupling constant of the semi-leptonic charged current.

It is my feeling that although the average properties (involving matrix elements) of our variational solutions, belonging to some specific models, may be closer to reality than should be expected, a more detailed description is required for comparison of the theoretical and experimental form factors. In particular, we should recall that the amplitudes ψ_0 are, strictly speaking, transition matrix elements between N and $N \pm 1$ fermion states, and not probability amplitudes, as in the case of weakly interacting fields. Also, we have considered here only "valence" quarks -- the sea quarks do also contribute significantly, in particular to the inelastic form factors.

In Fig. 1 we show some of the typical solutions associated with the Lagrangian (2.1) of the interacting fermion-scalar gluon fields³⁾ for $U(\phi^2) = \frac{1}{2}\mu^2\phi^2$ and $N = 3$. The self-consistent quasi-particle (bound quark) mass

$$m^* = m_q - g\varphi_c \quad (5.1)$$

is plotted for several values of the gluon mass μ/m_q and the coupling constant g^2 as a function of $r = |\vec{x}|$. For $\mu = 0.04m_q$ and $\mu = 0.02m_q$ we also show the vector

$$\varrho_v = \psi_0^\dagger \psi_0 \quad (5.2)$$

and scalar

$$\varrho_s = \bar{\psi}_0 \psi_0 \quad (5.3)$$

densities. We note that, in Fig. 1, ρ_s is enhanced by a factor of 10 in relation to ρ_v . For μ of the order of the bare quark mass m , $m^*(r)$ has a pronounced minimum. For coupling constants smaller than those considered here, i.e. for $g^2 \lesssim 10$, we have also found solutions *that were evenly distributed over the volume*. The transition from the volume to surface solution as a function of g^2 and μ/m is smooth. Qualitatively this behaviour can be easily understood in view of the particular nature of the scalar coupling. Disregarding the spin effects, the spectrum equivalent Schrödinger potential is given by

$$V_{\text{eff}} \sim \frac{(m - g\varphi_c)^2}{2m} \quad (5.4)$$

We notice that if $g\varphi_c(0) > m$, then a minimum of V_{eff} will occur at $m = g\varphi_c(R)$. Depending on the precise form of $g\varphi_c$, this minimum may be broad enough to support a wave function localized around R . Alternatively, only a minor perturbation of the volume solution may be found.

In Fig. 2 the mass density $\mathcal{M}$ is shown as a function of r , as defined by Eq. (2.16) for $N = 1$. We note, in particular, the existence of "surface" solutions for which $\mathcal{M}$ peaks at a certain distance R for large values of the coupling constant, while there are also volume solutions for smaller coupling constants.

In Fig. 3 the masses³⁾ of the bound states M_N are given as functions of g^2 for the gluon masses of 0.4, 0.1, 0.02 (m_q). We see that smaller gluon masses give considerably smaller bag masses. As g^2 rises we would find a point g_c^2 such that

$$2M_2 < \mu, \quad g^2 > g_c^2 (\mu/m_q). \quad (5.5)$$

For $g^2 > g_c^2$ it can be argued that no free gluons can exist owing to their strong coupling to the hadronic states. As g^2 is increased with a fixed gluon mass μ , M_N decreases monotonically.

However, weakly bound exotic states are found as shown in Fig. 3. These states are conglomerates of the allowed hadronic states. This lack of saturation is particular for the scalar interaction. When other (repulsive) interactions are included, such states may easily become unbound.

There is an easy way of understanding the functional form of the curves shown in Fig. 3. The mass consists of two terms, cf. Eq. (2.20),

$$M_N = M_f + M_g, \quad (5.6)$$

where M_g is the energy contained in the gluon field φ_c . For large g^2 the eigenfrequency of the Dirac equation solution is

$$\epsilon_0 = R^{-1}, \quad (5.7)$$

where R is the radius of the bag. This relation is confirmed by results shown in Fig. 4, where the product $\epsilon_0 \cdot R$ is shown as a function of g^2 , for large values of g^2 . Let us assume that

$$M_g = \gamma g^2 R^\beta, \quad (5.8)$$

where β may be considered as a slowly varying function of R . The constants γ and β depend also upon μ/m . Thus we have

$$M_N = \frac{N}{R} + g^2 \gamma R^\beta = \frac{N}{R} (1 + g^2/N \gamma R^{\beta+1}). \quad (5.9)$$

Upon minimization at fixed g^2 and μ/m with respect to R

$$\frac{\partial M_N}{\partial R} = 0 = -\frac{N}{R^2} + g^2 \gamma \beta R^{\beta-1}. \quad (5.10)$$

We obtain

$$g^2/N \gamma R^{\beta+1} = \beta^{-1}, \quad (5.11)$$

and for Eq. (5.9)

$$M_N = M_f (1 + \beta^{-1}). \quad (5.12)$$

Defining

$$R_a = M_f / M_g, \quad (5.13)$$

we find

$$R_a = \beta \quad (5.14)$$

Thus the ratio of the fermionic to the gluonic part of the hadron mass has a geometrical meaning, since the value of β , Eq. (5.8), determines the nature of the solution. For $\beta = 3$ we have a volume bag with the glue distributed over the whole hadron. For $\beta = 2$ we have a surface solution, and $\beta = 1$ indicates the possibility of a string-like solution. As shown in Fig. 5,

$$P = M_N R / N \approx 1 + 1/R_a \quad (5.15)$$

tends to the value 1.5 for large g^2 , consistent with $R_a \lesssim 2$. We recall that for the SLAC bag¹⁾ $R_a = 2$ (surface bag) and for the MIT bag¹¹⁾ $R_a = 3$ (volume bag).

We can use relation (5.11) to eliminate R completely from Eq. (5.9). Then, with R_a as the second variable apart from g^2 we find

$$M_N \sim N^{R_a / (R_a + 1)} (g^2)^{-(R_a + 1)^{-1}}, \quad (5.16)$$

where the dependence on N follows from Eq. (2.26) which can also be written as

$$M_N (g^2) = N M_1 (N g^2). \quad (5.17)$$

In passing we note that the lack of saturation of the scalar interaction is apparent in Eq. (5.16). The energy per quark M_N/N is proportional to $N^{(R_a+1)^{-1}}$ and is therefore decreasing with an increasing number of quarks in the bag (of course, only colour singlets are considered).

We consider the limit $g^2 \rightarrow \infty$, $m_q \rightarrow \infty$ in Eq. (5.16), while μ remains finite. If there is a relation between the values of m_q and g^2 such as

$$g^2 \sim \left(\frac{m_q}{\mu} \right)^{R_a + 1}, \quad (5.18)$$

then M_N stays constant in the limit $(m_q, g^2) \rightarrow \infty$ and we have permanent confinement in the sense that, though the free quark mass m_q is infinite, the mass of the bound state remains finite. It is apparent that the relation (5.18) between the parameters of the theory is rather arbitrary. The above remark is meant only to alert the reader to the fact that the ratio of the bound state mass M to the free quark mass m_q can be made arbitrarily small.

We now proceed to calculate other hadronic properties. We take wave functions with exact SU(6) symmetry of spin and flavour, and an extra internal quantum number, colour, in which the baryonic wave functions are antisymmetric, thus following

the concepts set forth in the SLAC bag paper¹). The baryon multiplet 56 has 56 states counting the spin states. Since the absolute value of the free quark mass m_q is not known, we choose to consider only quantities that are scale-independent. The simplest ones are the products of the r.m.s. radius of the baryons and mesons with their masses; $N \cdot P$, Eq. (5.15), which are shown in Figs. 6a and 6b. The experimental number to compare with for baryons is, most likely, the product of the proton charge radius, 0.8 fm, with the average mass of the 56 multiplet, $M_{56} = 1280$ MeV, which is $5.2\hbar c$. We see that this lies well within values spanned by our calculations. We note that 5.2 implies for $N = 3$ and $\epsilon_0 \sim 1/R$ that $R_a \sim 1.7$. Thus it would seem that hadrons are surface-like, or $\epsilon_0 \not\sim 1/R$.

We can also calculate the absolute value of the magnetic moment of the proton, using as the basic unit $e\hbar/2M_{56}c$, that is the Bohr magneton *in units of the computed mass* (rather than the experimental value). Therefore the unknown quark mass m_q cancels out from our result. For comparison with the experiment we must scale up the experimental value of μ_p to account for the larger multiplet mass than that of the proton³); we obtain for comparison with Fig. 6c a value $\mu_p^{\text{exp}} = 2.79 \times 1280/938 = 3.8$.

We see that the self-consistent bag results are of the right magnitude. We consider also the axial coupling constant g_A of the neutron decay process. The experimental value is $g_A^{\text{exp}} = 1.25$. The results shown in Fig. 6d are in the range $0.6 < g_A < 1$.

Finally, in Fig. 6e the ratio R_a as computed from the solutions is shown. We find $1 < R_a \lesssim 2$, indicating that the self-consistent quark bag is mostly surface-like. The small values of R_a for *small* g^2 cannot be as clearly associated with β , Eq. (5.8), since relation (5.7) is not satisfied in this range of g^2 , see Fig. 5.

As the final point on the subject, let us note that there is a simple relation between g_A , μ_p , and M_N , that can be derived using the Dirac equation (2.23) and which can be cast into the form

$$\begin{aligned} \mu_p &= (0.5 + 0.3 g_A) \cdot N \cdot (1 + R_a^{-1}) \\ &= 2.625 (1 + R_a^{-1}). \end{aligned} \tag{5.19}$$

In the second line of Eq. (5.19) we have used $g_A = 1.25$, $N = 3$. Thus if we were able to find a $U(\phi)$ such that μ_p would also have the desired value 3.8, R_a would necessarily turn out to be 2.2. This means that in a "perfect" scalar bag about 70% of the mass would be carried by the quarks. We further note that the bag could be volume-like. Since g_A must be 1.25, we know that a solution with $\epsilon_0 \langle r^2 \rangle^{1/2} > 1$ is needed. In this case the value $R_a = 2.23$ does not imply a surface-like solution. In view of the above remarks, it is quite possible that a theory based on scalar interaction with very good phenomenological properties could be found.

6. PROPERTIES OF SOLUTIONS OF A
SELF-COUPLED DIRAC FIELD

We have argued that localized, stable solutions of classical field equations are the basic ingredients in the description of extended objects in quantum field theory. We now turn to the discussion of the self-coupled Dirac field $\bar{\psi}$ in *three-dimensional* space governed by the action

$$L = \int d^4x [\bar{\psi} \gamma \cdot p \psi - m_q \bar{\psi} \psi + \frac{G}{2} (\bar{\psi} \psi)^2], \quad (6.1)$$

where m_q is the mass of the free ($G = 0$) fermion field. G , like the weak coupling constant, has the dimension of inverse mass squared.

The quantum field theory associated with Eq. (6.1) is known to lead to a non-renormalizable perturbation expansion in the vicinity of $G = 0$. I believe that the difficulties associated with the perturbation expansion do not provide a compelling argument for rejection of the Lagrangian (6.1). In particular it is quite possible that the properties of the theory change with increasing strength of the coupling G in a non-analytic fashion. I find⁶⁾ that for $G > G_{\min} = 4.47/(4\pi m_q^2)$ localized solutions of the classical field equation arise.

Another way of seeing the self-interaction invokes the infinite gluon mass limit of a scalar meson field discussed in the last section. We have considered the properties of the Hamiltonian (3.17) associated with the self-coupled field already in Section 3. We therefore turn now to the discussion of the properties of the solutions obtained by the methods introduced in Section 4, with the modification that the equation defining a field ϕ in terms of $\bar{\psi}_0 \psi_0$ is very simple now and reads

$$\phi = G \bar{\psi}_0 \psi_0. \quad (6.2)$$

From this point on the iteration procedure can proceed as described previously. Use of numerical continuation of the solutions as a function of the coupling constant increased the convergence speed of the iteration. The solutions ψ_0 that will be discussed below are normalized to unity. We note a similar scale invariance as already noted with meson fields, Eq. (2.26); given an arbitrary norm

$$Q = \int d^3x \psi_0^\dagger \psi_0, \quad (6.3)$$

we can introduce

$$\psi_0' = Q^{1/2} \psi_0. \quad (6.4)$$

Then ψ_0' is a solution of the problem with the parameter

$$G' = G/Q, \quad (6.5)$$

that leads to the total energy

$$H'(G/Q) = H(G) Q. \quad (6.6)$$

The dimensionless parameter of the model is

$$\beta = m_q^2 G / 4\pi. \quad (6.7)$$

The lowest mass solution associated with lowest positive frequency solution has been computed. The energy (H/m_q) and the frequency (ϵ_0/m_q) are shown in Fig. 7a. We recognize that both functions fall monotonically with increasing β . We also find a minimum value of G for which the mass of the bound state is smaller than m . It is $G_{\min} = 4.47/(4\pi m_q^2)$, as mentioned previously. Interestingly enough, the qualitative behaviour of $H(Gm_q^2)$ changes when H is plotted in units of $G^{-1/2}$, as shown in Fig. 7b. We find

$$(H/G^{-1/2}) = (4\pi\beta)^{1/2} (H/m_q), \quad (6.8)$$

which monotonically rises. The lines in Fig. 7a may be understood also as obtained for fixed m with changing G , while in Fig. 7b, G is fixed and m_q varies.

In Fig. 8 the forms of the solutions for some representative values of $\beta = 22, 475, 2125$ are shown. The effective mass m^* as a function of r , is shown in Fig. 8a. With increasing coupling strength the size of the solution increases significantly, while the value of m^* at origin decreases. Notably, it remains always negative. In Figs. 8b, c, and d, the vector ρ_v and scalar ρ_s densities belonging to the same solutions are shown. As β increases, the vector density becomes more and more localized around the point where $m^* = 0$. The scalar density remains distributed over the whole volume of the solution. In that respect the self-interacting field solutions differ from the solutions involving a gluon field. There we have seen that the scalar density, like the vector density, is localized at the surface of the solutions. The volume character of ρ_s in the present case is necessary to allow the relatively constant effective mass m^* over the space occupied by the solution.

In view of the results presented in Figs. 7 and 8, the self-interacting fermion field discussed here may also be taken as a prototype of a self-consistent quark bag *without gluons*. All we need to do in order to generate the bag solutions from the soliton solutions given above is to perform the transformation described in Eqs. (6.3) to (6.5) with $Q = N$. N is the number of quarks: $N = 3$ for baryons, $N = 2$ for mesons (the scalar self-coupling also does not distinguish between quarks and antiquarks). A self-interacting field provides a natural explanation of the phenomenon of quark confinement in that the free particle of mass m_q cannot be considered as a quantum of the fermion field for $G > G_{\min}$. Also, the absence of the gluons in the quark bag is satisfactory.

7. SUPERCRITICAL FIELDS IN QUANTUM FIELD THEORY

In this section we shall consider the supercritical binding ($\epsilon_0 < -m$) in the frame of the bag approach. Since some of the theoretical details concerning the bag approximation differ slightly from our previous considerations, we will briefly rederive the self-consistent c-number equations, with particular emphasis on the new case $\epsilon_0 < -m$. A simple and possibly relevant example of supercritical binding arises when considering strongly bound states in the (classical) field theory of *attractive* massive vector mesons interacting with fermion fields.

We consider a fermion field coupled to a vector meson field with the Hamiltonian⁹⁾ derived from a slight generalization of Eq. (3.25) (with $A \cdot A \equiv A^2 = A_0^2 - \vec{A} \cdot \vec{A}$)

$$H = \frac{1}{2} \left\{ \int d^3x \frac{1}{2} [\hat{\Psi}^\dagger (\vec{\alpha} \cdot \vec{p} + \beta m - g_v (\vec{\alpha} \cdot \vec{\hat{A}} - A_0)) \hat{\Psi}] - \frac{1}{2} \int d^3x [\hat{\pi}_\mu^2 + (\nabla \hat{A}_\mu)^2 + W(\hat{A} \cdot \hat{A}) - W(0)] + h.c. \right\}, \quad (7.1)$$

where $W(A^2)$ is an arbitrary potential energy of the vector field A_μ , which leads to the equations of motion

$$(\vec{\alpha} \cdot \vec{p} + \beta m - g_v (\vec{\alpha} \cdot \vec{\hat{A}} - A_0)) \hat{\Psi} = i \frac{\partial \hat{\Psi}}{\partial t} \quad (7.2)$$

$$\hat{\pi}_\mu - \Delta A_\mu + \hat{A}_\mu \frac{\partial W}{\partial A^2} = \frac{g_v}{2} [\hat{\Psi}, \gamma_\mu \Psi]. \quad (7.3)$$

We also have the auxiliary condition

$$\langle ps | \partial_\mu \hat{A}^\mu | ps \rangle = 0 \quad (7.4)$$

for any physical state $|ps\rangle$.

For strong interactions it is most convenient to choose as the basis for the expansion of the Fermi field ψ the complete set of solutions of the Dirac equation with an average field A_c (p stands for both discrete and continuous indices):

$$(\vec{\alpha} \cdot (\vec{p} - g_v \vec{A}_c) + \beta m + g_v A_c^0) \psi_p = \epsilon_p \psi_p. \quad (7.5)$$

The field operator is expanded in the quasi-particle Fock space as

$$\psi(\vec{x}, 0) = \int_m^\infty b_\epsilon \psi_\epsilon(\vec{x}) d\epsilon + \sum_{m > \epsilon_p > -m} b_p \psi_p(\vec{x}) + \int_{-\infty}^{-m} d_\epsilon^\dagger \psi_\epsilon(\vec{x}) d\epsilon, \quad (7.6)$$

whenever the field A_c^μ is undercritical. A different expansion is appropriate, when one of the bound states of Eq. (7.6) is embedded in the negative energy continuum and the total energy remains positive. We explain this remark further below. Then we have

$$\begin{aligned} \Psi(\vec{x}, 0) = & \int_{-m}^{\infty} b_{\epsilon} \Psi_{\epsilon} d\epsilon + \sum_{m > \epsilon_p > -m} b_p \Psi_p + b_0 \int_{-m}^{\infty} a(\epsilon) \Psi_{\epsilon} d\epsilon \\ & + \int_{-m}^{\infty} d\epsilon' \left[\int_{-m}^{\infty} h_{\epsilon'}(\epsilon) \Psi_{\epsilon} d\epsilon \right], \end{aligned} \quad (7.7)$$

where $a(\epsilon)$ and $h_{\epsilon'}(\epsilon)$ are suitable functions that describe a particle resonance in a negative energy continuum¹²). If more than one resonance is embedded in the continuum, a generalization of Eq. (7.7) is in order.

It is important to stress here the difference between the self-consistent field problem and the case of an external field. The neutral vacuum, characterized by $b_p|0\rangle = d_n|0\rangle = 0$, may be a *stable*, zero-energy state even should supercritical binding described by (7.7) occur, since the total energy of the excited state $b_0^{\dagger}|0\rangle$ may be positive upon consideration of the meson energy included in Eq. (7.1), in contradistinction to the situation with *external* potentials. To put it in another way, when fermions move in a *self-consistent* (shell) potential, then the total energy of the bound state is given by the sum of the Dirac eigenenergies and of the energy associated with the shell potential, sometimes also called "correlation energy". The stability condition of the conventional ground state characterized by $b_p|0\rangle = d_n|0\rangle = 0$ is

$$\epsilon_0 + E_{\text{meson}} > 0 \quad (7.8)$$

and not

$$\epsilon_0 > -m \quad (7.9)$$

as would be the case if external fields were considered. It must be expected that the critical potential strength at which $\epsilon_0 = -m$ can be exceeded by the *self-consistent* potential and a resonance be embedded in the negative energy continuum of the Dirac equation. In view of Eq. (7.8) we recognize that even if the self-consistent field is supercritical, the neutral vacuum may remain stable. This last remark can only be appreciated after a thorough discussion of the case of an *external* potential¹²).

For the sake of argument, let us assume now that we have found a theory such that Eq. (7.8) is violated. We then would find that a charged ground state $b_0^{\dagger}|0\rangle$ has lower energy than the neutral state $|0\rangle$ -- then the latter state is the false vacuum, which under certain circumstances could undergo a transition into the true vacuum. Fortunately, we will find below that it is difficult to violate condition (7.8). I will show here that Eq. (7.8) is satisfied in the model discussed, although $\epsilon_0 < -m$, beyond a certain coupling strength. This means that despite the resonance of the Dirac equation we do not have a resonance of the Hamiltonian.

Our choice of the mean field A_c^{μ} was directed by our desire to eliminate from the ground state most contributions from the virtual quasi-particle excitations.

Therefore $A_c^\mu = \langle ps | A^\mu | ps \rangle$ and we obtain

$$-\Delta A_c^\mu + A_c^\mu \left(\frac{\partial W}{\partial A^2} \right)_{A_c^2} = g_v \langle ps | \frac{1}{2} [\hat{\bar{\psi}}, \gamma^\mu \hat{\psi}] | ps \rangle. \quad (7.10)$$

We consider the state $b_0^\dagger | 0 \rangle = | p \rangle$ as a trial state. Then Eq. (7.10) becomes (neglecting virtual vacuum polarization charge density):

$$-\Delta A_c^\mu + A_c^\mu \left(-\frac{\partial W}{\partial A^2} \right)_{A_c^2} = g_v \psi_0 \gamma^\mu \psi_0. \quad (7.11)$$

Here ψ_0 is the coefficient of b_0 in Eq. (7.7) or the lowest energy particle wave function in Eq. (7.6).

At this point, the interaction must be modified⁹⁾ to allow bound fermion states, bound only by "electric" forces. That is to say the sign on the right-hand side of Eq. (7.11) has to be changed, in order to obtain localized solutions with $\vec{A}_c = 0$ [see Eqs. (3.25) to (3.28)]

$$-\Delta A_c^\mu + A_c^\mu \left(\frac{\partial W}{\partial A^2} \right)_{A_c^2} = -g_v \psi_0 \gamma^\mu \psi_0. \quad (7.12)$$

This change is also motivated by the consideration of possible internal symmetry of the fields of interest -- the sign may be understood in analogy with the different possible signs of the z-component of the isospin. This change also means that the energy of the classical bound state becomes

$$E_v = E_{\text{Dirac}} + E_{\text{meson}} = \int d^3x \psi_0^\dagger (\vec{\alpha} \cdot \vec{p} + \beta m - g_v (\vec{\alpha} \cdot \vec{A} - A^0)) \psi_0 + \frac{1}{2} \int d^3x [(\nabla A_c^\mu)^2 + W(A_c \cdot A_c) - W(0)] = \int d^3x \mathcal{H}, \quad (7.13)$$

where we find

$$E_{\text{Dirac}} = E_0 \quad (7.14)$$

In view of the arbitrary (but motivated) changes involved in writing (7.13), what follows should be understood only as illustrating a type of behaviour may be characteristic to the Yang-Mills fields.

Let us devote the rest of our discussion to the special case

$$W = \mu^2 A^2, \quad (7.15)$$

where μ is the vector meson mass. We recall from Eq. (3.28) that the total energy is given by

$$E_{\text{eff}}^A = \int d^3x (\mu_v^2 A^2 + m \bar{\psi}_0 \psi_0). \quad (7.16)$$

The energy E_{eff}^A is positive definite if the scalar integral $\int d^3x \bar{\psi}_0 \psi_0$ is positive. No lower bound for the scalar integral is known when vector fields are present, but in all examples studied numerically it has been found to be positive definite and very small for large g_v^2 .

Equations (7.5) and (7.12) have been solved numerically with $W = \mu_v^2 A^2$ for the lowest energy particle state such that the Dirac wave functions (either eigenstates or resonances) are normalized to one. In Fig. 9 we show the energy of the bound fermion, E_{Dirac} [Eq. (7.14)], and the total energy E_v [Eq. (7.13)], as a function of the coupling constant g_v for vector meson masses $\mu = 0.2$ and $0.4m$. The energy is measured as usual in units of the bare fermion mass m . In this example, we see that the total energy is always positive for coupling constants $g_v^2/4\pi \gtrsim 4$, while the Dirac eigenvalues E_{Dirac} take large negative values. The scalar integral is a negligible contribution to E_v , Eq. (7.13), for strong coupling. Exact agreement of numerical results with those expected from the virial theorem [Eq. (7.16)] (as long as $E_{\text{Dirac}} > -m$) has been found. This is also a good check of the computer codes involved in solving the non-linear equations (7.5) and (7.12).

The main conclusion that can be drawn from this exercise is the recognition that in this simple model *the total energy of the strongly bound state lies above that of the usual vacuum*, i.e. $E_v > 0$ even when the self-consistent potential is supercritical, i.e. $\epsilon_0 < -m$. Therefore the usual vacuum is the stable ground state of the theory. Further we note that essential in obtaining a proper solution of Eqs. (7.5) and (7.12) is a proper treatment of resonant states in the antiparticle continuum when they occur. The proper description of the system involves the amplitude:

$$\psi_0 \approx \int_{-\infty}^{-m} a(E) \psi_E dE. \quad (7.17)$$

Without the proper understanding of the resonant spectrum a solution would seem discontinuous at the critical points, since we would take the lowest *discrete* state in place of the resonance.

Another way of seeing the difference between the presently considered case and the external field problem is to consider the singularity structure of the time transform of the Green's function

$$\tilde{S}(\vec{x}, \vec{x}'; \epsilon) = \int d(t-t') e^{-i\epsilon(t-t')} S_p(x, x'), \quad (7.18)$$

where p indicates the dependence of S on the path chosen in the ϵ plane when inverting Eq. (7.18). Two such paths are shown in Fig. 10 implied by Feynman boundary conditions. The integration path C' of Fig. 10 should be taken to obtain the

Green's function in the case of the interacting fields, and not the path D used in the case of external fields¹²⁾. We note here that the pole in the complex ε plane shown in Fig. 10 is actually on the second ε -sheet. A quite different result would follow if we had found that E_v in Eq. (7.13) is actually a negative quantity at certain values of g_v^2 . Then the path D would have to be taken and we would expect that the vacuum state is the charged vacuum. We would then face the problem of a macroscopic phase transition. At every point in space the vacuum would be charged. Thus another procedure would be needed to determine the new ground state, since it is impossible to describe such a global change of the world in the single-mode approximation.

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Figure captions

- Fig. 1 : The effective quasi-particle mass $m^* = m_q - g\phi_c$ (in units of m_q) as a function of r (in units of m_q^{-1}) for $\mu/m_q = 0.02, 0.1, 0.4$ and $g^2 = 2260, 1072, 642, 777$, correspondingly. The dashed curves are the vector and scalar densities for $\mu = 0.4m_q$, $g^2 = 642$ and $\mu = 0.02m_q$, $g^2 = 2260$. The vector density is normalized to one.
- Fig. 2 : The mass density of the soliton solution as a function of r ; several values of the characteristic parameters g^2 and μ in a double-logarithmic plot.
- Fig. 3 : The total mass M_N in units of the quark mass of several quark states; baryons ($N = 3$), mesons ($N = 2$), and exotic baryons ($N = 6$) as a function of the coupling constant g^2 for several values of the gluon mass $\mu/m_q = 0.02, 0.1, 0.4$.
- Fig. 4 : The product of the lowest positive Dirac eigenvalue with r.m.s. radius of the amplitude ψ_0 as a function of $\ln g^2$.
- Fig. 5 : The product of the mass of the quark bag with the r.m.s. radius as a function of $\ln g^2$.
- Fig. 6 : The structure constants of SU(6) hadrons as a function of the coupling constant g^2 for several values of the gluon mass, $\mu/m_q = 0.02, 0.1, 0.4$. a) The product of the mass of the baryon with its size. b) The same as in (a) for mesons. c) The magnetic moment of protons. d) The axial-coupling constant g_A . e) The ratio R_a of the energy (momentum) carried by the quarks to that carried by the neutral glue.
- Fig. 7 : The energy H and the eigenfrequency ε_0 as a function of the dimensionless coupling constant β : a) in units of m_q ; b) in units of $G^{-1/2}$.
- Fig. 8 : The soliton solutions of self-coupled Dirac fields for $\beta = .22, 475, 2125$. a) The effective mass m^* ; b), c), d) the scalar and vector densities.
- Fig. 9 : The Dirac eigenvalue and total energies of interacting Dirac-vector meson fields.
- Fig. 10 : The contours and the singularities of the Green's function in over-critical potentials.

Fig. 1

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Fig. 3

Fig. 4

Fig. 5

Fig. 6

Fig. 7

Fig. 8

Fig. 9

Fig. 10

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