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WAVE PACKETS AND THE SPACE-TIME STRUCTURE OF PRODUCTION PROCESSES

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A B S T R A C T

Given an arbitrary production amplitude we derive expressions for the size and time dependence of the scattering region corresponding to this amplitude. This is achieved by studying the shifts and width changes caused in localized wave packets by the collision. These results are illustrated in a series of models including the space-time structure of multi-peripheral production amplitudes. They do not agree with the standard results.

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1. INTRODUCTION

The remarkable results obtained in high-energy collisions of nucleons and nuclei have led to great interest in the space-time structure of hadronic collisions. It is clear that two protons approaching each other with great energy and colliding to form many particles in the final state do not behave like two relativistically contracted pancakes undergoing a quasi-instantaneous interaction. If this were so we would expect a rapid rise in multiplicity with A in the high-energy collisions of protons and nuclei. This is due to the cascading effect which is clearly allowed if the interactions only take a short time. This is not seen experimentally¹⁾.

In simple multiperipheral models of the type shown in Fig. 1 attempts have been made to investigate the space-time structure by using old fashioned perturbation theory²⁾. The dominant time ordering is shown in Fig. 1. Moreover the energy denominator corresponding to the intermediate state A is $\sim m^2/p$ where p is the incident momentum in the lab. frame. Using the relation $\Delta E \times \Delta t \sim 1$ ³⁾ this implies that the particle $(p - q_1)$ is produced a distance $\sim ct \sim p/m^2$ upstream from the target. However it is associated with the virtual particles q_1, q_2 , which only turn into physical on-shell states at the last collision at time $t \sim 0$. Moreover the $\Delta E \times \Delta t$ relation is derived³⁾ from the expression for an $i \rightarrow f$ transition in lowest order perturbation theory with a constant potential switched on for a time Δt

$$P(i \rightarrow f) \simeq V_{if}^2 \frac{\sin^2(E_i - E_f)\Delta t}{(E_i - E_f)^2} \quad (1.1)$$

For a given Δt this clearly implies that only i, f such that $(E_i - E_f) \sim 1/\Delta t$ have significant transition rates. However, for a given $(E_i - E_f)$ the Δt is not restricted in the same way.

This analysis therefore leaves open several questions. For example, do we obtain the same qualitative result with a more general multiperipheral amplitude, say with Regge exchange? Do the fast particles $(p - q_1)$ actually appear far upstream of the geometrical collision point? If $(p - q_1)$ were a resonance of lifetime Γ , does its time delay correspond to production at $t = -p/m^2$ or $t = 0$?

Clearly to answer these questions we must study the scattering of wave packets. If the particle $(p - q_1)$ is produced far upstream then its outgoing wave-packet state should be displaced by the relevant amount. If the scattering takes place over a large volume of space then we would expect the size of the outgoing packets to be altered by the appropriate amount. In the simplest of examples if we scatter elastically in two small scatterers a distance b apart, then

the final wave packet is a superposition of two packets displaced, crudely speaking, by b and hence the over-all width is increased by b .

Thus the main content of this paper is a calculation, given an arbitrary T matrix, of the shifts and width changes induced in final state wave packets. This is a generalization of the elastic one-body scattering result of Froissart, Goldberger and Watson⁴⁾ who calculated wave packet shifts. In a study of one-dimensional production models, shifts were previously computed by one of the authors⁵⁾.

The results when applied to the amplitude of Fig. 1 are totally unexpected. The fast particle $p - q_1$ comes exactly from the geometrical collision point with no shift or time delay. The wave packet width undergoes a change which is exactly that given by old fashioned perturbation theory analysis, except that it has the wrong sign. In position space the wave packet gets narrower. In momentum space it gets wider of course. We have no simple classical explanation of this fact. It clearly corresponds to a coherent effect between the waves scattered at different times.

Gottfried⁶⁾ has proposed another view of the space-time development of production processes. Here the particles first interact at the geometrical point, but the large time scale is given by the time required for the final particles to separate themselves out from the general energy distribution. Since our formalism only deals with asymptotic states we are incapable of checking this picture. However, to the extent that both systems are consistent with zero shifts of the wave packets, we agree.

In Section 2 we calculate the shifts of the wave packets. In Section 3 we give a simple approximate treatment of the width changes for Gaussian packets. In Section 4 we give the rather complicated general analysis of the width changes. Section 5 then contains applications of the formalism to some interesting examples.

2. PACKET SHIFTS

The technique we shall use is to fire in two non-dispersive wave packets for the two initial-state particles (non-dispersive here means that the packets are sufficiently narrow in momentum space that they do not disperse in the time scale of our problem). These will be centred around the incoming momenta p_1, p_2 . Each such packet will contain a momentum spread which is typically much smaller than the range over which the T matrix varies. If the final state also consists of only two particles then the lengths of the momenta of these particles are essentially fixed if we know the scattering angle. Thus the final state is then also essentially non-dispersive in character. However, for a multi-body final state there is typically a large spread allowed in the momenta of the final-state particles both in length and direction. Thus the final state is essentially dispersive

in character. To circumvent this problem we project onto non-dispersive localized wave packets centred on the final-state momenta which interest us. The positions of these packets are then used to maximize the overlap with the outcoming final state which is, of course, given by the T matrix. This tells us the most likely shift in the position of a scattered particle of a given momentum. For potential scattering this trivially gives the results of Froissart, Goldberger and Watson⁴⁾. We shall also calculate the variations in the positions of the projecting packets and this will tell us the width changes in the final state due to the size of the scattering region.

At all stages we expect and check that our results are independent of the detailed shapes of our wave packets. We have also checked that our shifts, etc., have the expected transformation properties under Lorentz transformations.

Before starting our real problem, let us check that wave packet projection does what we expect. So now consider two packets

$$\begin{aligned}\Psi(\underline{x}, t) &= \int d^3k \, a(\underline{k}) e^{i(\underline{k} \cdot \underline{x} - \varepsilon(\underline{k})t)} \\ \Phi(\underline{x}, t) &= \int d^3q \, b(\underline{q}) e^{i(\underline{q} \cdot \underline{x} - \varepsilon(\underline{q})t)}\end{aligned}\tag{2.1}$$

where $a(\vec{k})$, $b(\vec{q})$ are centred at $\vec{p}_a$, $\vec{p}_b$, respectively, with widths W_a , W_b , respectively in momentum space. Then the overlap

$$\begin{aligned}O(\underline{a}, t) &= \int d^3x \, \Psi^* \Phi \\ &= \int d^3k \, a^*(\underline{k}) b(\underline{k}) e^{-i\underline{k} \cdot \underline{a}}\end{aligned}\tag{2.2}$$

This is time independent. Thus if a, b give rise to two localized wave packets of different velocity there is no increase in O as they pass through each other.

If p_a , p_b differ by an amount $\Delta \gg W_a, W_b$ then clearly

$$0 \leq e^{-\Delta^2/4W^2} \ll 1 \quad W \sim W_a \sim W_b\tag{2.3}$$

If we take a, b Gaussian we can calculate (2.2) explicitly and we see that there is an exponential cut-off in $(p_a - p_b)^2$. If $p_a = p_b$ then (2.2) clearly implies that $O(a)$ has a width $\sim W_a \sim W_b$ in a . Thus the overlap works correctly for two localized wave packets.

We start by rederiving the Froissart, Goldberger and Watson results, for single-body scattering on a fixed potential, in an almost trivial manner. We fire in a wave packet centred at $\vec{a}$ at $t = 0$

$$\Phi_i(x, t) = \int d^3k \, a(\vec{k} - \vec{p}_i) e^{i[\vec{k} \cdot (\vec{x} - \vec{a}) - \varepsilon(\vec{k})t]} \quad (2.4)$$

where $a(\vec{k} - \vec{p}_i)$ is some real function centred symmetrically around $\vec{k} = \vec{p}_i$ with a width W_i . The reader may consider $a = \exp[-(\vec{k} - \vec{p}_i)^2/W_i^2]$. In the no-dispersion approximation we may expand $\varepsilon(\vec{k})$ to first order in $(\vec{k} - \vec{p}_i)$ since $a(\vec{k} - \vec{p}_i)$ gives a rapid cut-off in $(\vec{k} - \vec{p}_i)^2$. Defining $\vec{v}_i = \nabla_{\vec{p}_i} \varepsilon(\vec{p}_i)$ we find

$$\begin{aligned} \Phi_i(x, t) &\approx e^{i\vec{p}_i \cdot (\vec{x} - \vec{a}) - i\varepsilon(\vec{p}_i)t} \int d^3u \, a(u) e^{i[\vec{x} - \vec{a} - \vec{v}_i t] \cdot \vec{u}} \\ &= e^{i\vec{p}_i \cdot (\vec{x} - \vec{a}) - i\varepsilon(\vec{p}_i)t} G_a(\vec{x} - \vec{a} - \vec{v}_i t) \end{aligned} \quad (2.5)$$

where $G_a(\vec{y})$ is a real function for real y centred about $y = 0$ with a width $\sim 1/W_i$. We believe this non-dispersive approximation for times t small enough compared to $1/W^2 \times \nabla^2 \varepsilon(p)$. In general the rate of dispersion given by these terms is negligible in laboratory scattering conditions.

Then the final-state wave function is

$$\begin{aligned} &\int d^3k \, d^3k' \, \delta(\varepsilon(\vec{k}) - \varepsilon(\vec{k}')) T(\vec{k}', \vec{k}) e^{i(\vec{k}' \cdot \vec{x} - \varepsilon(\vec{k}')t)} \\ &\quad a(\vec{k} - \vec{p}_i) e^{-i\vec{k} \cdot \vec{a}} \end{aligned} \quad (2.6)$$

We now overlap this with Φ_f given by (2.5) with $a \rightarrow b$, $i \rightarrow f$, and $\vec{k} \rightarrow \vec{q}$.

Then

$$\begin{aligned}
 O(\underline{a}, \underline{b}) &= \int d^3x \Phi_f^*(x, t) \Phi_{sc}(x, t) \\
 &= (2\pi)^2 \int d^3q d^3k d\alpha e^{i\alpha(\varepsilon(\underline{k}) - \varepsilon(\underline{q}))} a(\underline{k} - \underline{p}_i) \\
 &\quad T(\underline{q}, \underline{k}) e^{-i\underline{k} \cdot \underline{a} + i\underline{q} \cdot \underline{b}} b(\underline{q} - \underline{p}_f)
 \end{aligned} \tag{2.7}$$

where we have written a representation of the energy conserving δ -function. Now $\underline{a}, \underline{b}$ allow very little variation in $\vec{k}, \vec{q}$ so we may write

$$T(\underline{q}, \underline{k}) = T(\underline{p}_f, \underline{p}_i) \exp \left[(\underline{A}_1 + i\underline{A}_2) \cdot (\underline{k} - \underline{p}_i) + (\underline{B}_1 + i\underline{B}_2) \cdot (\underline{q} - \underline{p}_f) \right]$$

where

$$\begin{aligned}
 \underline{A}_1 &= \nabla_{\underline{p}_i} \log |T| & \underline{B}_1 &= \nabla_{\underline{p}_f} \log |T| \\
 \underline{A}_2 &= \nabla_{\underline{p}_i} \arg T & \underline{B}_2 &= \nabla_{\underline{p}_f} \arg T
 \end{aligned} \tag{2.8}$$

With this approximation the $\vec{k}, \vec{q}$ integrals in (2.7) are exactly of the form of (2.4) with α replacing t . Approximating the α -phase as before we find

$$\begin{aligned}
 O(\underline{a}, \underline{b}) &= (2\pi)^2 \int d\alpha e^{i\alpha(\varepsilon(\underline{p}_i) - \varepsilon(\underline{p}_f))} G_a(-\underline{a} - i\underline{A} + \alpha \underline{v}_i) \\
 &\quad G_b(\underline{b} - i\underline{B} - \alpha \underline{v}_f) \cdot T(\underline{p}_i, \underline{p}_f)
 \end{aligned} \tag{2.9}$$

Here for the moment we write $\underline{A} = \underline{A}_1 + i\underline{A}_2$. To maximize O we clearly must choose $\varepsilon(\underline{p}_i) = \varepsilon(\underline{p}_f)$ otherwise the first phase introduces multiple zeros into the integrand.

To maximize $O(\vec{a}, \vec{b})$ we must clearly use $\vec{b}, \vec{q}$ to superimpose the maxima of G_a, G_b . This may occur at an arbitrary real point of α since a translation of $\vec{a}$ along

the beam direction $\vec{v}_i$ merely means that the scattering does not occur at $t = 0$ and is easily compensated for by a shift of $\vec{b}$ along $\vec{v}_f$. Thus we use $-\vec{a}$ to cancel all the real parts of $-i\vec{A}$ and $\vec{b}$ to cancel all the real parts of $-i\vec{B}$.

Thus, for a given T , $\vec{p}_i$, $\vec{p}_f$, the best impact parameter for the incoming packet is

$$\underline{a}_{Ti} = \nabla_{\vec{p}_i} \arg T \big|_{Ti} \quad (2.10)$$

and the outgoing packet is preferentially displayed orthogonal to $\vec{p}_f$ by

$$\underline{b}_{Tf} = -\nabla_{\vec{p}_f} \arg T \big|_{Tf} \quad (2.11)$$

For motion along the beam then there is a time delay δ (assuming $|\vec{v}_i| = |\vec{v}_f|$)

$$v_i \delta = (\nabla_{\vec{p}_{iL}} + \nabla_{\vec{p}_{fL}}) \arg T \quad (2.12)$$

These choices guarantee that the maxima of G_a , G_b are superimposed. The imaginary parts merely skew the two distributions around the point of maximum overlap and add an over-all phase. Since only $|0\rangle$ is physically relevant we may ignore these effects.

Now $T(\vec{p}_i, \vec{p}_f) = T(\vec{p}_i^2, \vec{p}_f^2, \hat{\vec{p}}_i, \hat{\vec{p}}_f)$ and if $\varepsilon(\vec{p}_i) = \varepsilon(\vec{p}_f)$ clearly all quantities of physical interest must be independent of whether we write $T(\vec{p}_i^2, \vec{p}_f^2, z)$ or $T(\vec{p}_f^2, \vec{p}_i^2, z)$. Thus a little algebra leads to

$$\delta = \frac{1}{m} \frac{\partial T}{\partial \varepsilon}$$

$$\underline{a}_T = \left(\frac{\partial T}{\partial z} \right) \frac{1}{|\vec{p}_i|} \left[\vec{p}_f - (\hat{\vec{p}}_i \cdot \hat{\vec{p}}_f) \hat{\vec{p}}_i \right] \quad (2.13)$$

the standard results.

In the above we considered one-body scattering on a potential so that only energy and not three-momentum was conserved. We now generalize to two bodies scattering into two bodies. Thus we fire in two wave packets centred at $\vec{a}_1$, $\vec{a}_2$, at $t = 0$ with momenta centred at $\vec{p}_1^i$, $\vec{p}_2^i$, respectively.

$$\Phi_i(\underline{x}_1, \underline{x}_2, t) = \int d^3k_1 d^3k_2 a_1(\underline{k}_1 - \underline{p}_1^i) a_2(\underline{k}_2 - \underline{p}_2^i) \\ \exp[i \underline{k}_1 \cdot (\underline{x}_1 - \underline{q}_1) + i \underline{k}_2 \cdot (\underline{x}_2 - \underline{q}_2) - i(\varepsilon(\underline{k}_1) + \varepsilon(\underline{k}_2))t]$$

Then the scattered wave is

$$\Phi_{sc}(\underline{x}_1, \underline{x}_2, t) = \int d^3k_1 d^3k_2 a_1(\underline{k}_1 - \underline{p}_1^i) a_2(\underline{k}_2 - \underline{p}_2^i) \\ T(\underline{k}_1, \underline{k}_2, \underline{q}_1, \underline{q}_2) \delta(\varepsilon(\underline{k}_1) + \varepsilon(\underline{k}_2) - \varepsilon(\underline{q}_1) - \varepsilon(\underline{q}_2)) \delta^3(\underline{k}_1 + \underline{k}_2 - \underline{q}_1 - \underline{q}_2) \\ \exp[-i(\underline{k}_1 \cdot \underline{q}_1 + \underline{k}_2 \cdot \underline{q}_2) + i(\underline{q}_1 \cdot \underline{x}_1 + \underline{q}_2 \cdot \underline{x}_2) - i(\varepsilon(\underline{q}_1) + \varepsilon(\underline{q}_2))t] \quad (2.14)$$

Now we overlap this with a test state

$$\Phi_f = \int d^3k_1 d^3k_2 b_1(\underline{k}_1 - \underline{p}_1^f) b_2(\underline{k}_2 - \underline{p}_2^f) \\ \exp[i \underline{k}_1 \cdot (\underline{x}_1 - \underline{b}_1) + i \underline{k}_2 \cdot (\underline{x}_2 - \underline{b}_2) - i t (\varepsilon(\underline{k}_1) + \varepsilon(\underline{k}_2))] \quad (2.15)$$

Then

$$\hat{O}(\underline{q}_1, \underline{q}_2, \underline{b}_1, \underline{b}_2) = \int d^3k_1 d^3k_2 d^3q_1 d^3q_2 a_1(\underline{k}_1 - \underline{p}_1^i) a_2(\underline{k}_2 - \underline{p}_2^i) \\ b_1(\underline{q}_1 - \underline{p}_1^f) b_2(\underline{q}_2 - \underline{p}_2^f) T(\underline{k}_1, \underline{k}_2, \underline{q}_1, \underline{q}_2) \delta(\varepsilon(\underline{k}_1) + \varepsilon(\underline{k}_2) - \varepsilon(\underline{q}_1) - \varepsilon(\underline{q}_2)) \\ \delta^3(\underline{k}_1 + \underline{k}_2 - \underline{q}_1 - \underline{q}_2) \exp[-i(\underline{k}_1 \cdot \underline{q}_1 + \underline{k}_2 \cdot \underline{q}_2) + i(\underline{q}_1 \cdot \underline{b}_1 + \underline{q}_2 \cdot \underline{b}_2)] \quad (2.16)$$

Now introduce $\alpha_0, \vec{\alpha}$ as the Fourier transform variables for the two δ -functions and write T in the form

$$\begin{aligned}
 T(k_1, k_2, q_1, q_2) &= T(p_1^i, p_2^i, p_1^f, p_2^f) \exp \left[(k_1 - p_1^i) \cdot \nabla_1^i \log T \right. \\
 &\quad \left. + (k_2 - p_2^i) \cdot \nabla_2^i \log T + (q_1 - p_1^f) \cdot \nabla_1^f \log T + (q_2 - p_2^f) \cdot \nabla_2^f \log T \right] \\
 &= T_0 \exp \left[i \underline{A}_1 \cdot (k_1 - p_1^i) + i \underline{A}_2 \cdot (k_2 - p_2^i) + i \underline{B}_1 \cdot (q_1 - p_1^f) \right. \\
 &\quad \left. + i \underline{B}_2 \cdot (q_2 - p_2^f) \right] \quad (2.17)
 \end{aligned}$$

where we write $\nabla_1^f = \nabla_{p_1^f}$, etc.

Then

$$\begin{aligned}
 |O(q_1, q_2, b_1, b_2)| &= \frac{1}{(2\pi)^4} \int d\alpha_0 d^3\alpha \, G_1^i(-q_1 + \underline{A}_1, -\underline{\alpha} + \alpha_0 \underline{V}_1^i) \\
 &\quad G_2^i(-q_2 + \underline{A}_2, -\underline{\alpha} + \alpha_0 \underline{V}_2^i) G_1^f(\underline{b}_1 + \underline{B}_1 + \underline{\alpha} - \alpha_0 \underline{V}_1^f) G_2^f(\underline{b}_2 + \underline{B}_2 + \underline{\alpha} - \alpha_0 \underline{V}_2^f) \quad (2.18)
 \end{aligned}$$

where we have chosen the central momenta to satisfy four-momentum conservation. Here the $\vec{v}$'s are the appropriate velocities. They may all depend on their respective $\vec{p}$'s in different ways. Then O is clearly invariant under a uniform shift of $\vec{a}_1, \vec{a}_2, \vec{b}_1, \vec{b}_2$ as we would expect. There is now no preferred origin for the scattering. Thus we pick $\vec{\alpha} = 0$ as the point where the integrand is maximal. If we ignore the A, B for the moment we clearly get the maximum value of $|O|$ for $\vec{a}_1 = \vec{a}_2 = \vec{b}_1 = \vec{b}_2 = \alpha_0 = 0$. This is clear since, in order to get the maximum scattering from a point, structureless scatterer, all wave packets must be centred on the origin at $t = 0$. With A, B present we choose $\vec{a}, \vec{b}$ to cancel the real parts of the $\vec{A}, \vec{B}$ terms. We must now unravel the physical meaning. In the centre of mass we expect to have two impact parameters $\vec{a}_T = (\vec{a}_1 - \vec{a}_2)_{T_1}$ and $\vec{b}_T = (\vec{b}_1 - \vec{b}_2)_{T_f}$, in an obvious notation, which are equal in magnitude. Moreover $(\vec{a}_1 + \vec{a}_2) = (\vec{b}_1 + \vec{b}_2)$ for equal mass scattering since we do not expect the centre of mass to move.

Now we know that we may write $T = T[(\vec{p}_1^i + \vec{p}_2^i)^2, (\vec{p}_1^i - \vec{p}_1^f)^2] = T(s, t)$ in the non-relativistic equal mass case which we treat for simplicity here. Then in the centre-of-mass system the only components of $\vec{b}_T$ and $\vec{a}_T$ come from $(\vec{p}_1^i - \vec{p}_1^f)(\partial/\partial t)(\arg T)$. The $\partial/\partial s$ terms give zero contribution to the transverse components of a, b . Thus $|\vec{a}_T|, |\vec{b}_T|$ are equal. Moreover, $\vec{b}_1 + \vec{b}_2 = \vec{a}_1 + \vec{a}_2$ as is easily checked. We stress

that the centre-of-mass conditions may only be introduced after all differentiations. Our wave packets contain distributions of momenta and only a minute subset are explicitly in the centre-of-mass system. The time delay is calculated from the parallel displacements. If $v_1^i = v > 0$, $v_2^i = -v < 0$ along the z-axis, the impact time is $(a_1 - a_2)_z / 2v$. Similarly the apparent starting time of the final-state packets is $(b_1 - b_2)_L / 2v$ where L is defined as the direction of $\vec{v}_1^f$. Notice that $(b_1)_L$ may be positive or negative. Then the time delay τ is given by the apparent starting time minus the impact time. Clearly we could have made the impact time zero by picking the α_0 maximum at some $\alpha_0 \neq 0$.

Thus the time delay is given by

$$2v\tau = [a_1 - a_2]_{\parallel i} - [b_1 - b_2]_{\parallel f} \quad (2.19)$$

$$\begin{aligned} &= \left[2(p_1^i - p_2^i) \frac{\partial \phi}{\partial s} \right] + \left[(p_1^i - p_1^f) \frac{\partial \phi}{\partial t} \right]_{\parallel i} + \left[(p_1^i - p_1^f) \right]_{\parallel f} \frac{\partial \phi}{\partial t} \\ &= 4p \frac{\partial \phi}{\partial s} + 2p(1 - \cos \theta) \frac{\partial \phi}{\partial t} \end{aligned}$$

$$= p \frac{\partial}{\partial p^2} \phi(s, z) \quad (2.20)$$

$$\tau = \frac{\partial \phi(E, z)}{\partial E}, \quad E = \frac{(2p)^2}{2m} \quad \phi = \arg T$$

This agrees with the results we would have obtained by treating the two-body scattering as a single particle scattering from a fixed potential. The impact parameters also agree in an obvious manner.

Now we turn to the $2 \rightarrow n$ production process. Introducing wave packets for all n final-state particles we obtain

$$\begin{aligned} |O(a_1, a_2; b_1, \dots, b_n)| &= \left| \int d^4\alpha \, G(-a_1 + A_1 - \alpha + \alpha_0 v_1^i) \right. \\ &\quad \left. G(-a_2 + A_2 - \alpha + \alpha_0 v_2^i) \prod_1^n G(b_j + B_j + \alpha - v_j^f \alpha_0) T(p^i, p^f) \right| \end{aligned} \quad (2.21)$$

Here $A_j = +V_j^i \arg T$, $B_j = +V_j^f \arg T$. Thus the $\vec{A}_j$, $\vec{B}_j$ fix the best incoming and outgoing positions of the test wave packets. Thus the projections of the $\vec{A}_j$, $\vec{B}_j$ along their velocities determine the impact times and delays. The components perpendicular to these velocities determine the impact parameters. Again the $\vec{\alpha}$, α_0 freedom guarantees translation invariance in space and time. Again we see that the $\log |T|$ variations which give imaginary parts to the $\vec{A}$, $\vec{B}$ cannot be cancelled by $\vec{a}$, $\vec{b}$ choices and skew the G distributions.

We remark here that the amplitude of Fig. 1 is purely real and therefore trivially gives no shifts. Thus all the wave packets appear to come from the centre of the scattering region.

3. WIDTH CHANGES

In this section we treat with one major unchecked approximation the width changes for Gaussian packets in two simple examples. We will remove the unchecked assumption and derive a general formula in the next section. The following simple analysis is much more transparent than the rather involved analysis of the next section, so we include it for pedagogical reasons.

We first consider what we expect to happen when we overlap two packets of widths W_i , W_f in momentum space

$$O = \int d^3x \int d^3k d^3q e^{-\frac{(k-p_i)^2}{W_i^2}} e^{-\frac{(k-p_f)^2}{W_f^2}} e^{-i\vec{k} \cdot \vec{q}} e^{i(\vec{k}-\vec{q}) \cdot \vec{x}} \quad (3.1)$$

For $\vec{p}_i = \vec{p}_f = \vec{p}$ we obtain on doing the Gaussian $\vec{k}$, $\vec{q}$ integrals

$$O \propto e^{-i\vec{p} \cdot \vec{q}} \frac{e^{-\frac{q^2}{\left[\frac{1}{W_i^2} + \frac{1}{W_f^2}\right]}}}{\left[\frac{1}{W_i^2} + \frac{1}{W_f^2}\right]^{3/2}} \quad (3.2)$$

Thus if we overlap two packets of different widths, displaced by $\vec{a}$ relative to one another, then we get a Gaussian $\vec{a}$ cut-off with width $\left[\frac{1}{W_i^2} + \frac{1}{W_f^2}\right]$. Thus squares of widths add in position space.

Now consider scattering on a one-dimensional Gaussian distribution of weak point scatterers. For each scatterer we use the Born approximation and then $T(k,q) = e^{-\frac{(k-q)^2}{W^2}}$

$$0 = \int dk dq \delta(\varepsilon(k) - \varepsilon(q)) e^{-ika} e^{iqb} e^{-(k-p_i)^2/W_i^2} e^{-(q-p_f)^2/W_f^2} e^{-(k-q)^2/W^2} \quad (3.3)$$

Now in one dimension $\delta[\varepsilon(k) - \varepsilon(q)]$ implies $k = q$. For the non-relativistic case

$$0 = \int \frac{dk}{k} e^{ik(b-a)} e^{-(k-p_i)^2/W_i^2} e^{-(k-p_f)^2/W_f^2} \quad (3.4)$$

Thus in $(b - a)$ for $p_i = p_f$ we have a width $\sim \sqrt{1/W_i^2 + 1/W_f^2}$ independent of W . This agrees with our naive intuition since scattering at different points in the Gaussian distribution does not imply any time delays.

If we consider scattering on a three-dimensional Gaussian distribution of scatterers we have $T(\vec{k}, \vec{q}) = \exp[-(\vec{k} - \vec{q})^2/W^2]$. Then the overlap

$$0 = \int d^3k d^3q \delta(\varepsilon(k) - \varepsilon(q)) e^{-i\vec{k} \cdot \vec{a}} e^{i\vec{q} \cdot \vec{b}} e^{-(\vec{k}-\vec{p}_i)^2/W_i^2} e^{-(\vec{q}-\vec{p}_f)^2/W_f^2} e^{-(\vec{k}-\vec{q})^2/W^2} \quad (3.5)$$

We write our usual representation for the δ -function but only keep the linear, non-dispersive terms in the expansions of $\varepsilon(\vec{k})$, $\varepsilon(\vec{q})$

$$\delta(\varepsilon(\vec{k}) - \varepsilon(\vec{q})) \approx \int d\alpha e^{i\alpha[(\vec{k}-\vec{p}_i) \cdot \vec{v}_i - (\vec{q}-\vec{p}_f) \cdot \vec{v}_f]} \quad (3.6)$$

This is not justified at this stage. We will return to this problem in the next section. However, with this approximation the integrals in (3.5) are Gaussian in $\vec{k}$, $\vec{q}$.

$$O \propto \int d\alpha \frac{e^{-D/C}}{C^{3/2}} \quad (3.7)$$

$$C = \det \begin{vmatrix} \frac{1}{W_i^2} + \frac{1}{W^2} & -\frac{1}{W^2} \\ -\frac{1}{W^2} & \frac{1}{W_f^2} + \frac{1}{W^2} \end{vmatrix}$$

$$D = \det \begin{vmatrix} \frac{1}{W_i^2} + \frac{1}{W^2} & -\frac{1}{W^2} & \frac{i}{2} \frac{a}{W^2} - i\alpha v_i + \frac{(p_i - p_f)}{W^2} \\ -\frac{1}{W^2} & \frac{1}{W_f^2} + \frac{1}{W^2} & -\frac{i}{2} \frac{b}{W^2} + i\alpha v_f - \frac{(p_i - p_f)}{W^2} \\ \frac{i}{2} \frac{a}{W^2} - i\alpha v_i + \frac{(p_i - p_f)}{W^2} & -\frac{i}{2} \frac{b}{W^2} + i\alpha v_f - \frac{(p_i - p_f)}{W^2} & 0 \end{vmatrix} \quad (3.8)$$

This is now Gaussian in α and therefore computable. Then in D/C the coefficient of α^2 is

$$A = \left[v_i^2 \left(\frac{1}{W_f^2} + \frac{1}{W^2} \right) + v_f^2 \left(\frac{1}{W_i^2} + \frac{1}{W^2} \right) - 2 v_i \cdot v_f \left(\frac{1}{W^2} \right) \right] / C$$

The coefficient of 2α is

$$B = \left[- \left(\frac{1}{W_f^2} + \frac{1}{W^2} \right) v_i \cdot \left[\frac{a}{2} + \frac{p_i - p_f}{i W^2} \right] \right. \\ \left. - \left(\frac{1}{W_i^2} + \frac{1}{W^2} \right) v_f \cdot \left[\frac{b}{2} + \frac{p_i - p_f}{i W^2} \right] \right. \\ \left. + \frac{1}{W^2} v_f \cdot \left[\frac{a}{2} + \frac{(p_i - p_f)}{i W^2} \right] + \frac{1}{W^2} v_i \cdot \left[\frac{b}{2} + \frac{(p_i - p_f)}{i W^2} \right] \right] / C \quad (3.9)$$

The terms independent of α are given by

$$C' = \left[\left(\frac{1}{W_i^2} + \frac{1}{W^2} \right) \left(\frac{a}{2} + \frac{b_i - b_f}{i W^2} \right)^2 + \left(\frac{1}{W_f^2} + \frac{1}{W^2} \right) \left(\frac{b}{2} + \frac{(b_i - b_f)}{i W^2} \right)^2 - \frac{2}{W^2} \left(\frac{b}{2} + \frac{b_i - b_f}{i W^2} \right) \cdot \left(\frac{a}{2} + \frac{b_i - b_f}{i W^2} \right) \right] / C \quad (3.10)$$

Thus after the α -integral we now get an exponent $C' = B^2/A$ from which we can extract the coefficients of $\vec{a}^2$, $\vec{b}^2$. The different components of course behave in different ways. First consider the component of $\vec{b}$ parallel to $\vec{p}_f$ or $\vec{v}_f$. This occurs with coefficient

$$\frac{1}{CA} \left[\frac{A}{4} \left(\frac{1}{W_f^2} + \frac{1}{W^2} \right) - \left(-\frac{v_f}{2} \left(\frac{1}{W_i^2} + \frac{1}{W^2} \right) + \frac{v_i \cos \theta}{2W^2} \right)^2 \right] \quad (3.11)$$

where θ is the scattering angle between $\vec{v}_i$, $\vec{v}_f$. To leading order in $1/W^2$ we obtain for $v_i = v_f$

$$1 / \left[\left(\frac{1}{W_f^2} + \frac{1}{W^2} \right) + \left(\frac{1}{W_i^2} + \frac{1}{W^2} \right) - \frac{2}{W^2} \cos \theta \right] \quad (3.12)$$

Thus at $\theta = 0$ as before there is no broadening while at $\theta = 90^\circ$, the parallel broadening is $\sim 2/W^2$ as expected from the geometry of the problem.

If we consider the width in $\vec{b}$ orthogonal to $\vec{p}_i$ and $\vec{p}_f$ we get a rather different answer. We only have

$$\left(\frac{1}{W_f^2} + \frac{1}{W^2} \right) 1/4 / C \approx 1 / \left[\frac{1}{W_i^2} + \frac{1}{W^2} \right] \quad (3.13)$$

Thus in the directions orthogonal to the scattering plane the scattered wave packet size is independent of the incident packet size, and is given purely by the size of the scatterer. This again is exactly what we would expect from geometrical considerations. Thus parts of the incoming wave orthogonal to the scattering plane and a distance $\gg 1/W$ from the scattering plane will not be scattered.

The point of this exercise is to show, in a simple example where we know the size and shape of the scattering region, that our broadenings reproduce the crude

features of the scattering region. This gives us some confidence that, when we apply the technique to a T matrix whose space-time picture we wish to study, we will obtain correct information.

4. GENERAL CALCULATIONS OF SHIFTS AND WIDTHS

Here we develop a more general technique for computing widths without the approximations of the last section. The new method treats $|\hat{O}(\vec{a}_1, \vec{a}_2; \vec{b}_1 \dots \vec{b}_n)|^2$ as the probability that incoming packets centred at $\vec{a}_1, \vec{a}_2$ at $t = 0$ scatter into packets centred at $\vec{b}_1, \dots, \vec{b}_n$. Thus $|\hat{O}|^2$ gives a non-normalized probability distribution which can be used to calculate $\langle \vec{b}_i \rangle, \langle (\vec{b}_i - \langle \vec{b}_i \rangle)^2 \rangle$, etc., through

$$\langle \vec{b}_i \rangle = \int \vec{b}_i |\hat{O}|^2 \prod_1^n d^3 b_i / \int |\hat{O}|^2 \prod_1^n d^3 b_i \quad (4.1)$$

We write, as in (2.16),

$$0 = \int \prod_1^n d^3 q_i d^3 k_1 d^3 k_2 T(\vec{k}_1, \vec{k}_2; \vec{q}_1 \dots \vec{q}_n) \delta^4(k_1 + k_2 - \sum_1^n \vec{q}_i) \\ a_1(\vec{k}_1 - \vec{p}_1^i) a_2(\vec{k}_2 - \vec{p}_2^i) \prod_1^n b_i(\vec{q}_i - \vec{p}_i^f) e^{-i(\vec{k}_1 \cdot \vec{q}_1 + \vec{k}_2 \cdot \vec{q}_2)} e^{i \sum \vec{q}_i \cdot \vec{b}_i} \quad (4.2)$$

We write this as

$$0 \propto \int_{-\infty}^{\infty} d\alpha \int d^3 k_1 \prod_1^n d^3 q_i T(\vec{k}_1, \vec{q}_1 + \dots + \vec{q}_n - \vec{k}_1; \vec{q}_1 \dots \vec{q}_n) \\ a_1(\vec{k}_1 - \vec{p}_1^i) a_2(\vec{q}_1 + \dots + \vec{q}_n - \vec{k}_1 - \vec{p}_2^i) \prod_1^n b_i(\vec{q}_i - \vec{p}_i^f) \\ e^{-i \vec{k}_1 \cdot (\vec{q}_1 - \vec{q}_2)} e^{i \sum \vec{q}_i \cdot (\vec{b}_i - \vec{q}_2)} e^{i \alpha (\epsilon(k_1) + \epsilon(k_2) - \sum \epsilon(\vec{q}_i))} \quad (4.3)$$

Here we have eliminated $\vec{k}_2$ using the δ -function and put the energy δ -function into Fourier form.

In this formula we now assume that the dominant, sharpest cut-offs on $\vec{k}_1, \vec{q}_j$ come from a_1, b_j . In other words the target wave packet a_2 is much wider in momentum space than the projectile and the final-state trial packets. For simplicity we will ignore this dependence entirely. This then corresponds to scattering on a completely localized target. Our technique could clearly be used to reintroduce this dependence at the cost of a much longer calculation.

Because of the complexity of the problem we will introduce the other effects gradually. This will also enable us to pin down the origins of different terms. From (4.3) it is clear that the variable α has many of the properties of a time. Thus in the α exponential we first take the no-dispersion, linear, approximation in the variables $\vec{u} = \vec{k}_1 - \vec{p}_1^i, \vec{u}_j = \vec{q}_j - \vec{p}_j^f$

$$\varepsilon(\vec{k}_1) + \varepsilon(\vec{q}_1 + \vec{q}_2 + \dots + \vec{q}_n - \vec{k}_1) - \sum_i^n \varepsilon(\vec{q}_i) =$$

$$(\vec{v}_1^i - \vec{v}_1^f) \cdot \vec{u} - \sum_j (\vec{v}_j^f - \vec{v}_j^i) \cdot \vec{u}_j \quad (4.4)$$

where $\vec{v}_1^i, \vec{v}_1^f$ are the appropriate initial and final velocities defined by $\vec{v}^f = \nabla \varepsilon(\vec{p}^f)$. We will later keep the higher order terms in this expansion.

Then using the momentum shift operator we find

$$|\hat{\phi}\rangle = \int_{-\infty}^{\infty} G_0(-\vec{a}_0 + \alpha \vec{v}_0 - i \nabla_{\vec{p}_1^i}) \prod_i^n G_i(\vec{b}_i - \alpha \vec{v}_i - i \nabla_{\vec{p}_i^f})$$

$$T(\vec{p}_1^i, \sum_j \vec{p}_j^f - \vec{p}_1^i; \vec{p}_1^f) d\alpha \quad (4.5)$$

where $\vec{v}_0, \vec{v}_i$ are the relative velocities and $\vec{a}_0, \vec{b}_i$ are the relative displacements. The ∇ factors in G_0, G_i commute and only operate on the T matrix.

Now we restrict ourselves to Gaussian wave packets although our formalism applies equally well to superpositions of Gaussians

$$G_i(x) = e^{-w_i^2 x^2 / 4} \quad (4.6)$$

Then (4.5) is a Gaussian integral. To save writing we absorb the $W_i/2$ factors into $\vec{a}_i$, $\vec{v}_0$, ∇_{p_i} , etc., while carrying out formal manipulations. To further save writing we also put

$$\vec{y}_0 = -i \nabla_{p_i} i, \quad \vec{y}_j = -i \nabla_{p_j} f \quad (4.7)$$

Rather than calculating $\langle \vec{b}_i \rangle$, $\langle \vec{b}_i^2 \rangle$ we introduce a generating function. Also, rather than leaving the incoming impact parameter free when calculating the final states we average over the components perpendicular to the relative incoming velocity. The existence of components along the relative velocity vector merely corresponds to a redefinition of the collision time. So we define

$$F(\vec{b}_0, \vec{b}_1, \dots, \vec{b}_n) = \int d\vec{a}_0^\perp \prod_i^n db_i e^{\vec{a}_0^\perp \cdot \vec{b}_0} e^{\sum_j \vec{b}_j \cdot \vec{b}_j} |0|^2 \quad (4.8)$$

From (4.5) and (4.8) we see that the expression for F is a Gaussian integral. First we perform the α -integral in (4.5).

Here we use the formula

$$\int \prod_i^n dx_i e^{-[x \cdot A \cdot x + 2B \cdot x + c]} \propto \frac{e^{-D/C}}{C^{n/2}} \quad (4.9)$$

where

$$C = \det A \quad D = \det \begin{vmatrix} A & B \\ B & C \end{vmatrix}$$

Thus

$$|0| \propto \frac{e^{-D/C}}{\sqrt{C}} \cdot T$$

$$C = \underline{v}_0^2 + \sum_i^n \underline{v}_i^2 = \underline{v}^2 \quad (4.10)$$

$$D = \det \begin{vmatrix} \underline{v}^2 & \underline{v}_0 \cdot (\underline{a}_0 - \underline{t}_0) + \sum \underline{v}_j \cdot (\underline{b}_j + \underline{t}_j) \\ \underline{v}_0 \cdot (\underline{a}_0 - \underline{t}_0) + \sum \underline{v}_j \cdot (\underline{b}_j + \underline{t}_j) & (\underline{a}_0 - \underline{t}_0)^2 + \sum (\underline{b}_j + \underline{t}_j)^2 \end{vmatrix}$$

$$\frac{D}{C} = (\underline{a}_0 - \underline{t}_0)^2 + \sum (\underline{b}_j + \underline{t}_j)^2 - \frac{1}{\underline{v}^2} \left[\underline{v}_0 \cdot (\underline{a}_0 - \underline{t}_0) + \sum \underline{v}_j \cdot (\underline{b}_j + \underline{t}_j) \right]^2 \quad (4.11)$$

Then

$$F(\underline{t}_0, \underline{t}_1, \dots, \underline{t}_n) = \int e^{\underline{g}_0 \cdot \underline{t}_0} e^{\sum \underline{b}_j \cdot \underline{t}_j} e^{-\frac{D}{C}(\underline{t})} e^{-\frac{D}{C}(\underline{x})}^* \quad (4.12)$$

$$T \cdot T^* \quad d\underline{a}_0 \prod_i^n d\underline{b}_i$$

Here the $\vec{r}$'s operate on T and the $\vec{x}$'s operate on T^* . From (4.11) this is clearly Gaussian in $\vec{a}_0, \vec{b}_j$. We could integrate using (4.9) but we get a vast determinant which eventually collapses to an easy answer. Instead we finesse our way to this conclusion.

First we rewrite part of the exponent

$$e^{\frac{2}{\underline{v}^2} (\underline{v}_0 \cdot \underline{g}_0 + \underline{v}_j \cdot \underline{b}_j)^2} = \exp \left[\sqrt{\frac{2}{\underline{v}^2}} (\underline{v}_0 \cdot \underline{g}_0 + \underline{v}_j \cdot \underline{b}_j) \frac{d}{d\underline{r}} \right] e^{\underline{r}^2} \Big|_{\underline{r}=0} \quad (4.13)$$

by Maclaurin. Then calling

$$\underline{y} = 2 \left[\underline{v}_0 \cdot \underline{t}_0 - \sum_j \underline{v}_j \cdot \underline{t}_j \right]$$

$$\underline{y}' = 2 \left[\underline{v}_0 \cdot \underline{x}_0 - \sum_j \underline{v}_j \cdot \underline{x}_j \right]$$

$$F = e^{-\delta} \int da_0^\perp \pi db_j \exp[-2(q_0^2 + \sum b_j^2) + q_0 \cdot A_0 + \sum b_j \cdot B_j] \\ T \cdot T^* e^{\Gamma^2} \Big|_{\Gamma=0}$$

where

$$\delta = \underline{z}_0^2 + \sum_j \underline{z}_j^2 - \gamma^2/4\underline{v}^2 \\ + \underline{x}_0^2 + \sum_j \underline{x}_j^2 - \gamma'^2/4\underline{v}^2$$

$$A_0 = 2(\underline{z}_0 + \underline{x}_0^*) - \underline{v}_0(\gamma + \gamma'^*)/\underline{v}^2 + \sqrt{\frac{2}{\underline{v}^2}} \underline{v}_0 \frac{d}{d\Gamma} + \underline{l}_0$$

$$B_j = -2(\underline{z}_j + \underline{x}_j^*) - \underline{v}_j(\gamma + \gamma'^*)/\underline{v}^2 + \sqrt{\frac{2}{\underline{v}^2}} \underline{v}_j \frac{d}{d\Gamma} + \underline{l}_j \quad (4.14)$$

In this expression for F we only integrate over the components of a_0 perpendicular to the relative incoming central velocity $\vec{v}_0$. Thus in (4.14) we may ignore the $d/d\Gamma$ term in A_0 .

Then

$$F = e^{-\delta} e^{\sum_j B_j^2/8} e^{A_0^2/8} T \cdot T^* e^{\Gamma^2} \Big|_{\Gamma=0} \quad (4.15)$$

dropping inconsequential constants of proportionality. Now B_j^2 is quadratic in $d/d\Gamma$ so we need to know

$$e^{\alpha \frac{d}{d\Gamma} + \beta \frac{d^2}{d\Gamma^2}} e^{\Gamma^2} = e^{\beta \frac{d^2}{d\Gamma^2}} e^{(\Gamma + \alpha)^2} \\ = e^{\alpha^2} \int d\gamma e^{-\sqrt{\beta} \gamma \frac{d}{d\Gamma}} e^{-\gamma^2/4} / \sqrt{4\pi}$$

$$= e^{\alpha^2/(1-4\beta)/\sqrt{1-4\beta}} \quad (4.16)$$

all evaluated of course with $\Gamma = 0$.

In our application

$$1-4\beta = 1 - \sum_i^N v_j^2/v^2 = \frac{v_0^2}{v^2} \geq 0 \quad (4.17)$$

while

$$\alpha = -\frac{1}{4} \left[\sum_j v_j \cdot \left(2(\underline{v}_j + \underline{x}_j^*) + v_j(\gamma + \gamma'^*)/v^2 - \underline{e}_j \right) \right] \sqrt{\frac{2}{v^2}} \quad (4.18)$$

Then

$$F = e^{-\delta} e^{\alpha^2/(1-4\beta)} e^{\gamma} T \cdot T^*$$

where

$$\begin{aligned} 8\gamma &= \left[\underline{e}_0 + 2(\underline{v}_0 + \underline{x}_0^*) - \underline{v}_0(\gamma + \gamma'^*)/v^2 \right]^2 \\ &+ \sum_j \left[\underline{e}_j - 2(\underline{v}_j + \underline{x}_j^*) - \underline{v}_j(\gamma + \gamma'^*)/v^2 \right]^2 = A_0^2 + \underline{\beta}_j^2 \end{aligned} \quad (4.19)$$

Thus we now have a formal closed expression for our generating function involving only gradient operators acting on the two T matrices. We now expand in powers of gradients acting on the T matrices. Remembering that we originally scaled our $\vec{r}_j$ operators by factors of W_j this gives us an expansion in terms of gradients of the T matrix over gradients of the wave packets in momentum space. For the large wave packets relevant to scattering theory this should be a convergent expansion.

Thus let us now calculate

$$\langle \underline{b}_j \rangle = \nabla_{\underline{z}_j} F / F |_{\underline{z} = 0} \quad (4.20)$$

$$= \left[\frac{B_j}{4} + \frac{v_j}{4} \left[-2 \sum_l \left(\frac{v_l}{v_0} (\underline{z}_l + \underline{z}_l^*) + \frac{v_l^2}{v_0^2} (\gamma + \gamma'^*) / v^2 \right) \right] / v_0^2 \right] e^{-\delta} e^{\alpha^2 / (1-4\beta)} e^{-Y} T \cdot T^* / F |_{\underline{z} = 0} \quad (4.21)$$

where B_j is as in (4.14) without the $d/d\Gamma$ term. This is already linear in gradient terms so we drop δ , α , Y .

$$\begin{aligned} \langle \underline{b}_j \rangle &= \frac{1}{4} \left[(-2(\underline{z}_j + \underline{z}_j^*) - \frac{v_j}{v_0} (\gamma + \gamma'^*) / v^2) \right. \\ &\quad \left. - \frac{2v_j}{v_0^2} \left[\sum_l \frac{v_l}{v_0} (\underline{z}_l + \underline{z}_l^*) + \frac{v_l^2}{v_0^2} (\gamma + \gamma'^*) / v^2 \right] \right] T \cdot T^* / |T|^2 \\ &= \frac{1}{4} \left[-2(\underline{z}_j + \underline{z}_j^*) - 2v_j \left(\frac{v_0}{v_0^2} (\underline{z}_0 + \underline{z}_0^*) \right) / v_0^2 \right] T \cdot T^* / |T|^2 \end{aligned} \quad (4.22)$$

This is incidentally the same answer we would have obtained if we had simply set $a_0^T = 0$ in (4.12). If we now write $T = R e^{i\chi}$ we easily find that

$$\langle \underline{b}_j \rangle = - \frac{v_j}{v_0^2} \underline{v}_0 \cdot \nabla_0 \chi - \nabla_j \chi \quad (4.23)$$

This confirms our results of Section 2 that shifts are given purely by the phase variation of the T matrix. Notice, on scaling out our variables by W_0 , W_j , etc., that (4.23) is correct for the original variables of our problem. One may also check the far from obvious result that (4.23) has no corrections to order v^2 . We also stress that for wave packets big enough in position space there is no wave packet dependence in (4.23).

The first term in (4.23) is only non-zero if $\nabla_0 \chi$ has a component parallel to the incoming relative momentum and it alters the outgoing $\vec{b}_j$ by a term proportional to the velocity of $\vec{b}_j$. It therefore clearly corresponds to a shift in the time origin of the problem. The second $\nabla_j \chi$ term gives the total "impact" parameter of the final-state particle. This result then agrees totally with the results of Section 2.

We now calculate the variance of the variables $\vec{b}_j$ using our probability distribution. Thus for definiteness we pick an arbitrary fixed unit vector $\vec{e}$ and calculate

$$\sigma_j^2 = \langle (\underline{e} \cdot \underline{b}_j)^2 \rangle - \langle \underline{e} \cdot \underline{b}_j \rangle^2 \quad (4.24)$$

Now

$$\langle (\underline{e} \cdot \underline{b}_j)^2 \rangle = \left[(\underline{e} \cdot \underline{\nabla}_{\underline{r}_j})^2 F \right] / F \Big|_{\underline{r}=0} \quad (4.25)$$

$$= \left\{ \frac{1}{4} \left(1 + \frac{\underline{e} \cdot \underline{v}_j}{v_0^2} \right) + \frac{1}{16} \left[\underline{E}_j \cdot \underline{e} + (\underline{v}_j \cdot \underline{e}) \sum \underline{v}_j \cdot \underline{E}_j / v_0^2 \right]^2 \right\} F / F \quad (4.26)$$

where $\vec{E}_j = 2(\vec{r}_j + \vec{x}_j^*) + \vec{v}_j(\vec{\gamma} + \vec{\gamma}^*)/\vec{v}^2$. The first term in the curly bracket is a constant multiple and so gives rise to a contribution

$$\frac{1}{4} \left[1 + (\underline{e} \cdot \underline{v}_j)^2 / v_0^2 \right]$$

which on scaling all the variables back to their original values gives a contribution to $\langle (\vec{e} \cdot \vec{b}_j)^2 \rangle$ of

$$\frac{1}{W_j^2} + \frac{(\underline{e} \cdot \underline{v}_j)^2}{v_0^2} \frac{1}{W_0^2} \quad (4.27)$$

We stress that this involves no approximation in the calculation of numerator or denominator for this term.

This is the equivalent of the widths found in Section 3, which depend simply on the geometry of the problem and are T matrix independent. In this approximation $\langle \vec{b}_j \rangle = 0$ so that (4.28) really gives the variance. From (4.26) it is clear that this term is always there and adds to the "dynamical contribution" of the second term which always involves at least two derivatives of the T matrix. We ignore higher derivatives.

We now calculate the second term in (4.26). It is clear from the form of (4.26) that the explicit terms in (4.26) give rise to W_1^2 independent contributions to $\langle (\vec{e} \cdot \vec{b}_j)^2 \rangle$ provided F is then treated in the $W_1 \rightarrow \infty$ limit. In other words we calculate these terms with the F's evaluated as $T \cdot T^*$. This means that we must simply evaluate the second bracket in (4.26) applied to $T \cdot T^*$. However, on substituting for $\underline{E}_j, \underline{\gamma}$ in this bracket, we obtain simply

$$4 \left[\frac{\underline{e} \cdot \underline{v}_j}{v_0^2} \underline{v}_0 \cdot (\underline{v}_0 + \underline{x}_0^*) + \underline{e} \cdot (\underline{v}_j + \underline{x}_j^*) \right]^2 T \cdot T^* / R^2 \quad (4.28)$$

where we again write $T = R e^{i\chi}$, R, χ real.

Now introduce the notation $(\vec{v}_0 \cdot \vec{r}_0)R = -iR_0$, $(\vec{e} \cdot \vec{r}_j)R = -iR_j$, and similarly for χ . From (4.28) and (4.22) we see that these are the only derivatives actually required in $\langle (\vec{e} \cdot \vec{b}_j)^2 \rangle$ and $\langle \vec{e} \cdot \vec{b}_j \rangle^2$.

A tedious straightforward calculation then reveals that

$$\begin{aligned} \sigma_j^2 = & \frac{1}{w_j^2} + \frac{(\underline{e} \cdot \underline{v}_j)^2}{v_0^2} \frac{1}{w_0^2} - \frac{1}{2} \frac{(\underline{e} \cdot \underline{v}_j)^2}{v_0^2} \frac{R_{00}}{R} + \frac{1}{2} \frac{(\underline{e} \cdot \underline{v}_j)^2}{v_0^2} \frac{R_0^2}{R^2} \\ & - \frac{1}{2} \frac{R_{jj}}{R} + \frac{1}{2} \frac{R_j^2}{R^2} - \frac{\underline{e} \cdot \underline{v}_j}{v_0} \frac{R_{j0}}{R} + \frac{\underline{e} \cdot \underline{v}_j}{v_0} \frac{R_0 R_j}{R^2} \end{aligned} \quad (4.29)$$

Remarkably all the phase information in ϕ is now irrelevant because of cancellations.

We now return to the approximation introduced in (4.4). We have seen that our dispersion involves second derivatives of the T matrix and we may wonder at the effects neglected by ignoring the dispersive terms in the Fourier transform of the energy conservation δ function.

Thus we return to (4.3) and rewrite

$$\begin{aligned} & \exp \left(i \alpha \left(\varepsilon(\underline{k}_i) + \varepsilon(\underline{q}_i + \dots + \underline{q}_n - \underline{k}_i) - \sum_i^n \varepsilon(\underline{q}_i) \right) \right) \\ & = \exp \left[i \alpha (\underline{v}_0 \cdot \underline{u}_0 - \sum \underline{v}_j \cdot \underline{u}_j) \right] \exp \left(i \alpha \frac{\partial}{\partial \underline{r}} \right) \exp \left(\sum_0^n \underline{u}_i \cdot \nabla_{\underline{r}_i} \right) \\ & \exp \left[\gamma \left(f^B(\eta_0) - \sum_i^n f^j(\eta_j) + f^T(\underline{u}_1 + \dots + \underline{u}_n - \underline{u}_0) \right) \right] \Big|_{\gamma, \underline{r}=0} \end{aligned} \quad (4.30)$$

Here f^B , f^T , f^j are the terms beyond the first two in the Taylor series expansions of the beam, target, and final-state energies about their central momenta. In principle, these terms are at least cubic in α , $\vec{u}_i$ and so wreck our Gaussian integrals. Hence we rewrite them using the above operator identities. Thus our overlap integral is altered to the form

$$\begin{aligned} & \int d\alpha e^{i\alpha \frac{\partial}{\partial \underline{r}}} G_0(-\underline{q}_0 + \alpha \underline{v}_0 - i \nabla_{\underline{p}_i} - i \nabla_{\underline{r}_0}) \prod_i^n G_j(\underline{b}_j - \alpha \underline{v}_j - i \nabla_{\underline{p}_j} - i \nabla_{\underline{r}_j}) \\ & e^{\gamma(f^B + f^T - \sum_i^n f^j(\eta_j))} \cdot T \cdot T^* \end{aligned} \quad (4.31)$$

Thus we have an extra linear term in the α -integral and we join $\nabla_p + \nabla_\eta$ and write only ∇_p for the moment.

Thus in (4.10) there is no change, but D becomes

$$D' = D + \frac{1}{4} \frac{\partial^2}{\partial \tau^2} - i \left[\frac{v_0}{v} (a_0 - t_0) + \sum \frac{v_j}{v} (b_j + t_j) \right] \frac{\partial}{\partial \tau} \quad (4.32)$$

Then in $|0\rangle^2$ the coefficient of $\vec{b}_j$ picks up a term

$$\frac{i v_j}{v^2} \left(\frac{\partial}{\partial \tau} - \frac{\partial}{\partial \tau'} \right) \quad (4.33)$$

and B_j of (4.14) picks up corresponding corrections. In the $\vec{b}_j$ integrals (4.33) clearly acts as a constant and the correct answer is obtained on letting

$$\underline{b}_j \rightarrow \underline{b}_j + \frac{i v_j}{v^2} \left(\frac{\partial}{\partial \tau} - \frac{\partial}{\partial \tau'} \right)$$

in (4.18), (4.19). Thus finally we obtain

$$\begin{aligned} F = & \exp \left[-\frac{1}{4v^2} \left(\frac{\partial^2}{\partial \tau^2} - \frac{\partial^2}{\partial \tau'^2} \right) \right] \exp \left[-\frac{i}{2v^2} \left(\tau \frac{\partial}{\partial \tau} - \tau' \frac{\partial}{\partial \tau'} \right) \right] \\ & e^{-\delta} \exp \left[\frac{1}{8} \sum_i \dot{c}_i^2 + \frac{1}{8v_0^2} \left(\sum_i \underline{c}_i \cdot \underline{v}_i \right)^2 \right] \exp \left[\tau (f^B + f^T - \sum_j f_j) \right] \\ & \exp \left[\tau' (f^{B'} + f^{T'} - \sum_j f'_j) \right] T \cdot T^* \Big|_{\tau=\eta=\eta'=0} \end{aligned} \quad (4.34)$$

Now in the above we have implicitly scaled $\vec{r}_0$, ∇_{p_i} , ∇_{η_i} by $W_0/2$, $W_1/2$, etc. Thus if we only keep the quadratic term in $f^B(\eta) = \varepsilon_{\alpha\beta}^B \eta_\alpha \eta_\beta \cdot W_0^2/4$, we must now explicitly show the W_0^2 dependence. From this and our subsequent calculations it is clear that higher order terms in the expansions of the energies will not contribute to our order W^2 answer. We therefore ignore them.

Then if we are only interested in calculating the $O(W^2)$ corrections to F from these terms, we may expand the last two exponents in (4.34) to first order only. Then from the previous exponents we must extract one τ derivative and two η derivatives. We may treat the f^B , f^T , etc., terms independently one at a time. Moreover, since we are only interested in moments up to the second, we shall only keep track of terms up to second order in the $\vec{b}_j$.

We keep track of the switch from primed to unprimed η 's and τ by a factor σ ($\sigma^2 = 1$) for each term odd in the switch. For ease we rewrite

$$\begin{aligned}
 \underline{C}_i &= 2i(\nabla_i - \nabla'_i) + \underline{f}_i + \frac{i \underline{v}_i}{v^2} \left(\frac{\partial}{\partial \tau} - \frac{\partial}{\partial \tau'} \right) - \frac{1}{v^2} \underline{v}_i (\gamma - \gamma') \\
 \delta &= \sum_0^n (\nabla_i^2 + \nabla_i'^2) + \frac{\gamma^2 + \gamma'^2}{4v^2} \\
 \gamma &= 2i \left(\sum_1^n \underline{v}_i \cdot \nabla_i - \underline{v}_0 \cdot \nabla_0 \right), \quad \nabla_i = \nabla_{p_i} + \nabla_{q_i} = \bar{\nabla}_i + \nabla_i
 \end{aligned} \tag{4.35}$$

Then we may write for the $\nabla_{\eta_0} = \nabla_0$ terms

$$\begin{aligned}
 \gamma &= \gamma - 2i \underline{v}_0 \cdot \nabla_0 \\
 \underline{C}_i &= \underline{C}_i + \frac{2i \underline{v}_i}{v^2} (\underline{v}_0 \cdot \nabla_0) - (\underline{v}_0 \cdot \nabla_0)^2 \\
 \delta &= \delta + 2 \bar{\nabla}_0 \cdot \nabla_0 - i \frac{\gamma \underline{v}_0 \cdot \nabla_0}{v^2} - (\underline{v}_0 \cdot \nabla_0)^2
 \end{aligned} \tag{4.36}$$

Thus we obtain the linear $\partial/\partial \tau$ term with coefficient

$$\begin{aligned}
 &\left[-\frac{i\gamma}{2v^2} + \frac{1}{4} \frac{i\sigma A}{v^2} + \frac{1}{4} \frac{iAB\sigma}{v^2 v_0^2} \right] + \left[-\frac{1}{v^2} - \frac{(\sum v_i^2)^2}{2v_0^2 v^4} - \frac{1}{2} \frac{(\sum v_i^2)}{v^4} \right] \underline{v}_0 \cdot \nabla_0 \\
 &= X + Y (\underline{v}_0 \cdot \nabla_0)
 \end{aligned} \tag{4.37}$$

where $A = \sum \vec{C}_i \cdot \vec{v}_i$, $B = \sum_1^n v_i^2 = v^2 - v_0^2$. The first term must then pick up two η_0 derivatives, the second one from

$$\begin{aligned}
 (\sum C_i^2) &= (\sum C_i'^2) + \frac{4i\sigma}{v^2} A \underline{v}_0 \cdot \nabla_0 - \frac{4B}{v^4} (\underline{v}_0 \cdot \nabla_0)^2 \\
 (\sum \underline{C}_i \cdot \underline{v}_i)^2 &= (\sum \underline{C}_i' \cdot \underline{v}_i')^2 + \frac{4i\sigma AB}{v^2} (\underline{v}_0 \cdot \nabla_0) - \frac{4B^2}{v^4} (\underline{v}_0 \cdot \nabla_0)^2 \\
 \delta &= \delta + 2 \bar{\nabla}_0 \cdot \nabla_0 + \nabla_0^2 - \frac{(\underline{v}_0 \cdot \nabla_0)^2}{v^2} - i \frac{\gamma (\underline{v}_0 \cdot \nabla_0)}{v^2}
 \end{aligned} \tag{4.38}$$

Thus collecting the linear and quadratic terms

$$\begin{aligned}
 E^\alpha \nabla_0^\alpha &= \left[\frac{i\sigma A \underline{v}_0 \cdot \nabla_0}{2v_0^2} + 2 \bar{\nabla}_0 \cdot \nabla_0 - i \frac{\gamma \underline{v}_0 \cdot \nabla_0}{v^2} \right] \\
 F^{\alpha\beta} \nabla_0^\alpha \nabla_0^\beta &= \left[-\frac{1}{2} \frac{B (\underline{v}_0 \cdot \nabla_0)^2}{v^2 v_0^2} + \nabla_0^2 - \frac{(\underline{v}_0 \cdot \nabla_0)^2}{v^2} \right]
 \end{aligned} \tag{4.39}$$

Before computing this in ghastly detail, we first explore what we need to know. We have schematically

$$F = [1 + w^2 [A \ell_i^2 + B \ell_i + C]] F_{old} \quad (4.40)$$

since in the above only C_i is linear in $\vec{\ell}_i$. We write $F_{old} = \exp(\Omega)$

$$\begin{aligned} \langle \underline{b}_i \rangle &= \frac{\frac{\partial \Omega}{\partial \ell_i} (1 + w^2 c) F_{old}}{(1 + w^2 c) F_{old}} + \frac{B w^2 F_{old}}{(1 + w^2 c) F_{old}} \\ &= \langle \underline{b}_i \rangle_{old} + o(w^2) \\ \langle \underline{b}_i^2 \rangle &= \frac{[\frac{\partial^2 \Omega}{\partial \ell_i^2}] (1 + w^2 c) F_{old}}{(1 + w^2 c) F_{old}} + \frac{(\frac{\partial \Omega}{\partial \ell_i})^2 (1 + w^2 c) F_{old}}{(1 + w^2 c) F_{old}} \\ &\quad + \frac{2 B w^2 (\frac{\partial \Omega}{\partial \ell_i}) F_{old}}{(1 + w^2 c) F_{old}} + \frac{A w^2 F_{old}}{(1 + w^2 c) F_{old}} \\ &= \frac{\partial^2 \Omega}{\partial \ell_i^2} + \left(\frac{\partial \Omega}{\partial \ell_i} \right)^2 [1 + o(w^2)] + w^2 \left(\frac{\partial \Omega}{\partial \ell_i} \right) B + \frac{A w^2 F_{old}}{F_{old}} \end{aligned} \quad (4.41)$$

Thus on scaling out the ℓ_i , etc., by W^2 , we find

$$\langle \underline{b}_i^2 \rangle \cong \frac{X}{w^2} + \left[\left(\frac{\partial \Omega}{\partial \ell_i} \right)^2 + A \right] + o(w^3) \quad (4.42)$$

Thus we only need to know A, i.e. the quadratic $\vec{\ell}_i$ terms in our expansion. The crucial point is that $\partial^2 \Omega / \partial k_i^2$ is a number, *not* an operator, and so in the first term $(1 + W^2 C) F_{old}$ cancels between top and bottom. Thus in (4.37) and (4.39) only the A terms contain $\vec{\ell}_i$. Now we have schematically terms

$$X (F + \frac{1}{2} E^2) + Y E \quad (4.43)$$

E is only linear in $\vec{\ell}_i$ and Y is independent so we drop the last term. Similarly F is independent of $\vec{\ell}_i$ so the only important terms come from $+(1/2) X E^2$. Keeping only terms independent of σ , since odd terms cancel, we obtain

$$\frac{-i \gamma}{2 v^2} \left[- \frac{A^2 v_0^\alpha v_0^\beta}{8 v_0^4} - \frac{A^2 v_0^\alpha v_0^\beta}{4 v_0^4} \right] - \frac{1}{4} \frac{A^2}{v_0^4} \bar{\nabla}_0^\beta v_0^\alpha \quad (4.44)$$

This then gives a contribution to $\langle (\vec{e} \cdot \vec{b}_1)^2 \rangle$ of

$$\frac{W^2}{4} \left[\frac{3}{4} \frac{(i\gamma)(\underline{e} \cdot \underline{v}_i)^2 v_0^\alpha v_0^\beta \varepsilon_{\alpha\beta}^0}{v^2 v_0^4} - \frac{(\underline{e} \cdot \underline{v}_i)^2}{v_0^4} \varepsilon_{\alpha\beta}^0 v_0^\alpha \bar{\nabla}_0^\beta \right] \quad (4.45)$$

For the terms with $\varepsilon_{\alpha\beta}^m \eta_m^\alpha \eta_m^\beta$, where α, β are Lorentz labels and m labels the m 'th particle

$$\begin{aligned} \gamma &= \gamma + 2i \underline{v}_m \cdot \nabla_m \\ \underline{C}_i &= \underline{C}_i + 2i\sigma \nabla_m \delta_m^i - \frac{2i\sigma \underline{v}_i (\underline{v}_m \cdot \nabla_m)}{v^2} + \frac{i \underline{v}_i}{v^2} \sigma \frac{\partial}{\partial t} \\ \delta &= \delta + 2\bar{\nabla}_m \cdot \nabla_m + \nabla_m^2 + i\gamma (\underline{v}_m \cdot \nabla_m) - \frac{(\underline{v}_m \cdot \nabla_m)^2}{v^2} \end{aligned} \quad (4.46)$$

The linear τ derivatives give

$$\begin{aligned} & -\frac{i}{2v^2} (\gamma + 2i \underline{v}_m \cdot \nabla_m) + \frac{1}{4} \sum_i \frac{i\sigma \underline{v}_i}{v^2} \left[\underline{C}_i + 2i\sigma \nabla_m \delta_m^i - \frac{2i\sigma \underline{v}_i (\underline{v}_m \cdot \nabla_m)}{v^2} \right] \\ & + \frac{1}{8v_0^2} \left[\sum \underline{C}_i \cdot \underline{v}_i + 2i\sigma \underline{v}_m \cdot \nabla_m - \frac{2i\sigma (\sum \underline{v}_i^i)}{v^2} \underline{v}_m \cdot \nabla_m \right] \left[\frac{2i(\sum \underline{v}_i^i)\sigma}{v^2} \right] \end{aligned} \quad (4.47)$$

The exponents are now expanded in terms of the η derivatives

$$\begin{aligned} (\sum \underline{C}_i^2) &= (\sum \underline{C}_i^1) - 4\sigma i \sum \frac{(\underline{v}_i \cdot \underline{C}_i) \underline{v}_m \cdot \nabla_m}{v^2} + 4i \underline{C}_m \cdot \nabla_m \sigma \\ & - 4 \frac{(\sum \underline{v}_i^i)}{v^4} (\underline{v}_m \cdot \nabla_m)^2 + \frac{8(\underline{v}_m \cdot \nabla_m)^2}{v^2} - 4 \nabla_m^2 \\ (\sum \underline{v}_i \cdot \underline{C}_i)^2 &= (\sum \underline{v}_i \cdot \underline{C}_i)^2 + 2(\sum \underline{v}_i \cdot \underline{C}_i) \left[2i\sigma \underline{v}_m \cdot \nabla_m - \frac{2i\sigma (\sum \underline{v}_i^i)}{v^2} \underline{v}_m \cdot \nabla_m \right] \\ & - 4(\nabla_m \cdot \underline{v}_m) - 4 \frac{(\sum \underline{v}_i^i)^2}{v^4} (\underline{v}_m \cdot \nabla_m)^2 + 8(\underline{v}_m \cdot \nabla_m)^2 \frac{(\sum \underline{v}_i^i)}{v^2} \end{aligned} \quad (4.48)$$

As before, we may drop the ∇_m terms in (4.47) since they are $\vec{x}_i$ independent. The quadratic ∇ terms in δ and (4.48) may also be dropped as before.

Thus the equivalent of A in (4.40) is contained in

$$\begin{aligned} & \left[-\frac{i\gamma}{2v^2} + \frac{1}{4} \frac{iA\sigma}{v_0^2} \right] \left[\frac{1}{2} i\sigma C_m^\alpha + 2\nabla_m^\alpha + i\gamma \frac{v_m^\alpha}{v^2} \right] \\ & \cdot \left[\frac{1}{2} i\sigma C_m^\beta + 2\bar{\nabla}_m^\beta + i\gamma \frac{v_m^\beta}{v^2} \right] \end{aligned} \quad (4.49)$$

Then remembering that $A = \sum_1^n \vec{C}_i \cdot \vec{v}_i = \sum_1^n \vec{C}_i \cdot \vec{v}_i + \sum_1^n \vec{v}_i \cdot \vec{\ell}_i$, and keeping only the $\vec{\ell}_i$ terms, we see that the only quadratic terms have $m = 1$. Thus we finally get a contribution $\langle (\vec{e} \cdot \vec{b}_m)^2 \rangle$ of

$$\left[\frac{1}{2} i \gamma \epsilon^\alpha \epsilon^\beta - \frac{1}{v_0^2} (\sum v_i \cdot \ell_i) \epsilon^\alpha \left[2 \bar{v}_m^\beta + i \frac{\gamma v_m^\alpha}{v^2} \right] \right] \epsilon_{\alpha\beta}^m \quad (4.50)$$

Here γ is the operator defined in (4.35), and (4.50) operates on $T \cdot T^*$ with a similar term coming from the η primed contribution.

In (4.34) the target contribution $f^T(\eta_1 + \eta_2 + \dots + \eta_n - \eta_0)$ gives rise to various contributions. For $f^T(x) = \epsilon_{\alpha\beta}^T x^\alpha x^\beta$, we clearly obtain diagonal and off-diagonal terms in the various η 's. The diagonal terms $\epsilon_{\eta_m \eta_m}^T$ we absorb in $\epsilon_{\alpha\beta}^m$ of (4.50) and $\epsilon_{\eta_0 \eta_0}^T$ in the $\epsilon_{\alpha\beta}^0$ of (4.45). We must now calculate the effect of the off-diagonal terms.

We shall compute the effects of these terms only in the limit where $W_0 \ll W_j$. This is purely to simplify our rather complex formulae. Using our scaled variables this means that $|\vec{v}_0| \gg |\vec{v}_j|$, $v_0 \gg v_j$, etc. Notice that since the W_0, W_j have scales which are completely independent of all the scales given by the T matrices, we can always insist on this result.

As before, since we are only interested in the $\langle b_i^2 \rangle$ terms, we need only look for $\vec{\ell}_i^2$ terms in our corrections. Thus we only need to keep the terms

$$\left[-\frac{i\gamma}{2v^2} + \frac{1}{4} \frac{iA}{v_0^2} \right] E_\alpha^\nu E_\beta^s \quad (4.51)$$

where

$$\begin{aligned} E_\alpha^0 &= \left(\frac{i\sigma A}{2v_0^2} v_0^\alpha + 2\bar{v}_0^\alpha - \frac{2(v_0 \cdot \bar{v}_0) v_0^\alpha}{v_0^2} \right) \\ E_\alpha^\nu &= \left(\frac{1}{2} i\sigma C_\nu^\alpha + 2\bar{v}_\nu^\alpha + \frac{2(v_\nu \cdot \bar{v}_0) v_\nu^\alpha}{v_0^2} \right) \end{aligned} \quad (4.52)$$

We consider the $\vec{\ell}_i^2$ terms but coming in principle from arbitrary r, s in (4.51) corresponding to $\epsilon_{\eta^r \eta^s}^T$ terms in f^T . First we note that $\vec{C}_r$ only contains $\vec{\ell}_r$. Thus, if neither r nor s is zero, from (4.52) we see that one at least must equal i . Then we consider $r = i, s \neq i$, which implies that we need an $\vec{\ell}_i$ from the A in the first bracket of (4.51). Then the answer

$$\approx \frac{v_i v_i}{v_0^2} \cdot 1 \cdot C_s \quad (4.53)$$

If $r = 0$, $s \neq i$, then we must pick up ℓ_i from the first two brackets in (4.51)

$$\approx \frac{v_i}{v_0^2} \frac{v_i}{v_0^2} v_0 \bar{\nabla}_s \quad (4.54)$$

If $r = 0$, $s = i$, then we pick up ℓ_i from the third term of (4.51) and the first or second

$$- \frac{v_0 \cdot \bar{\nabla}_0}{v_0^2} \cdot i \frac{v_i^\gamma v_0^\alpha}{v_0^2} \frac{1}{2} i \varepsilon_{\alpha\gamma}^T \quad (4.55)$$

$$+ \frac{1}{4} \frac{i v_i^\gamma}{v_0^2} \left(\frac{1}{2} i \right) \left(2 \bar{\nabla}_0^\alpha + 2 \frac{(v_0 \cdot \bar{\nabla}_0) v_0^\alpha}{v_0^2} \right) \varepsilon_{\alpha\gamma}^T$$

Counting powers of W_j versus W_0 , it is clear that this last term dominates over (4.53) and (4.54).

Then for $W_T \gg W_B \gg W_\ell$ we find

$$\begin{aligned} \langle (\underline{e} \cdot \underline{b}_j)^2 \rangle - \langle \underline{e} \cdot \underline{b}_j \rangle^2 &= \frac{4}{W_j^2} + \frac{4}{W_B^2} \frac{[\underline{e} \cdot (\underline{v}_j - \underline{v}_T)]^2}{(\underline{v}_B - \underline{v}_T)^2} \\ &- \frac{1}{2} \left[\frac{\underline{e} \cdot (\underline{v}_j - \underline{v}_T)}{(\underline{v}_B - \underline{v}_T)^2} (\underline{v}_B - \underline{v}_T) \cdot (\underline{v}_B - \underline{v}_T) + \underline{e} \cdot \underline{v}_j + \underline{e} \cdot \underline{v}_T \right]^2 \ln R \\ &- \frac{(\underline{v}_B - \underline{v}_T) \cdot (\underline{v}_B - \underline{v}_T) \ln R}{2 (\underline{v}_B - \underline{v}_T)^2} \left[\underline{e} \cdot [\underline{\varepsilon}^j - \underline{\varepsilon}^T] \cdot \underline{e} \right] \\ &+ \frac{3}{2} \frac{[\underline{e} \cdot (\underline{v}_j - \underline{v}_T)]^2}{(\underline{v}_B - \underline{v}_T)^6} \left[(\underline{v}_B - \underline{v}_T) \cdot [\underline{\varepsilon}^T + \underline{\varepsilon}^B] \cdot (\underline{v}_B - \underline{v}_T) \right] (\underline{v}_B - \underline{v}_T) \cdot (\underline{v}_B - \underline{v}_T) \ln R \\ &- \frac{[\underline{e} \cdot (\underline{v}_j - \underline{v}_T)]^2}{(\underline{v}_B - \underline{v}_T)^4} (\underline{v}_B - \underline{v}_T) \cdot [\underline{\varepsilon}^B + \underline{\varepsilon}^T] \cdot (\underline{v}_B - \underline{v}_T) \ln R \\ &- 2 \frac{[\underline{e} \cdot (\underline{v}_j - \underline{v}_T)]}{(\underline{v}_B - \underline{v}_T)^4} \left[(\underline{v}_B - \underline{v}_T) \cdot (\underline{v}_B - \underline{v}_T) \ln R \right] (\underline{v}_B - \underline{v}_T) \cdot \underline{\varepsilon}^T \cdot \underline{e} \\ &+ \frac{\underline{e} \cdot (\underline{v}_j - \underline{v}_T)}{(\underline{v}_B - \underline{v}_T)^2} \underline{e} \cdot \underline{\varepsilon}^T \cdot (\underline{v}_B - \underline{v}_T) \ln R \end{aligned} \quad (4.56)$$

Here B,T refer to beam and target velocities or momentum derivatives. We have written the dispersion in dyadic form.

The wicked-looking complexity of this formula is necessary to guarantee that a reshuffling of the three momenta and energies in T, using four-momentum conservation, leaves our widths, etc., unaltered. This is verified in Appendix A. The formula must also guarantee that multiplying the T matrix by a factor equal to one on energy conservation shell leaves the shifts and widths unchanged.

We can also derive the results for the conceptually simpler problem of scattering on a fixed massive scattering centre by letting $\vec{v}_T \rightarrow 0$ and $\epsilon^T \rightarrow 0$.

5. TWO APPLICATIONS

We would first like to apply our formalism to the simple multiperipheral graph of Fig. 1. We evaluate this in old-fashioned perturbation theory and insist on keeping track of the contribution to $\log R$ coming from the intermediate state A

$$- \log [E(p) - E(p-q_1) - E(q_1)] \quad (5.1)$$

Since this does not involve the target we feel justified in pretending that the target is a fixed massive scattering centre so that in (4.56) we drop the ϵ^T and $\vec{v}^T$ terms. We emphasize that the argument of the log is supposed to be $1/t$ where t is the time scale of the problem. To further simplify the problem we assume that we have only forward scattering, or, in other words, we have only a one-dimensional problem.

Before starting we have, of course, the knowledge that at high energies the covariant T matrix of diagram 1 is a constant. Thus (4.56) gives zero in the leading expansion of the T matrix. As we will see, this is expected because of relativistic effects even for a large time scale being involved in the problem.

In order to see this, let us stress the type of contributions that we expect on scattering on a target of large size. As in Section 3, we expect simple geometrical size changes. However, even in cases where there is only forward scattering, we expect size changes at a naive level. Thus if an incoming packet of velocity v_i changes into a packet of velocity v_f , we expect the position space size to be scaled by a factor v_i/v_f . This is seen in formula (4.56) as the factor multiplying $1/W_B^2$. This is, of course, independent of the T matrix and only depends on the final-state momenta which we are studying.

Moreover, if the transition from $v_i \rightarrow v_f$ takes place somewhere in a region of size $(v_I \cdot t)$, where we guess v_I to be the velocity of the intermediate-state, virtual particle q_1 in Fig. 1, then we expect a positive contribution to the width

$$\simeq \left(\frac{v_i - v_f}{v_i} \right) v_I t \quad (5.2)$$

This arises from the fact that the particle v_f may be produced at the front or back of a region of size $v_I \cdot t$. Thus we expect this to be an additive width.

This should show up in our formalism for $\langle (\vec{e} \cdot \vec{b})^2 \rangle$ as a positive contribution proportional to t^2 . Thus in (4.56) it will only occur in the terms where two ∇ 's operate on separate R factors. Thus inserting (5.1) in (4.56) and ignoring all $\nabla^2 E \approx \epsilon$ terms

$$\begin{aligned} \langle (\vec{e} \cdot \vec{b}_i)^2 \rangle &= \frac{4}{W_i^2} + \frac{4}{W_B^2} \frac{(\vec{e} \cdot \vec{v}_i)^2}{v_B^2} \\ &\quad - \frac{t^2}{2} \left(1 - \frac{v_i}{v_B} \right)^2 v_I^2 \end{aligned} \quad (5.3)$$

This is precisely what we expect, except that the sign is wrong. Thus although the $v_i - v_f$ velocity difference gives rise to an effect associated with a large time scale, the sign is exactly wrong for a simple geometrical interpretation.

Now we turn to the terms where t only occurs linearly. These come from terms where the E 's of (5.1) are differentiated twice. There are some rather remarkable cancellations hidden in (4.56).

First of all we note that

$$\nabla_B \ln R = - [v_B - v_I] / R = - t (v_B - v_I) \quad (5.4)$$

Now we compute all the terms in (4.56) where this t is not differentiated

$$\begin{aligned} \langle b^2 \rangle - \langle b \rangle^2 &= \frac{4}{W_f^2} + \frac{4}{W_B^2} \frac{v_f^2}{v_B^2} - \frac{t^2}{2} \left(1 - \frac{v_f}{v_B} \right)^2 v_I^2 \\ &\quad + \frac{t}{2} \left[\frac{v_f}{v_B} \nabla_B + \nabla_f \right] \left[\frac{v_f}{v_B} (\bar{v}_B - \bar{v}_I) - (\bar{v}_f - \bar{v}_I) \right] \\ &\quad + \frac{t}{2} \frac{\epsilon^f}{v_B} (\bar{v}_B - \bar{v}_I) - \frac{3}{2} t \frac{v_f^2}{v_B^2} \frac{\bar{v}_B - \bar{v}_I}{v_B} \epsilon^\beta \\ &\quad + t \frac{v_f^2}{v_B^2} \frac{\bar{v}_B - \bar{v}_I}{v_B} \epsilon^\beta \end{aligned} \quad (5.5)$$

Here $\bar{v}_B = \nabla_B E(B)$, $\bar{v}_I = \nabla_B E(p_B - p_f) = -\nabla_f E(p_B - p_f)$. Thus the linear t terms are

$$\begin{aligned} & \frac{t}{2} \left[\frac{v_f}{v_B} \left\{ \frac{v_f}{v_B} (\varepsilon^B - \varepsilon^I) + \varepsilon^I \right\} + \left\{ \frac{v_f}{v_B} \varepsilon^I - \varepsilon^f - \varepsilon^I \right\} \right] \\ & + \frac{t}{2} \frac{\varepsilon^f}{v_B} (v_B - v_I) - \frac{3}{2} t \frac{v_f^2}{v_B^2} \frac{(v_B - v_I)}{v_B} \varepsilon^B \\ & + t \frac{v_f^2}{v_B^2} \frac{(v_B - v_I)}{v_B} \varepsilon^B \end{aligned} \quad (5.6)$$

We stress again that in (4.56) the ∇ do not operate on the v 's. We have used $\varepsilon^I = \nabla_B^2 E(p_B - p_f) = -\nabla_f \nabla_B E(p_B - p_f)$. Thus we finally obtain

$$-\frac{t}{2} \varepsilon^I \left(1 - \frac{v_f}{v_B}\right)^2 - \frac{v_I t}{v_B} \frac{\varepsilon^f}{2} + \frac{v_f^2}{v_B^2} \frac{v_I t}{v_B} \frac{\varepsilon^B}{2} \quad (5.7)$$

In our example $v_f < v_B$ and $\varepsilon^f > \varepsilon^B$, so the sum of the last two terms is negative and we have over-all shrinkage due to the intermediate state denominator of the T matrix.

We should stress that genuine dispersive terms D appear in the form $\exp[-b^2/(1/W^2 + iDt)]$, so they should appear in the combination $1/W^4 + D^2 t^2$ if they were genuinely real dispersive terms, not in the form $1/W^2 + Dt$.

The other terms in the T matrix coming from $1/2E$ factors either in the intermediate states or in transforming the covariant T matrix to our non-covariant normalizations clearly give terms dependent on v_B , ε^B , etc., but lack the large time scale associated with t .

Notice that all the terms we have calculated tend to zero for large momenta since $\varepsilon(p) \rightarrow p$ as $p \rightarrow \infty$ relativistically and $(v_i - v_f) \rightarrow 0$ as both tend to C .

The last two factors in (5.1) clearly are connected to the fact that in the collision distance $v_I \cdot t$, which is traversed by the beam packet in time $v_I \cdot t / v_B$, we do not have the usual dispersion using ε^B , but a dispersion using ε^f since we go from $i \rightarrow f$. The interpretation of the first term is less clear, but it is negative and is related to the propagation of the virtual particle.

Thus our widths again behave in a rather different way to that expected on the naivest pictures. The velocity rather than dispersion dependent terms pick up the wrong sign exactly. Thus although the picture of scattering in a fixed scattering centre of a definite size is ruled out, it is clear that there are large time scales associated with the problem.

We have been unable to construct a naive space-time picture of the scattering which gives the above results. The shrinkage rather than growth of our packet widths leads one to the weak conclusion that perhaps momentum space is a much better place to think of these processes than position space.

After the problems of the above application, we reassure ourselves by studying a problem where we believe we understand the relevant time scales. Thus we study a simple two-body elastic Breit-Wigner S-wave resonance

$$T = \frac{m_0 \Gamma}{(S - m_0^2) + i m_0 \Gamma} \quad (5.8)$$

where we expect a lifetime of $\tau = 2/\Gamma$. We take $S = (q_1 + q_2)^2$ so that T is independent of p_i . A quick application of our formula gives a shift of q_1 of

$$\tau'(\mathbf{v}_R - \mathbf{v}_1) \quad (5.9)$$

where τ' is the time-dilated lifetime, $\vec{v}_R = (\vec{q}_1 + \vec{q}_2)/[E(\vec{q}_1) + E(\vec{q}_2)]$ is the resonance velocity and $\vec{v}_1$ is the velocity of $\vec{q}_1$. This is, of course, the naively expected answer at resonance.

Similarly at resonance for the packet broadening squared, we obtain the width-independent contribution

$$\frac{1}{2} \tau'^2 [(\mathbf{v}_R - \mathbf{v}_1) \cdot \hat{\epsilon}]^2 \quad (5.10)$$

of the anticipated general form. If $\vec{v}_R$ and $\vec{v}_1$ are parallel and $\hat{\epsilon}$ is along $\vec{v}_R$, this is just our expected velocity change shape alteration. If $\vec{v}_R = 0$ it is just the varying radial width change due to the varying times of emission of $\vec{q}_1$ from the decaying resonance.

6. CONCLUSIONS

We have given a general derivation of the wave packet shifts and width changes caused by an arbitrary $2 \rightarrow n$ T matrix in an arbitrary kinematic region. We believe that these expressions may be useful in checking the general form of models. Here we have applied the technique to scattering on a Gaussian sphere of weak point scatterers and a simple multiperipheral model. Even at this level it leads to unexpected and interesting results. In particular, the Gribov-Mueller picture of $2 \rightarrow n$ peripheral scattering is open to some doubt.

In a paper with similar intent to this one, Gottfried and Low⁷⁾ have studied the final classical trajectories of the wave packets and obtained centring results purely from knowledge of the inclusive cross-sections. Since we insist that phase information is crucial to compute shifts there is a technical disagreement. However, they ignore the variation of T across the packets [cf. Eq. (2.12)] which of course automatically gives no shifts or delays even in cases such as a Breit-Wigner or semi-classical scattering where there is known to be a delay or shift of the outgoing particle⁴⁾.

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APPENDIX

In our formalism of Section 4, in order to calculate variances, we have deliberately rewritten four-momentum conservation. Thus in (4.3) we have explicitly removed the $\vec{k}_2$ dependence and have rewritten the energy conservation δ -function. The form of the $\vec{k}_2$ replacement is equivalent to the use in (4.56) of derivatives of all three momenta, but, of course, only in the forms $\vec{v}_B - \vec{v}_T$ or $\vec{v}_j + \vec{v}_T$. It is then trivially clear that substitutions of the form $\vec{p}_j = \vec{p}_B + \vec{p}_T - \sum_{\neq i} \vec{p}_j$ leave (4.56) invariant.

What is not so clear is that substitutions of $E_i = E_B + E_T - \sum_{\neq i} E_j$ leave (4.56) invariant, although clearly this must be so if our calculation is correct. Consider, for example, what happens if we use energy conservation to fix the length of $\vec{q}_i$ in terms of the other vectors. Then we see that

$$\begin{aligned} (\vec{v}_j + \vec{v}_T)' &= (\vec{v}_j + \vec{v}_T + (\bar{v}_T - \bar{v}_j) \vec{v}_i'' / \bar{v}_i) \\ (\vec{v}_B - \vec{v}_T)' &= (\vec{v}_B - \vec{v}_T + (\bar{v}_B - \bar{v}_T) \vec{v}_i'' / \bar{v}_i) \end{aligned} \quad (A1)$$

i.e. the j,T momenta occur in the radial or parallel component of the i' momentum. The i' momentum derivatives of course then have no radial component. Thus (A1) is correct for all j. Then the above substitution should leave (4.56) invariant. Here we have placed bars over the velocities since, in principle, they may be differentiated unlike the velocities in (4.56).

$$\begin{aligned} &\left[\underline{e} \cdot (\vec{v}_j + \vec{v}_T) + \frac{\underline{e} \cdot (\vec{v}_j - \vec{v}_T)}{(\vec{v}_B - \vec{v}_T)^2} (\vec{v}_B - \vec{v}_T) \cdot (\vec{v}_B - \vec{v}_T) \right]' \\ &= [\dots] + \left[\underline{e} \cdot (\bar{v}_T - \bar{v}_j) + \frac{\underline{e} \cdot (\vec{v}_j - \vec{v}_T)}{(\vec{v}_B - \vec{v}_T)^2} (\vec{v}_B - \vec{v}_T) \cdot (\bar{v}_B - \bar{v}_T) \right] \frac{\vec{v}_i''}{\bar{v}_i} \end{aligned} \quad (A2)$$

For $\bar{v} = v$ the last bracket is zero. However, when we square we pick up $\nabla_{\vec{v}}$ terms

$$\begin{aligned} [\dots]'^2 &= [\dots]^2 + \left\{ -\underline{e} \cdot \underline{\xi}^j \underline{e} + \underline{e} \cdot \underline{\xi}^T \underline{e} - \frac{\underline{e} \cdot (\vec{v}_j - \vec{v}_T)}{(\vec{v}_B - \vec{v}_T)^2} (\vec{v}_B - \vec{v}_T) \cdot \underline{\xi}^T \underline{e} \right. \\ &\quad \left. + \frac{[\underline{e} \cdot (\vec{v}_j - \vec{v}_T)]^2}{(\vec{v}_B - \vec{v}_T)^4} (\vec{v}_B - \vec{v}_T) \cdot (\underline{\xi}^T + \underline{\xi}^B) \cdot (\vec{v}_B - \vec{v}_T) + \frac{\underline{e} \cdot (\vec{v}_j - \vec{v}_T)}{(\vec{v}_B - \vec{v}_T)^2} (\vec{v}_B - \vec{v}_T) \cdot \underline{\xi}^T \underline{e} \right\} \frac{\vec{v}_i''}{\bar{v}_i} \end{aligned} \quad (A3)$$

The last five terms of (4.56) also pick up extra terms of the form

$$\begin{aligned}
 & \left\{ -\frac{1}{2} \underline{e} \cdot (\underline{\xi}^j - \underline{\xi}^T) \cdot \underline{e} + \frac{3}{2} \frac{(\underline{e} \cdot (\underline{v}_j - \underline{v}_T))^2}{(\underline{v}_B - \underline{v}_T)^4} (\underline{v}_B - \underline{v}_T) \cdot (\underline{\xi}^T + \underline{\xi}^B) \cdot (\underline{v}_B - \underline{v}_T) \right. \\
 & \quad - \frac{(\underline{e} \cdot (\underline{v}_j - \underline{v}_T))^2}{(\underline{v}_B - \underline{v}_T)^4} (\underline{v}_B - \underline{v}_T) \cdot (\underline{\xi}^B + \underline{\xi}^T) \cdot (\underline{v}_B - \underline{v}_T) \\
 & \quad - 2 \frac{[\underline{e} \cdot (\underline{v}_j - \underline{v}_T)]}{(\underline{v}_B - \underline{v}_T)^2} (\underline{v}_B - \underline{v}_T) \cdot \underline{\xi}^T \cdot \underline{e} \\
 & \quad \left. + \frac{\underline{e} \cdot (\underline{v}_j - \underline{v}_T)}{(\underline{v}_B - \underline{v}_T)^2} \underline{e} \cdot \underline{\xi}^T \cdot (\underline{v}_B - \underline{v}_T) \right\} \frac{\nabla_i''}{v_i}
 \end{aligned} \tag{A4}$$

Multiplying the extra terms in (A3) by $-1/2$ and adding to (A4) we obtain zero identically. Thus we see the invariance of our answer under the use of four-momentum conservation. The terms which involve second derivatives of T are separately unaltered with corrections linear in T derivatives. Thus the complexity of (4.56) is required to preserve our formula under variations due to four-momentum conservation.

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Figure caption

Fig. 1 : The dominant time ordering for the simple ϕ^3 multiperipheral model.

FIG. 1