INSTANTON GREEN'S FUNCTIONS AND TENSOR PRODUCTS

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ABSTRACT

The structure of the Green's function for a scalar field, transforming under the adjoint representation of a gauge group, in the background field of an arbitrary self-dual instanton field, is considered. It is shown that it can be written in the same elegant form as the Green's function for the fundamental vector representation of the group, given the introduction of a certain conformally invariant matrix. The same method solves the more general problem of finding the Green's function for a field transforming under the direct product of two groups. Simple expressions for the solutions of the corresponding massless Dirac equation are obtained. The results on the products of representations may be applied, iteratively, to construct the Green's function and massless solutions of the Dirac equation for any higher representation of the group.

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1. INTRODUCTION

Our understanding of the self-dual solutions to the Euclidean Yang-Mills field equations (self-dual instantons) has been completely transformed by the work of Atiyah, Drinfeld, Hitchin and Manin (ADHM)^{1,2)}. They have given a construction for the general self-dual solution in terms of constant parameters constrained by certain quadratic equations. This construction may be used, without any essential variations in the formalism, for any compact classical group, involves only elementary matrix algebra, and leads to expressions for the gauge potentials which are rational functions of the spatial coordinates (in a suitable gauge). Further, it provides a framework within which the previously accumulated results on problems associated with self-dual solutions may be understood and their structure elucidated and simplified. In particular, using the techniques implicit in the construction of Atiyah et al., simple and elegant forms have been found for the Green's function for a (spatially) scalar field transforming under the fundamental vector representation of the gauge group^{3,4)}, and for the massless solutions of the corresponding Dirac equation^{3,5)}. [By the "fundamental vector representation" we mean that representation with the same dimension as the matrices forming the group, e.g. isospin $\frac{1}{2}$ for SU(2); we shall simply call it the fundamental representation henceforth.] These results not only generalize the results found previously^{6,7)}, for the special case of the 't Hooft SU(2) solutions⁸⁾, but clarify their essentially simple structure.

In addition to their intrinsic mathematical interest, such Green's functions are of practical importance in heuristic estimations of instanton effects in quantum chromodynamics. For example, work has begun on calculating instanton effects in electron-positron annihilation^{9,10)} and on studying effects of instantons in bound state mass calculations¹¹⁾. The latter problem requires knowledge of the adjoint representation Green's function, as does the former, once one proceeds beyond the lowest order. Once the scalar Green's function is known for a given representation, the corresponding (spatially) spinor and vector Green's functions may be found easily using the techniques of Brown et al.⁶⁾. However, a good knowledge of the corresponding zero frequency modes is desirable¹²⁾.

Brown et al.⁶⁾ showed how to construct the adjoint representation (isospin 1) Green's function for the 't Hooft SU(2) solutions. Christ et al.⁴⁾ pointed out that these methods could be applied immediately to the general self-dual SU(2) solution. Superficially, the resulting expression shows little of the elegance so striking in the isospin $\frac{1}{2}$ case^{3,4)}, but we shall see that this appearance is rather deceptive. The results described in this present paper have grown out of an attempt to understand this expression for the adjoint Green's function, its structure and generality. We feel that these attributes are somewhat obscured in

Ref. 4 by the notation and explicit symmetrization. Suitably recast, the expression reveals an intricate elegance intimately connected with the conformal invariance of the theory, and is completely general: it works for any compact classical group. Further, it solves the wider problem of constructing the Green's function for a scalar field transforming under the direct product of two groups, $G_1 \times G_2$, with respect to the fundamental representation of each, in the background field obtained by taking a self-dual solution for each of the groups. In other words, it constructs the Green's function for the fundamental representation of a non-simple Lie group consisting of two simple compact factors. [We shall refer to the solution for $G_1 \times G_2$ obtained by taking particular (self-dual) solutions for G_1 and G_2 as the *tensor product* of those solutions. The adjoint representation occurs as a special case when we take $G_1 = G_2^*$ and the solutions to be conjugate.]

From another point of view it would seem that working out the Green's function for the tensor product and, hence, the adjoint representation should be easy. We can always regard $G_1 \times G_2$ as a subgroup of some larger simple group, G . The tensor product of solutions for G_1 and G_2 provides us with a solution for G and, if G is suitably chosen, the Green's function for the fundamental representation of G will be what is required. The reason that this approach is not as trivial as it sounds is that to implement it we have to describe the solution for G in the terms of the linear algebra of the ADHM construction¹⁾ for G , given the corresponding descriptions for G_1 and G_2 . This is not easy because taking tensor products destroys linearity. In fact the additional structure in the adjoint representation, or tensor product, Green's function is really there to solve this problem. By elaborating this structure we have discovered how to describe the tensor product of instantons within the framework of the linear ADHM construction, thus unifying the descriptions of the fundamental and adjoint representation Green's functions. In addition, this enables us to immediately extend results obtained for the fundamental representation (for example, the massless solutions of the Dirac equation) to tensor products and so, in particular, the adjoint representation. Going further, we may use this new description of the adjoint representation, or any tensor product, as an input and calculate the Green's function for *its* tensor product with some other instanton. In this way we may, at least in principle, obtain the Green's function, etc., for any representation of the group by an iterative process of taking tensor products of representations and reducing.

Our paper is arranged as follows. In Section 2 we review the formalism we shall use and summarize the results we have obtained. In Section 3 we establish our results on the Green's function for the adjoint representation and, more generally, the tensor product of instantons. Section 4 is devoted to understanding the tensor product of instantons within the ADHM framework. As an illustration of

the power of this point of view, in Section 5 we show how the expression for the massless solutions of the Dirac equation in the fundamental representation yields those for the adjoint representation. Finally, in Section 6, we discuss the significance of our results and further problems.

2. A SUMMARY OF RESULTS

We shall begin this section by reviewing briefly the construction of Atiyah et al.¹⁾ and the results on Green's functions for the fundamental representation^{3,4)}. Throughout this paper we shall describe our results with specific reference to $Sp(n)$ which may conveniently be regarded as the group of $n \times n$ unitary matrices with quaternion entries. In particular $Sp(1) \cong SU(2)$, the group of quaternions with unit modulus. Any compact classical group may be thought of as a subgroup of $Sp(n)$ for sufficiently large n , and it is simple to deduce the corresponding results for the orthogonal and unitary groups in this way. To be quite explicit we shall restate some of our results for $SU(n)$.

In the ADHM construction for $Sp(n)$, the vector potential, A_α , represented as a traceless $n \times n$ anti-Hermitian matrix of quaternions, is given in the form

$$A_\alpha = v^\dagger \partial_\alpha v, \quad (2.1)$$

where $v \equiv v(x)$ is a $(k+n) \times n$ matrix of quaternions which must satisfy the equations,

$$v^\dagger v = 1, \quad (2.2)$$

$$v(x)^\dagger \Delta(x) = 0. \quad (2.3)$$

Here $\Delta(x)$ is an $(n+k) \times k$ matrix of quaternions, linear in the position coordinates x , which has the form

$$\Delta_{\lambda i}(x) = a_{\lambda i} + b_{\lambda i} x, \quad 1 \leq i \leq k, \quad (2.4)$$

$$1 \leq \lambda \leq n+k,$$

with x being the quaternionic representation of the position coordinates,

$$x = x_0 - i \vec{x} \cdot \vec{\sigma} = \begin{pmatrix} x_0 - i x_3 & -x_2 - i x_1 \\ x_2 - i x_1 & x_0 + i x_3 \end{pmatrix} \quad (2.5)$$

The quaternionic matrices a, b are constants, independent of x . Provided they satisfy the constraints that $a^\dagger a$, $b^\dagger b$ and $a^\dagger b$ all be symmetric as $k \times k$ matrices of quaternions, Eq. (2.1) yields a self-dual solution of the field equations, determined up to gauge equivalence by Eq. (2.2-2.3). Further this solution is non-singular at points where $\Delta(x)$ has maximal rank. All these conditions may be summarized by saying that one obtains a solution which is non-singular everywhere if the $k \times k$ matrix of quaternions $\Delta(x)^\dagger \Delta(x)$ is actually real and invertible for all x . [We shall denote its inverse by $f(x)$.]

It is not difficult to give simple and elementary demonstrations that the resulting field strength,

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha + [A_\alpha, A_\beta] \quad (2.6)$$

is self-dual,

$$F_{\alpha\beta} = {}^* F_{\alpha\beta} = \frac{1}{2} \epsilon_{\alpha\beta\gamma\delta} F_{\gamma\delta} \quad (2.7)$$

with instanton number k ,

$$k = \frac{1}{16\pi^2} \int d^4x \operatorname{Tr} \{ F_{\alpha\beta} {}^* F_{\alpha\beta} \} \quad (2.8)$$

see Refs. 2, 3 or 4. (For this last statement it is also necessary that the solution be regular at infinity, which is ensured by $b^\dagger b$ being non-singular.) A deeper statement¹⁾ is that the ADHM construction yields all the self-dual solutions which are regular everywhere including infinity.

For further details such as the form of a and b for the 't Hooft $SU(2)$ solutions see Ref. 3.

The Green's functions we shall be concerned with satisfy equations of the form

$$\Box^2 G(x, y) = -\delta(x-y) \quad (2.9)$$

where, in the case of the fundamental representation

$$\mathfrak{D}_\alpha = \partial_\alpha + A_\alpha = \partial_\alpha + v^\dagger \partial_\alpha v. \quad (2.10)$$

In the case of the fundamental representation, a form has been found for G which is valid for any self-dual solution for any compact-classical group^{3,4)}

$$G(x, y) = \frac{v(x)^\dagger v(y)}{4\pi^2(x-y)^2} \quad (2.11)$$

This could hardly be simpler. The Green's function for the ordinary Laplacian, corresponding to putting $A_\alpha = 0$ in Eq. (2.9), is just $[4\pi^2(x-y)^2]^{-1}$. Thus 1 has just been replaced by $v(x)^\dagger v(y)$, or δ_{ab} by $[v(x)^\dagger v(x)]_{ab}$ to be more explicit.

This result might tempt one into hoping that this prescription would work for other representations too: that the Green's functions for products of the fundamental representation, q , could be obtained by judiciously replacing 1 by $v(x)^\dagger v(y)$ in appropriate places. In particular the adjoint representation can be obtained by decomposing $q \otimes \bar{q}$, for which

$$\mathfrak{D}_\alpha = \partial_\alpha + A_\alpha \otimes 1 + 1 \otimes A_\alpha^* \quad (2.12)$$

A_α^* being the complex conjugate of A_α . The prescription would lead to a result of the form,

$$G_{ab,cd}(x, y) = \frac{[v(x)^\dagger v(y)]_{ac} [v(x)^\dagger v(y)]_{bd}^*}{4\pi^2(x-y)^2} + \frac{1}{4\pi^2} C_{ab,cd}(x, y) \quad (2.13)$$

in which $C = 0$, and the labels a, b, c, d take $2n$ values. Unfortunately this does not quite work; the C term really is necessary, as was first pointed out by Brown *et al.*⁶⁾ in the case of the 't Hooft $SU(2)$ solutions⁶⁾. The tempting prescription provides the correct δ -function singularity as $x \rightarrow y$, so that $C(x, y)$ is non-singular in this limit. In their pioneering papers, Brown *et al.* found a form for $C(x, y)$ which was extended to the general self-dual $SU(2)$ solution by Christ *et al.*⁴⁾. Both these sets of authors explicitly performed the simple reduction from $q \otimes \bar{q}$ to the adjoint representation, which tends to obscure the structure of $C(x, y)$. Leaving this undone, it becomes apparent that this structure is independent of the group and actually solves a wider class of problems.

The structure of $C(x, y)$ is contained in the general formula,

$$C_{ab,cd}(x,y) = M_{ij,lm} \cdot [v_1(x)^\dagger b_1]_{a, iI} [v_2(x)^\dagger b_2]_{b, jJ} \varepsilon_{IJ} [b_1^\dagger v_1(y)]_{iL, c} [b_2^\dagger v_2(y)]_{mM, d} \varepsilon_{LM} \quad (2.14)$$

where $M_{ij,lm}$ is defined by the equation

$$M_{rs,ij} \{ [a_1^\dagger a_1]_{ii} [b_2^\dagger b_2]_{jm} + [b_1^\dagger b_1]_{ii} [a_2^\dagger a_2]_{jm} - [a_1^\dagger b_1]_{iI, iL} [a_2^\dagger b_2]_{jJ, mM} \cdot \varepsilon_{IJ} \varepsilon_{LM} \} = \delta_{ri} \delta_{sm} \quad (2.15)$$

[See Eq. (3.20) for a diagrammatic representation of M^{-1}]. In Eqs. (2.14-15), ε is the usual two-dimensional antisymmetric tensor,

$$\varepsilon_{12} = -\varepsilon_{21} = 1, \quad \varepsilon_{11} = \varepsilon_{22} = 0 \quad (2.16)$$

The upper case labels I, J, L, M run over the values 1, 2 and label the rows and columns of quaternions; we shall refer to these as the quaternionic labels.

To construct the Green's function for $q \otimes \bar{q}$ we take

$$a_1 = a, \quad b_1 = b, \quad a_2 = a^* \varepsilon, \quad b_2 = b^* \varepsilon \quad (2.17)$$

where a, b are the matrices used to define the potential A_a via Eqs. (2.1-4). In this case i, j, l, m, r, s run over k values. [The multiplication by ε in Eq. (2.17) is on the quaternionic labels, of course.]

To extract the adjoint representation Green's function, $G_{\rho\tau}(x, y)$, from that for $q \otimes \bar{q}$, we multiply Eq. (2.13) by $\Lambda_{ab}^\rho, \Lambda_{cd}^\tau$, and sum over a, b, c, d , where $\{\Lambda_{ab}^\rho, \rho = 1, \dots, n(2n+1)\}$ is a suitably normalized basis of Hermitian generators for $Sp(n)$, i.e. for Hermitian $2n \times 2n$ matrices satisfying $\varepsilon \Lambda \varepsilon^{-1} = -\Lambda^T$. It follows from this property of the generators that only the antisymmetric part of $M_{ij,lm}$, $M_{[ij], [lm]}$, contributes in the adjoint representation. For $Sp(1) \cong SU(2)$ the $\{\Lambda^\rho\}$ are just the Pauli σ matrices.

The motivation for writing Eqs. (2.14-15) as we have, is that with $C(x, y)$ so defined, Eq. (2.13) solves a more general problem. Consider a non-simple gauge group $Sp(n_1) \times Sp(n_2)$ and a scalar field $\phi_{ab}(x)$ which transforms under the fundamental representation of each factor. For such a field the covariant derivative takes the form

$$\mathbb{D}_\alpha = \partial_\alpha + A_{1\alpha} \otimes 1 + 1 \otimes A_{2\alpha} \quad (2.18)$$

The self-dual solutions for $Sp(n_1) \times Sp(n_2)$ are given by taking self-dual solutions for each of the factors. These can be specified by equations of the form of Eqs. (2.1-2.4):

$$A_{m\alpha} = v_m^\dagger \partial_\alpha v_m, \quad m = 1, 2 \quad (2.19)$$

where $v_m^\dagger v_m = 1$, $v_m^\dagger \Delta_m = 0$ and $\Delta_m = a_m + b_m x$, $m = 1, 2$. The quaternionic matrices a_m, b_m have dimensions $(n_m + k_m) \times k_m$ for $m = 1, 2$, respectively. The solution has two instanton numbers k_1, k_2 . Eq. (2.19) will provide a self-dual solution if $\Delta_m(x)^\dagger \Delta_m(x)$ is everywhere a real and invertible $k_m \times k_m$ matrix, $f_m(x)^{-1}$, for $m = 1, 2$. Then,

$$G_{ab,cd}(x,y) = \frac{[v_1(x)^\dagger v_1(y)]_{ac} [v_2(x)^\dagger v_2(y)]_{bd}}{4\pi^2 (x-y)^2} + \frac{1}{4\pi^2} C_{ab,cd}(x,y) \quad (2.20)$$

with $C(x,y)$ given by Eqs. (2.14-2.15) solves the Green's function equation (2.9) in which the covariant derivative is given by Eq. (2.18). In other words, Eqs. (2.13-2.15) solve the Green's function problem for the direct product of two compact simple classical groups. This our first main result which will be proved in Section 3, where we shall also introduce a convenient diagrammatic notation to make the structure of equations like Eqs. (2.14-2.15) clearer.

We can use Eqs. (2.13-2.15) and (2.20) to go further and consider non-simple groups with more than two factors, for example, $G_1 \times G_2 \times G_3$, if we can find a description of the solution for $G_1 \times G_2$ which is strictly analogous to Eq. (2.1), and which would mean we could amalgamate the two terms in Eq. (2.20) to give a precise analogue of Eq. (2.11). Having achieved this we may use the solution for $G_1 \times G_2$ as an input, in the places we used the solution for G_1 , and replace G_2 by G_3 , to obtain the Green's function for the fundamental representation of $G_1 \times G_2 \times G_3$ and so on. Making identifications of the sort made in Eq. (2.17), we may particularize obtaining Green's functions for the higher dimensional representations of a given group, working iteratively.

The main task of this paper is to demonstrate how the program described in the last paragraph may be carried out. The first step is to rewrite Eq. (2.20) in the form of Eq. (2.11). In Section 3 we shall show that we can rewrite Eq. (2.20) as

$$G(x, y) = \frac{[v_1(x) \otimes v_2(x)]^\dagger (1 - \mathcal{M}) [v_1(y) \otimes v_2(y)]}{4\pi^2 (x - y)^2} \quad (2.21)$$

where

$$\begin{aligned} \mathcal{M}_{ab,cd} = & \{ a_{1a, iI} b_{2b, jJ} a_{1cI, c}^\dagger b_{2mJ, d}^\dagger + b_{1a, iI} a_{2b, jJ} b_{1cI, c}^\dagger a_{2mJ, d}^\dagger \\ & - a_{1a, iI} b_{2b, jJ} b_{1cJ, c}^\dagger a_{2mI, d}^\dagger - b_{1a, iI} a_{2b, jJ} a_{1cJ, c}^\dagger b_{2mI, d}^\dagger \\ & - [a_{1a, iI} a_{2b, jJ} b_{1cL, c}^\dagger b_{2mM, d}^\dagger + b_{1a, iI} b_{2b, jJ} a_{1cL, c}^\dagger a_{2mM, d}^\dagger] \delta_{IJ} \delta_{LM} \} \\ & \cdot M_{ij, lm} \end{aligned} \quad (2.22)$$

Whilst a diagrammatic notation is helpful for discerning the structure of Eqs. (2.14-2.15) it is essential for Eq. (2.22); the reader should compare this equation with Eq. (3.29).

The matrix $1 - \mathcal{M}$ has a number of significant properties which we shall establish and use in the remainder of this paper. Briefly, it is conformally invariant, its eigenvalues are non-negative and at least $4k_1 k_2$ of them are zero. As a consequence of the second statement we may define a positive square root $(1 - \mathcal{M})^{1/2}$, which is Hermitian, and set

$$\tilde{v}(x) = (1 - \mathcal{M})^{1/2} v_1(x) \otimes v_2(x) \quad (2.23)$$

The non-singularity of $C(x, y)$ as $x \rightarrow y$ implies that $\tilde{v}(x)^\dagger \tilde{v}(x) = 1$. Consequently, if we can find a $\tilde{\Delta}(x)$, linear in x such that $\tilde{v}(x)$ is defined by $\tilde{v}(x)^\dagger \tilde{\Delta}(x) = 0$ and $\tilde{\Delta}(x)^\dagger \tilde{\Delta}(x)$ satisfies the usual constraints, we shall have achieved our objective. In Section 4 we show how to define such a $\tilde{\Delta}(x)$. Note that the Green's functions of Eq. (2.20) can now be obtained by replacing v by $\tilde{v}$ in Eq. (2.11); the validity of this expression then follows merely from the existence of some $\tilde{\Delta}(x)$ with the properties just described and the previous proof of the formula for the Green's function^{3,4)}.

Having thus unified the treatments of the product of the adjoint and fundamental representations, we may immediately take results from one to the other. In particular we may use the formula for the massless solutions of the Dirac equation for the fundamental representation^{3,5)} to write them down for the adjoint

representation or, more generally, a tensor product of instantons. In a sense this is a slightly circular point of view because the construction of $\tilde{\Delta}(x)$ is intimately linked with these solutions of the Dirac equation. However, in Section 5, we show how they may be obtained immediately from the simple result for the fundamental representation, given $\tilde{\Delta}(x)$. Here we state the results in their simplest form; for further details of the formalism for the Dirac equation see Section 5.

Consider the massless Dirac equation corresponding to the covariant derivative of Eq. (2.18). It has only negative chirality solutions described by two spinors $\psi_L \equiv \psi_{a,b,I}$ satisfying

$$e_\alpha^\dagger \not{\partial}_\alpha \psi_L = 0 \quad (2.24)$$

Here $e_\alpha^\dagger = \partial_\alpha x^\dagger$ acts on the spinor index, I . There are $2(n_1 k_2 + n_2 k_1)$ linearly independent solutions to Eq. (2.24) and these are of the form,

$$\psi_{ab,I} = [v_1^\dagger b_1 z f_1]_{a,iI} [v_2^\dagger d]_{b,i} + [v_1^\dagger c]_{a,i} [v_2^\dagger b_2 z f_2]_{b,iI} \quad (2.25)$$

where c, d are (constant) $2(k_1 + n_1) \times k_2$ and $2(k_2 + n_2) \times k_1$ dimensional matrices, respectively satisfying the linear equations

$$\begin{aligned} [a_1^\dagger c]_{iI,j} &= [a_2^\dagger d]_{jI,i} \\ [b_1^\dagger c]_{iI,j} &= [b_2^\dagger d]_{jI,i} \end{aligned} \quad (2.26)$$

for $1 \leq i \leq k$, $1 \leq j \leq k$, $I = 1, 2$. Again we may particularize to the $q \otimes \bar{q}$ representation by using Eq. (2.17), and extract the adjoint representation solutions, $\psi_{\rho,I}$, by multiplying by Λ_{ab}^0 and summing. In the latter case the symmetry of Λ^0 implies that only those c, d with $d = \epsilon c$ contribute.

We first obtained Eq. (2.26) by studying the so called "supersymmetric" solutions^{13,14)} to the massless Dirac equation for the adjoint representation and generalizing. These solutions, which in general have the form

$$\begin{aligned} \psi_L &= (e_\alpha e_\beta^\dagger - e_\beta e_\alpha^\dagger) \chi F_{\alpha\beta} \\ \psi_L &= (e_\alpha e_\beta^\dagger - e_\beta e_\alpha^\dagger) x \chi F_{\alpha\beta} \end{aligned} \quad (2.27)$$

where χ is a fixed two-spinor, here correspond to $c = b\chi$ and $c = a\chi$, respectively. Eqs. (2.25-2.26) generalize the results obtained by Jackiw and Rebbi¹⁴⁾ for the 't Hooft SU(2) solutions in the adjoint representation.

The results we have described may be transcribed immediately so that they apply directly to SU(n). The ADHM construction for SU(n) may be described³⁾ using Eqs. (2.1)-(2.3), where now v is a $(2k + n) \times n$ complex matrix and $\Delta_{\lambda, iI} = \alpha_{\lambda, iI} + b_{\lambda, iJ} x_{JI}$, $1 \leq \lambda \leq 2k + n$, $1 \leq i \leq k$, $I, J = 1, 2$, is a $(2k + n) \times 2k$ complex matrix, such that $\{\Delta(x)^\dagger \Delta(x)\}_{iJ, jJ}$ is proportional to δ_{IJ} and invertible for all x (including, in an appropriate sense, infinity). Given solutions for SU(n_m) with topological charge k_m , described by $\Delta_m(x)$, $m = 1, 2$, we may use Eq. (2.15) to construct an Hermitian $k_1 k_2$ -dimensional matrix M , hence obtaining the $n_1 n_2$ -dimensional complex matrix $c(x, y)$ from Eq. (2.14), and the Green's function for the tensor product. Similarly we may define $\mathcal{M}$, which has dimensions $(2k_1 + n_1)(2k_2 + n_2)$ and $4k_1 k_2$ unit eigenvalues, using Eq. (2.22). [The instanton number of the tensor product, regarded as a solution for SU($n_1 n_2$) is now $k_1 n_2 + k_2 n_1$.] Subsequent formulae and equations may be reinterpreted similarly.

3. GREEN'S FUNCTIONS

As was emphasised in Ref. 3, the covariant derivative is most easily understood and calculated with if it is thought of, in a larger space, as differentiation followed by orthogonal projection¹⁾. We define a map between $2n$ -dimensional fields, ϕ , in the fundamental representation and $2(n + k)$ dimensional fields, $\hat{\phi}$, using matrix multiplication by $v(x)$.

$$\hat{\phi}(x) = v(x) \phi(x) \quad (3.1)$$

Then the covariant derivative takes the form

$$\mathcal{D}_\alpha \hat{\phi} \equiv \widehat{\mathcal{D}_\alpha \phi} = P \partial_\alpha \hat{\phi} \quad (3.2)$$

where the orthogonal projection operator

$$P(x) = v(x) v(x)^\dagger \quad (3.3)$$

Because of the normalization condition on v , Eq. (2.2), there is one to one correspondence between $2n$ -dimensional fields ϕ and $2(n+k)$ dimensional fields $\hat{\phi}$ satisfying $P\hat{\phi} = \phi$. As in Ref. 3 we shall employ this correspondence to lift equations into the larger space and use the simple form for the covariant derivative.

For a field in the $q \otimes \bar{q}$ representation $\phi_{ab}(x)$ we may define

$$\hat{\phi}(x) = V(x) \phi(x) V(x)^\dagger \quad (3.4)$$

regarding ϕ as a $2n \times 2n$ matrix, and the covariant derivative of Eq. (2.12) then takes the form

$$\mathcal{D}_\alpha \hat{\phi} \equiv \widehat{\mathcal{D}_\alpha \phi} = P(\partial_\alpha \hat{\phi})P \quad (3.5)$$

More generally, for the tensor product case of Eq. (2.18) we define

$$\hat{\phi}_{\lambda\mu}(x) = V_1(x)_{\lambda a} V_2(x)_{\mu b} \phi_{ab}(x) \quad (3.6)$$

and the covariant derivative is given by

$$\mathcal{D}_\alpha \hat{\phi} = (P_1 \otimes P_2) \partial_\alpha \hat{\phi} \quad (3.7)$$

where $P_m(x) = v_m(x) v_m(x)^\dagger$, $m = 1, 2$.

The Green's function equation for the tensor product may thus be written in the form,

$$\mathcal{D}^2 \hat{G}_{\lambda\mu, \nu\rho}(x, y) = -P_{1\lambda\nu}(x) P_{2\mu\rho}(x) \delta(x-y) \quad (3.8)$$

The first guess at a solution, to which Eq. (2.11) might tempt us, is $\hat{G} = \hat{H}$, where

$$\hat{H}_{\lambda\mu, \nu\rho}(x, y) = \frac{[P_1(x)P_1(y)]_{\lambda\nu} [P_2(x)P_2(y)]_{\mu\rho}}{4\pi^2(x-y)^2} \quad (3.9)$$

Before discussing the extent to which Eq. (3.9) fails to yield a solution to Eq. (3.8) and how Eqs. (2.14-2.15) and (2.20) rectify this, we shall introduce a diagrammatic notation which we find convenient for representing and manipulating such expressions. We shall use lines to indicate the contraction of indices. The

matrices being multiplied typically have dimensions $2k_m$ or $2(k_m + n_m)$, $m = 1, 2$; we shall treat the two-dimensional quaternionic labels separately from those of dimension k_m or $k_m + n_m$, which label the different quaternions. We shall use broken lines to stand for sums over the former and solid lines for sums over the latter. The matrix ϵ occurs so frequently that it is convenient to have a special symbol for it, and we shall use an arrow. Thus,

$$\begin{aligned}
 \overline{i} \text{---} j &= \delta_{ij} \\
 \overline{i} \text{---} j &= \delta_{ij} \\
 \overline{i} \text{---} \rightarrow j &= \epsilon_{ij} \\
 \overline{i} \text{---} \leftarrow j &= -\epsilon_{ij} = \epsilon_{ji}
 \end{aligned} \tag{3.10}$$

where $1 \leq i, j \leq k_m$ or $n_m + k_m$ and $I, J = 1, 2$. We shall write those operators P_m , a_m , b_m , etc., with a suffix 2 below those with suffix 1, and omit the suffices for brevity. In this notation Eq. (3.9) becomes

$$\hat{H} = \frac{1}{4\pi(x-y)^2} \frac{[P(x)] \text{---} [P(y)]}{[P(x)] \text{---} [P(y)]} \tag{3.11}$$

The suffices λ, μ, ν, ρ are left implicit at the four corners.

In subsequent calculations we shall often need to use the identity,

$$\epsilon_{AB} \epsilon_{CD} = \delta_{AC} \delta_{BD} - \delta_{AD} \delta_{CB} \tag{3.12}$$

which diagrammatically appears as

$$\begin{array}{c} A \\ \downarrow \\ B \end{array} \begin{array}{c} C \\ \downarrow \\ D \end{array} = \begin{array}{c} A \text{---} C \\ B \text{---} D \end{array} - \begin{array}{c} A \text{---} D \\ B \text{---} C \end{array} \tag{3.13}$$

Another useful identity is

$$[e_\alpha]_{AC} [e_\alpha]_{BD} = 2 \varepsilon_{AB} \varepsilon_{CD} \quad (3.14)$$

which appears as

$$\begin{array}{c} A \text{---} e_\alpha \text{---} C \\ B \text{---} e_\alpha \text{---} D \end{array} = 2 \begin{array}{c} A \\ B \end{array} \begin{array}{c} C \\ D \end{array} \quad (3.15)$$

In evaluating $\mathcal{D}^2 \hat{H}$ we may divide the terms into four groups. We have a term involving $\partial^2 \{(x-y)^{-2}\}$ which gives the desired δ -function as on the right-hand side of Eq. (3.8). We have a term involving

$$P_1 \partial_\alpha \{P_1 \partial_\alpha P_1\} (x-y)^{-2} + 2 \{P_1 \partial_\alpha P_1\} \partial_\alpha \{(x-y)^{-2}\}$$

which vanishes as in the proof of the Green's function for the fundamental representation³⁾. We have a similar term involving P_2 instead of P_1 . Finally we have a term involving $P_1 \partial_\alpha P_1$ and $P_2 \partial_\alpha P_2$, which we might like to vanish, but does not. Thus

$$\mathcal{D}^2 \hat{H} = - \frac{[P(x)]}{[P(x)]} \delta(x-y) + \frac{2}{4\pi(x-y)^2} \frac{[P(x) b e_\alpha f(x) \Delta(x)^\dagger P(y)]}{[P(x) b e_\alpha f(x) \Delta(x)^\dagger P(y)]} \quad (3.16)$$

We use $\Delta(x)^\dagger P(y) = (x-y)^\dagger b^\dagger P(y)$ and Eq. (3.15) to rewrite the last term in Eq. (3.16), obtaining that $\hat{G} = \hat{H} + \hat{C}$ will solve Eq. (3.8) provided that

$$\mathcal{D}^2 \hat{C} = - \frac{1}{\pi^2} \frac{[P(x) b f(x)] [b^\dagger P(y)]}{[P(x) b f(x)] [b^\dagger P(y)]} \quad (3.17)$$

If we set

$$4\pi^2 \hat{C} = \begin{array}{c} [P(x) b] \\ [P(x) b] \end{array} M \begin{array}{c} [b^\dagger P(y)] \\ [b^\dagger P(y)] \end{array} \quad (3.18)$$

where $M_{ij,lm}$ is a constant $k_1 k_2 \times k_1 k_2$ matrix, $\hat{e}$ will satisfy Eq. (3.17) provided that M satisfies the equation,

$$\left\{ \begin{array}{c} \overline{[\Delta^\dagger \Delta]} \\ \overline{[b^\dagger b]} \end{array} + \begin{array}{c} \overline{[b^\dagger b]} \\ \overline{[\Delta^\dagger \Delta]} \end{array} - \begin{array}{c} \overline{[\Delta^\dagger b]} \\ \overline{[\Delta^\dagger b]} \end{array} \right\} [M] = \begin{array}{c} \text{---} \\ \text{---} \end{array} \quad (3.19)$$

where $\Delta \equiv \Delta(x)$ and we have used $P\partial_\alpha\{P\partial_\alpha P\} = -4Pbfb^\dagger$. At first sight Eq. (3.19) offers no hope for a solution since it appears to state that the constant matrix M is the inverse of an x -dependent matrix. But, as was first noted by Brown et al.⁶⁾ in what is effectively a special case of this equation, the appearance is deceptive because the x -dependence cancels out; Eq. (3.19) merely states that M is the inverse of

$$\begin{array}{c} \overline{[a^\dagger a]} \\ \overline{[b^\dagger b]} \end{array} + \begin{array}{c} \overline{[b^\dagger b]} \\ \overline{[a^\dagger a]} \end{array} - \begin{array}{c} \overline{[a^\dagger b]} \\ \overline{[a^\dagger b]} \end{array} \quad (3.20)$$

To show this, note that

$$\begin{array}{c} \overline{[x]} \\ \overline{[a^\dagger b]} \end{array} = \overline{[a^\dagger b x]} \quad (3.21)$$

We conclude that, defining M as the inverse of expression (3.20), which is equivalent to Eq. (2.15), the solution to Eq. (3.8) is $\hat{G} = \hat{H} + \hat{C}$. This constitutes a generalization of a result obtained by Christ et al.⁴⁾ for the adjoint representation of $SU(2)$.

We shall now discuss the properties of M . Firstly the expression (3.20), and so M , is invariant under the interchange of a and b . This is obvious from the symmetry of $a^\dagger b$ once one notes that this can be expressed as

$$\overline{[a^\dagger b]} = \overline{[b^\dagger a]} \quad (3.22)$$

The x-independence of Eq. (3.19) is equivalent to the statement that expression (3.20) is invariant under the transformations

$$a_m \rightarrow a_m + b_m x, \quad b_m \rightarrow b_m, \quad m = 1, 2. \quad (3.23)$$

Because of the symmetry under interchange of a and b it is also invariant under the transformations

$$a_m \rightarrow a_m, \quad b_m \rightarrow b_m + a_m x, \quad m = 1, 2, \quad (3.24)$$

whilst under the transformations

$$a_m \rightarrow a_m x, \quad b_m \rightarrow b_m y, \quad m = 1, 2, \quad (3.25)$$

it changes by a factor $|x|^2 |y|^2$ for any quaternions x, y. Taken together transformations (3.23-2.25) generate a sixteen parameter group

$$a_m \rightarrow a_m \delta + b_m \beta, \quad b_m \rightarrow b_m \alpha + a_m \gamma \quad (3.26)$$

where $\alpha, \beta, \gamma, \delta$ are quaternions such that

$$\kappa = \det \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \neq 0. \quad (3.27)$$

We deduce that under these transformations expression (3.20) is scaled by $|\kappa|^2$ so that M is merely multiplied by $|\kappa|^{-2}$. Actually the transformation (3.26) corresponds to performing a conformal transformation

$$x \rightarrow (\alpha x + \beta)(\gamma x + \delta)^{-1} \quad (3.28)$$

on the solutions specified by Δ_1, Δ_2 . In such a transformation we may as well take $\kappa = 1$ and, with this restriction, M is conformally invariant.

We have implicitly assumed the real $k_1 k_2 \times k_1 k_2$ matrix (3.20) is invertible. We have no precise set of conditions under which this is true, but it is easily seen to be true for particular tensor products or $q \otimes \bar{q}$ representations, using the 't Hooft solutions. Therefore, because the whole construction is algebraic, it can only be singular for certain special tensor products, or representations q , which form a manifold of smaller dimension within the whole space of solutions. In other words typically our procedure will work, though we have not proved rigorously that it will work everywhere.

To conclude this section, we shall discuss how we may amalgamate the two parts of G given by Eqs. (3.11) and (3.18). Consider the square matrix $\mathcal{M}$ of dimension $4(n_1 + k_1)(n_2 + k_2)$ defined by

$$\begin{aligned} \mathcal{M} = & \begin{array}{|c|} \hline [a] \text{---} [a^\dagger] \\ \hline [b] \text{---} [b^\dagger] \\ \hline \end{array} \mathcal{M} \begin{array}{|c|} \hline [a^\dagger] \\ \hline [b^\dagger] \\ \hline \end{array} + \begin{array}{|c|} \hline [b] \text{---} [b^\dagger] \\ \hline [a] \text{---} [a^\dagger] \\ \hline \end{array} \mathcal{M} \begin{array}{|c|} \hline [b^\dagger] \\ \hline [a^\dagger] \\ \hline \end{array} - \begin{array}{|c|} \hline [a] \text{---} [b^\dagger] \\ \hline [b] \text{---} [a^\dagger] \\ \hline \end{array} \mathcal{M} \begin{array}{|c|} \hline [b^\dagger] \\ \hline [a^\dagger] \\ \hline \end{array} \\ & - \begin{array}{|c|} \hline [b] \text{---} [a^\dagger] \\ \hline [a] \text{---} [b^\dagger] \\ \hline \end{array} \mathcal{M} \begin{array}{|c|} \hline [a^\dagger] \\ \hline [b^\dagger] \\ \hline \end{array} - \begin{array}{|c|} \hline [a] \text{---} [b^\dagger] \\ \hline [a] \text{---} [b^\dagger] \\ \hline \end{array} \mathcal{M} \begin{array}{|c|} \hline [b^\dagger] \\ \hline [b^\dagger] \\ \hline \end{array} - \begin{array}{|c|} \hline [b] \text{---} [a^\dagger] \\ \hline [b] \text{---} [a^\dagger] \\ \hline \end{array} \mathcal{M} \begin{array}{|c|} \hline [a^\dagger] \\ \hline [a^\dagger] \\ \hline \end{array} \end{aligned} \quad (3.29)$$

The matrix $\mathcal{M}$ is conformally invariant. The simplest way to prove this is by checking invariance under transformations (3.23-3.25), using the invariance of M , and Eq. (3.13). In particular we may replace a everywhere by $\Delta(x)$ in Eq. (3.29) and, thus,

$$\begin{array}{|c|} \hline [P(x)] \text{---} [P(y)] \\ \hline [P(x)] \text{---} [P(y)] \\ \hline \end{array} \mathcal{M} \begin{array}{|c|} \hline [P(y)] \\ \hline [P(y)] \\ \hline \end{array} = - \begin{array}{|c|} \hline [P(x)b] \text{---} [\Delta(x)^\dagger P(y)] \\ \hline [P(x)b] \text{---} [\Delta(x)^\dagger P(y)] \\ \hline \end{array} \mathcal{M} \begin{array}{|c|} \hline [\Delta(x)^\dagger P(y)] \\ \hline [\Delta(x)^\dagger P(y)] \\ \hline \end{array} \quad (3.30)$$

$$= -4\pi^2 \hat{C}(x,y) (x-y)^2, \quad (3.31)$$

since only the last term of Eq. (3.29) contributes after the conformal transformation, and using $\Delta(x)^\dagger P(y) = (x-y)^\dagger b^\dagger P(y)$ again. As a result we have the compact formula

$$\hat{G}(x, y) = \frac{\{P_1(x) \otimes P_2(x)\} \{1 - \pi_0\} \{P_1(y) \otimes P_2(y)\}}{4\pi^2 (x-y)^2} \quad (3.32)$$

Eq. (3.32) goes some way towards unifying the tensor product with the fundamental representation and we shall pursue this further in the next section.

4. THE TENSOR PRODUCTS OF INSTANTONS

As we have indicated in Sections 2 and 3 the existence of the form for the tensor product Green's function given in Eq. (3.32) is not really surprising. The self-dual $Sp(n_1) \times Sp(n_2)$ solution specified by Eq. (2.18) may be considered as a solution for a larger simple group, e.g. $SU(4n_1 n_2) \supset Sp(n_1) \times Sp(n_2)$. Because of the completeness of their results, there must exist a $\Delta(x)$, $\tilde{v}(x)$, etc., which will give us this solution directly through the ADHM construction. The complex dimension of $\tilde{\Delta}$ must be $(4n_1 n_2 + 2\tilde{k}) \times 2\tilde{k}$, where $\tilde{k}$ is the instanton number of the field tensor

$$F_{\alpha\beta} = F_{1\alpha\beta} \otimes 1_{2n_2} + 1_{2n_1} \otimes F_{2\alpha\beta} \quad (4.1)$$

regarded as an $SU(4n_1 n_2)$ solution. Using Eq. (2.8) we find

$$\tilde{k} = 2(n_1 k_2 + n_2 k_1)$$

This shows that we may not identify $\tilde{v}$ with $v_1 \otimes v_2$, since the former must have dimension $4(n_1 n_2 + k_1 n_2 + n_1 k_2) \times 4n_1 n_2$ whilst the latter has dimension $4(n_1 + k_1)(n_2 + k_2) \times 4n_1 n_2$. This fact is enough in itself to tell us that we need the extra (x, y) term in the tensor product Green's function because this Green's function would be given by Eq. (2.13) with $C = 0$ if and only if $\tilde{v} = v_1 \otimes v_2$. Indeed, looking at the form we have found, we see that

$$\tilde{v}(x)^\dagger \tilde{v}(y) = \{v_1(x) \otimes v_2(x)\}^\dagger \{1 - \pi_0\} \{v_1(y) \otimes v_2(y)\} \quad (4.3)$$

for all values of x and y . Now, at least for a typical tensor product, the columns of $v_1(x) \otimes v_2(x)$ will span the whole $4(n_1 + k_1)(n_2 + k_2)$ dimensional complex space. In consequence, the rank of $1 - \mathcal{M}$ cannot exceed that of the left-hand side, i.e. $4(n_1 n_2 + k_1 n_2 + k_2 n_1)$, and this must be true for all tensor products since the rank can only decrease at atypical points. For the same reason $1 - \mathcal{M}$ must be non-negative. Thus $\mathcal{M}$ has at least $4k_1 k_2$ unit eigenvalues and the rest must be less than or equal to unity.

The discussion would be simpler if $\mathcal{M}^2 = \mathcal{M}$, so that $1 - \mathcal{M}$ was a projection operator. This is possible *a priori* since Eqs. (3.19) and (3.29) imply that $\text{tr } \mathcal{M} = 4k_1 k_2$, consistent with all its eigenvalues being zero or unity. We have studied the eigenvalues of $\mathcal{M}$ numerically on a computer for a large number of particular cases (all corresponding to 't Hooft solutions for ease of computation) and explicitly checked the statement we have made about the eigenvalues and that $\mathcal{M} \neq \mathcal{M}^2$ in general. In Appendix A, we explicitly find $4k_1 k_2$ linearly independent unit eigenvectors of $\mathcal{M}$.

Because of the properties of its spectrum, $1 - \mathcal{M}$ has a unique positive square root and, as we indicated in Eq. (2.23), we may identify $\tilde{v}$ with $(1 - \mathcal{M})^{1/2} v_1 \otimes v_2$. We may see that this identification and Eq. (4.3) are consistent with the normalization of $\tilde{v}$ either from the non-singularity of $C(x,y)$ or, more directly, from the conformal invariance of $\mathcal{M}$; if we use the latter property to change all the a 's in Eq. (3.29) to Δ 's, we see that none of the terms in $\mathcal{M}$ contribute in Eq. (4.3) when we put $x = y$. Further we may see directly that

$$\tilde{A}_\alpha = A_{1\alpha} \otimes 1 + 1 \otimes A_{2\alpha} = \tilde{v}^\dagger \partial_\alpha \tilde{v} \quad (4.4)$$

by differentiating,

$$\{v_1(y) \otimes v_2(y)\}^\dagger \{v_1(x) \otimes v_2(x)\} = \tilde{v}(y)^\dagger \tilde{v}(x) + O(|x - y|^2) \quad (4.5)$$

with respect to x and then setting $x = y$.

Equations (2.23) and (4.4,4.5) together with the normalization condition $\tilde{v}^\dagger \tilde{v} = 1$ would be enough in themselves to establish the formula for the tensor product Green's function if we could also find a matrix $\tilde{\Delta}(x)$, linear in x such that

$$\tilde{v}(x)^\dagger \tilde{\Delta}(x) = 0 \quad (4.6)$$

determines $\tilde{v}$ up to normalization, and $\tilde{\Delta}(x)^\dagger \tilde{\Delta}(x)$ is an invertible matrix, proportional to the unit matrix in the quaternionic labels. One might look for a $\tilde{\Delta}(x)$ of the form $(1 - \mathcal{M})^{1/2} \{\Delta_1(x) \otimes \Delta_2(x)\}$ but this is quadratic in x . It seems that there is no obvious candidate for $\tilde{\Delta}(x)$. One would expect an arbitrariness corresponding to $GL(\tilde{k}, \mathbb{C})$ for reasons which are explained, for example, in Ref. 3. At first sight it will appear that the arbitrariness is even greater but this appearance is deceptive.

In attempting to construct a suitable $\tilde{\Delta}(x)$ we can be guided by the fact that it can be viewed as a mapping from the space of massless solutions of the Dirac equation in the background field in question. In Eqs. (2.25, 2.26) we summarized the form we found (by inductive guess work) for these solutions. The essential new ingredients needed to construct these solutions are matrices c and d satisfying the linear Eqs. (2.26) which have the diagrammatic form,

$$\begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \end{array} \begin{array}{c} \nearrow \\ \text{---} \end{array} [a^\dagger c] \quad = \quad \begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \end{array} \begin{array}{c} \nearrow \\ \text{---} \end{array} [a^\dagger d] \quad (4.7a)$$

$$\begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \end{array} \begin{array}{c} \nearrow \\ \text{---} \end{array} [b^\dagger c] \quad = \quad \begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \end{array} \begin{array}{c} \nearrow \\ \text{---} \end{array} [b^\dagger d] \quad (4.7b)$$

Eqs. (4.7) from the $4k_1k_2$ linear equations for the $2(n_1 + k_1)k_2 + 2(n_2 + k_2)k_1$ unknown elements of c and d . Thus we expect $\tilde{k} = 2n_1k_2 + 2n_2k_1$ linearly independent solutions which we label (c_r, d_r) . (There must be at least $\tilde{k}$ independent solutions and we shall argue in Appendix B that there can be no more if M exists.)

In constructing $\tilde{\Delta}(x)$, we first attempted to find as general as possible $\tilde{\Delta}(x)$, linear in x satisfying Eq. (4.6). In doing so we were led to introduce a $4(n_1 + k_1)(n_2 + k_2) \times k_1k_2$ dimensional matrix $\mathcal{N}$ of the form

$$\begin{aligned} \begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \end{array} \mathcal{N} \begin{array}{c} \text{---} \\ \text{---} \end{array} &= \begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} N_1 \begin{array}{c} \text{---} \\ \text{---} \end{array} + \begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} N_2 \begin{array}{c} \text{---} \\ \text{---} \end{array} \\ &+ \begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} Q_\alpha \begin{array}{c} \text{---} \\ \text{---} \end{array} + \begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} Q_\alpha \begin{array}{c} \text{---} \\ \text{---} \end{array} \end{aligned} \quad (4.8)$$

and define $\tilde{\Delta}(x)$ by

$$\tilde{\Delta}(x) = (1 - \sigma\tau)^{\frac{1}{2}} \tilde{\Gamma}(x) \quad (4.9)$$

where $\Gamma_{\lambda\mu, rR}(x)$, $1 \leq \lambda \leq 2(k_1 + n_1)$, $1 \leq \mu \leq 2(k_2 + n_2)$, $1 \leq r \leq k$, $1 \leq R \leq 2$, is defined by

$$\left[\begin{array}{c} \tilde{\Gamma}_{rR} \end{array} \right] = \left[\begin{array}{c} \Delta^{\dagger} c_r \\ \text{---} \\ \text{---} \end{array} \right] + \left[\begin{array}{c} \Delta \\ \text{---} \\ d_r \end{array} \right] + \left[\begin{array}{c} c_r \\ \text{---} \\ \Delta \end{array} \right] \quad (4.10)$$

In Eqs. (4.8-4.10) N_1 , N_2 and Q_{α} , $\alpha = 0, \dots, 3$, are square matrices of dimension $k_1 k_2$ and L_1 , L_2 are square matrices of dimension k_1 and k_2 , respectively.

It is reasonably straightforward to show, using the definitions and conformal invariance of $\mathcal{M}$ and $\mathcal{M}$, that $\tilde{\Delta}(x)$, as defined by Eqs. (4.8-4.10), satisfies Eq. (4.6), that is

$$\{v_i(x) \otimes v_j(x)\}^{\dagger} \{1 - \sigma\tau\} \tilde{\Gamma}(x) = 0, \quad (4.11)$$

provided that N_1 , N_2 , Q_{α} , L_1 , L_2 satisfy

$$\left[\begin{array}{c} \text{---} \\ b^{\dagger} b \end{array} \right] N_1 + \left[\begin{array}{c} \text{---} \\ a^{\dagger} a \end{array} \right] N_2 + \left[\begin{array}{c} \text{---} \\ a^{\dagger} b e_{\alpha} \end{array} \right] Q_{\alpha} + \left[\begin{array}{c} \text{---} \\ L_1 \end{array} \right] = 0$$

$$\left[\begin{array}{c} b^{\dagger} b \end{array} \right] N_1 + \left[\begin{array}{c} a^{\dagger} a \end{array} \right] N_2 + \left[\begin{array}{c} a^{\dagger} b e_{\alpha} \end{array} \right] Q_{\alpha} - \left[\begin{array}{c} L_2 \end{array} \right] = 0 \quad (4.12)$$

Further details of the proof of this statement are given in Appendix B. It is for deriving statements like this that a diagrammatic notation seems almost obligatory to us. Merely relying on suffices obscures the essential simplicity of the algebra; in any case one quickly runs out of letters.

Although constrained by Eqs. (4.12) a vast amount of arbitrariness remains in $\tilde{\Delta}$. It would appear to have $6k_1^2k_2^2 + k_1^2 + k_2^2$ degrees of freedom subject to $2k_1^2k_2^2$ constraints, leaving, apparently, $4k_1^2k_2^2 + k_1^2 + k_2^2$. Some of this freedom may be quickly shown to be illusory; clearly we may change $\tilde{\Gamma}$, using the $4k_1k_2$ linearly independent with vectors of $1 - \mathcal{M}$, without affecting $\tilde{\Delta}$ or the validity of Eqs. (4.6) or (4.11). (See Appendix A for further details.) In fact this accounts for $4k_1^2k_2^2$ of the degrees of freedom and we may think of the remaining $k_1^2 + k_2^2$ effective degrees of freedom as corresponding to the ability to choose L_1 and L_2 arbitrarily. This last point is reinforced by the explicit calculations of $(1 - \mathcal{M})\tilde{\Gamma}(x)$ and

$$\tilde{\Delta}(x)^\dagger \tilde{\Delta}(x) = \tilde{\Gamma}(x)^\dagger (1 - \mathcal{M}) \tilde{\Gamma}(x) \quad (4.13)$$

which are reported in Appendix B, and show that these quantities depend only on L_1 and L_2 , the dependence on N_1 , N_2 and Q_α cancelling out. This is important because whilst *in general* Eqs. (4.12) have solutions for N_1 , N_2 , Q_α for any given L_1, L_2 , for specific tensor products (in particular, for the $q \otimes \bar{q}$ representation) we cannot choose L_1, L_2 freely and obtain a solution. On the other hand what we need to construct is $\tilde{\Delta}(x) = (1 - \mathcal{M})^{1/2} \tilde{\Gamma}(x)$ which will depend only on L_1, L_2 because $(1 - \mathcal{M})\tilde{\Gamma}(x)$ does. We may regard the calculations of Appendix B as being performed for a typical tensor but use the results, which depend only on L_1, L_2 , for an arbitrary tensor product. Having realized this, we see that N_1, N_2, Q_α and Eqs. (4.12) are really irrelevant, their only function having been to enable us to calculate $\tilde{\Delta}(x)^\dagger \tilde{\Delta}(x)$.

The calculations of Appendix B also show that $\tilde{\Delta}(x)^\dagger \tilde{\Delta}(x)$ is diagonal in and independent of the quaternionic labels, and so may be regarded as an Hermitian matrix of dimension $\tilde{k}$ [see the ADHM formalism for $SU(n)$ as described in Section 2 and Ref. 3]. Further it is established there, using properties of the c 's and d 's, that it has the following simple expression

$$[\tilde{\Delta}(x)^\dagger \tilde{\Delta}(x)]_{rR, sS} = [Z^\dagger \Omega^{-1} Z]_{rs} \delta_{RS} \quad (4.14)$$

where

$$Z_{rs} = \begin{bmatrix} [c_r^\dagger c_s] \\ [L_2] \end{bmatrix} + \begin{bmatrix} [L_1] \\ [d_r^\dagger d_s] \end{bmatrix} \quad (4.15)$$

and

$$\Omega_{rs} = \begin{bmatrix} [c_r^\dagger c_s] \\ [f] \end{bmatrix} + \begin{bmatrix} [f] \\ [d_r^\dagger d_s] \end{bmatrix} - \begin{bmatrix} [c_r^\dagger \Delta] & [f] \\ [f] & [\Delta^\dagger d_s] \end{bmatrix} \quad (4.16)$$

Note that the last term in Eq. (4.16) may be written as

$$- \begin{bmatrix} [c_r^\dagger \Delta] & [f] & [\Delta^\dagger c_s] \\ [f] \end{bmatrix} \quad (4.17)$$

or one of two other symmetrical possibilities.

The non-singularity of Ω follows from that of f_1, f_2 using Eqs. (4.16-4.17). Also Z is clearly non-singular for suitable choices of L_1, L_2 , for example, $L_1 = L_2 = 1$. [It is here that the irrelevance of N_1, N_2, Q_α becomes important because they do not always exist in this case. In the obvious cases in which they do exist like $L_1 = -(a_1^\dagger a_1)^{-1}$, $L_2 = (a_2^\dagger a_2)^{-1}$, $N_1 = Q_\alpha = 0$, $(N_2)_{ij,lm} = (a_1^\dagger a_1)^{-1}_{ie} (a_2^\dagger a_2)^{-1}_{jm}$, Z is not always non-singular.]

Equation (4.14) shows completely explicitly that the $\tilde{f}(x) = [\tilde{\Delta}(x)^\dagger \tilde{\Delta}(x)]^{-1}$ for the various $\tilde{\Delta}(x)$ we have found, i.e. for the various L_1, L_2 , are related to $\Omega(x)$ by the *constant* matrices $Z \in GL(k, \mathbb{C})$, and so to one another, as had to be the case from the nature of the ADHM construction.

Because of the non-singularity of $\tilde{\Delta}(x)^\dagger \tilde{\Delta}(x)$ Eq. (4.6) defines $\tilde{v}(x)$ and we have found a construction of the ADHM type for the tensor product. The form for the tensor product Green's function now follows from the argument used for the fundamental representation^{3,4)} and in the next section we shall apply the formula for the massless solutions of the Dirac equation^{3,5)} to obtain Eq. (2.25).

5. THE MASSLESS SOLUTIONS OF THE DIRAC EQUATION

In this section we shall consider the massless Dirac equation for a tensor product of instantons,

$$\gamma_\alpha \tilde{\partial}_\alpha \psi = 0, \quad (5.1)$$

where the covariant derivative, $\mathbb{D}_\alpha$ is as in Eq. (2.18) and, as in Ref. 3, we take as a representation of the γ matrices

$$\gamma_\alpha = \begin{pmatrix} 0 & e_\alpha \\ e_\alpha^\dagger & 0 \end{pmatrix} \quad (5.2)$$

Equation (6.1) possesses no normalizable positive chirality ($\gamma_5 \psi \equiv \gamma_0 \gamma_1 \gamma_2 \gamma_3 \psi = +\psi$) solutions and exactly $\tilde{k}$ negative chirality solutions. For these

$$\psi = \begin{pmatrix} \psi_L \\ 0 \end{pmatrix} \quad (5.3)$$

with

$$e_\alpha^\dagger \gamma_\alpha \psi_L = 0. \quad (5.4)$$

For the corresponding Dirac equation for the fundamental representation, the k independent solutions are given by^{3,5)}

$$(\psi_L)_{a,i} = (v^\dagger b f \varepsilon)_{a,i} \quad (5.5)$$

for each fixed i , $1 \leq i \leq k$, i being the spinor index. Following the formulation of the last section the solution to Eq. (5.4) will be given by

$$\tilde{v}^\dagger \tilde{b} \tilde{f} \tilde{\varepsilon} = (v_1 \otimes v_2) (1 - \gamma_5) \hat{b} \varepsilon \tilde{z}^{-1} \mathcal{Q} \tilde{z}^{\dagger-1} \quad (5.6)$$

where $\tilde{\Gamma}(x) = \hat{a} + \hat{b}x$. Eq. (5.6) looks formidable but simplifies greatly. In fact Eq. (5.6) reduces to

$$\left\{ \begin{array}{|c|} \hline [v^\dagger b] \quad [f] \\ \hline \quad \quad \quad \nearrow^R \\ \hline [v^\dagger d_r] \end{array} \right\} + \left\{ \begin{array}{|c|} \hline [v^\dagger c_r] \\ \hline \quad \quad \quad \nearrow^R \\ \hline [v^\dagger b] \quad [f] \end{array} \right\} (Z^{\dagger-1})_{rs} \quad (5.7)$$

To establish this we first note, from Appendix B, that $(v_1 \otimes v_2)(1 - \mathcal{M})\delta\epsilon$ simplifies to

$$\begin{aligned}
 & \left[\begin{array}{c} [v^\dagger c_r] \\ [v^\dagger b][L_2] \end{array} \right] + \left[\begin{array}{c} [v^\dagger b][L_1] \\ [v^\dagger d_r] \end{array} \right] + \\
 & \left[\begin{array}{c} [v^\dagger b] \\ [v^\dagger b] \end{array} \right] M \left\{ \left[\begin{array}{c} [b^\dagger c_r] \\ [\Delta^\dagger \Delta L_2] \end{array} \right] - \left[\begin{array}{c} [\Delta^\dagger \Delta L_1] \\ [b^\dagger d_r] \end{array} \right] + \left[\begin{array}{c} [\Delta^\dagger c_r] \\ [\Delta^\dagger b][L_2] \end{array} \right] + \left[\begin{array}{c} [\Delta^\dagger b][L_1] \\ [\Delta^\dagger d_r] \end{array} \right] \right\}
 \end{aligned} \tag{5.8}$$

It remains to show that we obtain expression (5.8) if we multiply expression (5.7) by $Z^\dagger \Omega^{-1} Z$. To do this we use the relations for $c_r(\Omega^{-1})_{rs}^\dagger$, $c_r(\Omega^{-1})_{rs} d_s^\dagger$, etc. proved in Appendix B, together with Eq. (3.13).

The form for the Dirac equation solutions given is just that we claimed in Eq. (2.25) apart from the (irrelevant) constant matrix $Z^{\dagger-1}$. It is straightforward to check directly that these satisfy Eq. (5.4) using the results for the fundamental representation and Eqs. (4.7). Indeed this was the way we established this result first, but now we see it fitting into the same framework as the previous results.

6. CONCLUSIONS AND OUTLOOK

Our aim in beginning this study was to understand the structure of the adjoint representation Green's function, whose form at first seems rather complicated and enigmatic compared to that for the fundamental representation. We have seen that this structure enables us to solve a much wider class of problems, involving tensor products of instantons. Further, we have seen how to unify our treatment of these problems with the treatment of the fundamental representation. Results proved for the fundamental representation can be immediately extended to this more general situation. In principle, we can obtain results for higher dimensional representations iteratively.

On the other hand, since we have often obtained elegant results at the end of extensive and tedious algebra, we conclude that there must be better concepts in terms of which to frame the discussion. Presumably there should be some natural geometrical way to approach these problems.

We hope that our results go some small way to indicate the way the work of Atiyah et al.^{1,2)} has increased one's computational power in dealing with self-dual gauge fields. The solutions to the problems of the Green's function and the massless Dirac wave functions for the fundamental and adjoint representation, which were obtained before their work with considerable ingenuity for the special case of the 't Hooft SU(2) solutions, can now be unified and completely generalized using a few simple formulae.

The surprising simplicity of what has so far been discovered leads one to hope that there may be comparable results, for example, for the determinants of the covariant Laplacian and the covariant Dirac operator. Another related problem which is of considerable interest is the possible relation of functional integrals over gauge fields to the (gauge invariant) description of solutions, and other fields, provided by the ADHM construction.

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ZERO EIGENVECTORS OF $1 - \mathcal{M}$

In this Appendix we construct $4k_1k_2$ linearly independent eigenvectors of $\mathcal{M}$ with unit eigenvalues and discuss their relation to the construction of $\tilde{\Delta}(x)$. We consider vectors of the form

$$\begin{bmatrix} \chi \end{bmatrix} = \begin{bmatrix} [a] \\ [a] \end{bmatrix} A + \begin{bmatrix} [b] \\ [b] \end{bmatrix} B + \begin{bmatrix} [be_\alpha] \\ [a] \end{bmatrix} C_\alpha + \begin{bmatrix} [a] \\ [be_\alpha] \end{bmatrix} C_\alpha \quad (A1)$$

where $A, B, C_\alpha, \alpha = 0, \dots, 3$ are arbitrary vectors of dimension k_1k_2 . Using the definitions of $\mathcal{M}$, M and Eq. (3.13) one may show by straightforward algebra (facilitated by the diagrammatic notation) that

$$\mathcal{M} \chi = \chi \quad (A2)$$

provided that

$$\begin{bmatrix} [a^\dagger a] \\ \text{---} \end{bmatrix} A + \begin{bmatrix} [b^\dagger b] \\ \text{---} \end{bmatrix} B + \begin{bmatrix} [a^\dagger b e_\alpha] \\ \text{---} \end{bmatrix} C_\alpha = 0 \quad (A3)$$

and

$$\begin{bmatrix} \text{---} \\ [a^\dagger a] \end{bmatrix} A + \begin{bmatrix} \text{---} \\ [b^\dagger b] \end{bmatrix} B + \begin{bmatrix} \text{---} \\ [a^\dagger b e_\alpha] \end{bmatrix} C_\alpha = 0 \quad (A4)$$

Since A, B, C_α possess $6k_1k_2$ degrees of freedom and Eq. (A3-4) provides $2k_1k_2$ equations we are left with (at least) $4k_1k_2$ linearly independent solutions for A, B, C_α . We should show that these yield linearly independent solutions for χ . This will follow if we show that $\chi = 0$ implies A, B, C_α vanish. To show this consider multiplying Eq. (A1) by

$$\begin{bmatrix} \text{---} [b^\dagger] \\ \vdots \\ \text{---} [b^\dagger] \end{bmatrix} \quad (A5)$$

Using Eqs. (A3-A4) we thus see that $\chi = 0$ implies $(M^{-1})A$ vanishes and so that A vanishes. It may be shown similarly that B, C_α vanish if $\chi = 0$.

Comparing Eq. (A1) with Eqs. (4.8-4.10) we see that given one solution of Eqs. (4.12) for given L_1, L_2 , we may use the vectors χ to construct new N_1, N_2 , Q_α , and hence a new $\tilde{\Gamma}$, which also solve the equation. In fact it is clear that given one such solution we may obtain $4k_1^2 k_2^2$ linearly independent solutions in this way. But changing $\tilde{\Gamma}$ in this way does not effect $\tilde{\Delta} = (1 - \mathcal{M})^{1/2} \tilde{\Gamma}$ because the vectors χ are null vectors of $1 - \mathcal{M}$. Thus we have understood $4k_1^2 k_2^2$ of the $4k_1^2 k_2^2 + k_1^2 + k_2^2$ degrees of freedom of $\tilde{\Delta}$; they are not effective. Really everything is fixed by the choice of L_1, L_2 as we show in Appendix B.

PROPERTIES OF $\tilde{\Delta}(x)$

We begin this appendix by discussing the derivation of Eqs. (4.12). We begin by noting that $\mathcal{N}$, as defined by Eq. (4.8) is unchanged if we replace a by Δ and replace N_1 by N_1' , N_2 by N_2' and Q_α by Q_α' , where

$$\begin{aligned} N_1' &= N_1 - 2x_\alpha Q_\alpha + x^2 N_2 \\ N_2' &= N_2 \\ Q_\alpha' &= Q_\alpha - x_\alpha N_1 \end{aligned} \tag{B1}$$

Further, Eqs. (4.12) are equivalent to the same Eqs. for N_1', N_2', Q_α' with a everywhere replaced by Δ . Using this new expression for $\mathcal{N}$ and the new form of Eqs. (4.12) it is straightforward to show, using the definition of M , that $\{v_1 \otimes v_2\}[1 - \mathcal{M}]\mathcal{N}$ is given by

$$- \begin{array}{|c|} \hline [v^\dagger b] \\ \hline [v^\dagger b] \\ \hline \end{array} M \begin{array}{|c|} \hline [\Delta^\dagger \Delta L_1] \\ \hline \end{array} + \begin{array}{|c|} \hline [v^\dagger b] \\ \hline [v^\dagger b] \\ \hline \end{array} M \begin{array}{|c|} \hline [\Delta^\dagger \Delta L_2] \\ \hline \end{array} \tag{B2}$$

Eq. (4.11) now follows immediately from Eq. (4.10).

Using similar techniques [together with Eq. (3.13)], but working rather harder, one may express $\{1 - \mathcal{M}\}\mathcal{N}$ entirely in terms of L_1 and L_2 obtaining

$$\begin{aligned} & - \begin{array}{|c|} \hline [b] \\ \hline [b] \\ \hline \end{array} \begin{array}{|c|} \hline [a^\dagger a L] \\ \hline \end{array} - \begin{array}{|c|} \hline [a] \\ \hline [a] \\ \hline \end{array} \begin{array}{|c|} \hline [b^\dagger b L] \\ \hline \end{array} - \begin{array}{|c|} \hline [a] \\ \hline [b] \\ \hline \end{array} \begin{array}{|c|} \hline [a^\dagger b L] \\ \hline \end{array} - \begin{array}{|c|} \hline [b] \\ \hline [a] \\ \hline \end{array} \begin{array}{|c|} \hline [b^\dagger a L] \\ \hline \end{array} \\ & + \begin{array}{|c|} \hline [b] \\ \hline [b] \\ \hline \end{array} \begin{array}{|c|} \hline [a^\dagger a L] \\ \hline \end{array} + \begin{array}{|c|} \hline [a] \\ \hline [a] \\ \hline \end{array} \begin{array}{|c|} \hline [b^\dagger b L] \\ \hline \end{array} + \begin{array}{|c|} \hline [a] \\ \hline [b] \\ \hline \end{array} \begin{array}{|c|} \hline [b^\dagger a L] \\ \hline \end{array} + \begin{array}{|c|} \hline [b] \\ \hline [a] \\ \hline \end{array} \begin{array}{|c|} \hline [a^\dagger b L] \\ \hline \end{array} \end{aligned} \tag{B3}$$

where we have further developed our diagrammatic notation by abbreviating $[M]$ to two vertical lines: $||$ and by suppressing the indices on L_1 and L_2 . Note that (B3) is conformally invariant. Going further one may calculate $\mathcal{N}^\dagger(1 - \mathcal{M})\mathcal{N}$ to obtain

$$\begin{aligned}
 & \begin{array}{c} [l^+ a^+ a] \\ \hline \end{array} \begin{array}{c} [b^+ b L] \\ \hline \end{array} + \begin{array}{c} [l^+ b^+ b] \\ \hline \end{array} \begin{array}{c} [a^+ a L] \\ \hline \end{array} - \begin{array}{c} [l^+ b^+ a] \\ \hline \end{array} \begin{array}{c} [a^+ b L] \\ \hline \end{array} \\
 & + \begin{array}{c} \hline \\ [l^+ a^+ a] \end{array} \begin{array}{c} \hline \\ [b^+ b L] \end{array} + \begin{array}{c} \hline \\ [l^+ b^+ b] \end{array} \begin{array}{c} \hline \\ [a^+ a L] \end{array} - \begin{array}{c} \hline \\ [l^+ b^+ a] \end{array} \begin{array}{c} \hline \\ [a^+ b L] \end{array} \\
 & - \begin{array}{c} [l^+ a^+ a] \\ \hline \end{array} \begin{array}{c} \hline \\ [b^+ b L] \end{array} - \begin{array}{c} [l^+ b^+ b] \\ \hline \end{array} \begin{array}{c} \hline \\ [a^+ a L] \end{array} + \begin{array}{c} [l^+ b^+ a] \\ \hline \end{array} \begin{array}{c} \hline \\ [b^+ a L] \end{array} + \begin{array}{c} [l^+] \\ \hline \end{array} \begin{array}{c} [L] \\ \hline \end{array} \\
 & - \begin{array}{c} \hline \\ [l^+ a^+ a] \end{array} \begin{array}{c} [b^+ b L] \\ \hline \end{array} - \begin{array}{c} \hline \\ [l^+ b^+ b] \end{array} \begin{array}{c} [a^+ a L] \\ \hline \end{array} + \begin{array}{c} \hline \\ [l^+ b^+ a] \end{array} \begin{array}{c} [b^+ a L] \\ \hline \end{array} + \begin{array}{c} [L] \\ \hline \end{array} \begin{array}{c} [l^+] \\ \hline \end{array}
 \end{aligned}
 \tag{B4}$$

Working even harder one may proceed to calculate $(\tilde{\Delta}^+ \tilde{\Delta})_{rR, sS}$; using (B4) we obtain the following expression

$$\begin{aligned}
 & \begin{array}{c} [c^+ c] \\ \hline \end{array} \begin{array}{c} [l^+ \Delta^+ \Delta L] \\ \hline \end{array} - \begin{array}{c} [c^+ \Delta] \\ \hline \end{array} \begin{array}{c} [\Delta^+ c] \\ \hline \end{array} + \begin{array}{c} [c^+ b] \\ \hline \end{array} \begin{array}{c} [\Delta^+ c] \\ \hline \end{array} + \begin{array}{c} [c^+ \Delta] \\ \hline \end{array} \begin{array}{c} [b^+ c] \\ \hline \end{array} - \begin{array}{c} [c^+ b] \\ \hline \end{array} \begin{array}{c} [b^+ c] \\ \hline \end{array} \\
 & + \begin{array}{c} [l^+ \Delta^+ \Delta L] \\ \hline \end{array} \begin{array}{c} [d^+ d] \\ \hline \end{array} - \begin{array}{c} [l^+ \Delta^+ b] \\ \hline \end{array} \begin{array}{c} [b^+ \Delta L] \\ \hline \end{array} + \begin{array}{c} [l^+ \Delta^+ \Delta] \\ \hline \end{array} \begin{array}{c} [b^+ \Delta L] \\ \hline \end{array} + \begin{array}{c} [l^+ \Delta^+ b] \\ \hline \end{array} \begin{array}{c} [\Delta^+ \Delta L] \\ \hline \end{array} - \begin{array}{c} [l^+ \Delta^+ \Delta] \\ \hline \end{array} \begin{array}{c} [\Delta^+ \Delta L] \\ \hline \end{array} \\
 & + \begin{array}{c} [l^+ \Delta^+ c] \\ \hline \end{array} \begin{array}{c} [d^+ \Delta L] \\ \hline \end{array} + \begin{array}{c} [l^+ \Delta^+ b] \\ \hline \end{array} \begin{array}{c} [\Delta^+ c] \\ \hline \end{array} - \begin{array}{c} [l^+ \Delta^+ \Delta] \\ \hline \end{array} \begin{array}{c} [\Delta^+ c] \\ \hline \end{array} - \begin{array}{c} [l^+ \Delta^+ b] \\ \hline \end{array} \begin{array}{c} [b^+ c] \\ \hline \end{array} + \begin{array}{c} [l^+ \Delta^+ \Delta] \\ \hline \end{array} \begin{array}{c} [b^+ c] \\ \hline \end{array} \\
 & + \begin{array}{c} [c^+ \Delta L] \\ \hline \end{array} \begin{array}{c} [d^+ \Delta L] \\ \hline \end{array} + \begin{array}{c} [c^+ \Delta] \\ \hline \end{array} \begin{array}{c} [b^+ \Delta L] \\ \hline \end{array} - \begin{array}{c} [c^+ b] \\ \hline \end{array} \begin{array}{c} [b^+ \Delta L] \\ \hline \end{array} - \begin{array}{c} [c^+ \Delta] \\ \hline \end{array} \begin{array}{c} [\Delta^+ \Delta L] \\ \hline \end{array} + \begin{array}{c} [c^+ b] \\ \hline \end{array} \begin{array}{c} [\Delta^+ \Delta L] \\ \hline \end{array}
 \end{aligned}
 \tag{B5}$$

where we have suppressed the labels r, s on the c and d matrices, multiplied by δ_{rs} . This expression has a certain elegance but is rather unwieldy. It is important to realise that it may be simplified by writing the "central parts" of each line in terms of c and d . Let us define

$$\mathcal{E} = \begin{array}{c} \text{---} \\ \text{---} \end{array} - \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} + \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} + \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} - \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \quad (B6)$$

and define $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$ similarly using the second, third and fourth rows of (B5), respectively, in the same way we have used the first row to define $\mathcal{E}$. Now it is straightforward to show that

$$\left[\begin{array}{c} \text{---} \\ \text{---} \end{array} \right] \begin{array}{c} \text{---} \\ \text{---} \end{array} = \left[\begin{array}{c} \text{---} \\ \text{---} \end{array} \right] \begin{array}{c} \text{---} \\ \text{---} \end{array} \quad \left[\begin{array}{c} \text{---} \\ \text{---} \end{array} \right] \begin{array}{c} \text{---} \\ \text{---} \end{array} = \left[\begin{array}{c} \text{---} \\ \text{---} \end{array} \right] \begin{array}{c} \text{---} \\ \text{---} \end{array} \quad (B7a)$$

$$\left[\begin{array}{c} \text{---} \\ \text{---} \end{array} \right] \begin{array}{c} \text{---} \\ \text{---} \end{array} = \left[\begin{array}{c} \text{---} \\ \text{---} \end{array} \right] \begin{array}{c} \text{---} \\ \text{---} \end{array} \quad \left[\begin{array}{c} \text{---} \\ \text{---} \end{array} \right] \begin{array}{c} \text{---} \\ \text{---} \end{array} = \left[\begin{array}{c} \text{---} \\ \text{---} \end{array} \right] \begin{array}{c} \text{---} \\ \text{---} \end{array} \quad (B7b)$$

Using the definition of M we can show that Eqs. (B7) hold with Δ replaced by b . It follows from this and the fact that the matrices (c_r, d_r) form a complete set of solutions to Eqs. (4.7) that we must be able to write

$$\mathcal{E} = c_r X_{rs} c_s^\dagger, \quad \mathcal{F} = d_r X_{rs} d_s^\dagger, \quad \mathcal{G} = d_r X_{rs} c_s^\dagger, \quad \mathcal{H} = c_r X_{rs} d_s^\dagger \quad (B8)$$

for some matrix X . It is not difficult to determine X ; consider

$$\left[\begin{array}{c} \text{---} \\ \text{---} \end{array} \right] \begin{array}{c} \text{---} \\ \text{---} \end{array} + \left[\begin{array}{c} \text{---} \\ \text{---} \end{array} \right] \begin{array}{c} \text{---} \\ \text{---} \end{array} - \left[\begin{array}{c} \text{---} \\ \text{---} \end{array} \right] \begin{array}{c} \text{---} \\ \text{---} \end{array} \quad (B9)$$

This simplifies to give just c_t . Since the c_r 's are linearly independent this implies

$$X_{\Omega} \Omega_{st} = \delta_{st}$$

with Ω defined as in Eq. (4.16). So $X = \Omega^{-1}$ and using this in expression (B5) gives Eq. (4.14). [We first found the relations of Eq. (B8) by studying the relationship of the Dirac equation Green's function to the complete sum over the solutions of the Dirac equation; compare the calculation of Ref. 5.]

The manifest positive definiteness of $\tilde{\Delta}^{\dagger}\tilde{\Delta}$ for suitable L_1, L_2 shows that $\tilde{\Delta}$ may have as many linearly independent columns as there are independent solutions for (c,d), multiplied by two for quaternionic label. Since the number of columns, being each orthogonal to the $4k_1k_2$ columns of $\tilde{v}$ in a space of dimension $4k_1k_2 + 4n_1k_2 + 4n_2k_1$, can not exceed $4n_1k_2 + 4k_1n_2$, we conclude that there are precisely $\tilde{k}$ linearly independent solutions (c,d).

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