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ON DUALITY BETWEEN RESONANCES AND FREE
QUARK-ANTIQUARK PAIRS

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ABSTRACT

The semi-classical formula due to Krammer and Leal-Ferreira, and Quigg and Rosner, for the S wave bound state wave function at the origin, is generalized to all partial waves, and the relation to duality is traced to a common dependence on short time behaviour.

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Consider a reaction such as

$$e^+e^- \rightarrow c\bar{c} \quad (1)$$

where c is the charmed quark. The final state may be considered as just a free quark-antiquark pair and the cross section calculated in Born approximation. But more realistically, the quark and the antiquark form a hadronic resonance, or are distributed over multi-hadron states. The hypothesis¹⁾ that these two descriptions are dual to one another, and give the same cross-section when resonance structure is averaged over energy, has proved fruitful²⁾.

The average over resonances can also be calculated in potential models either by numerical computation³⁾ for various potentials or by using approximate formulae for the resonance wave functions⁴⁾. We are here concerned with the latter approach. The formulae of Krammer and Leal-Ferriera⁵⁾ and of Quigg and Rosner⁶⁾ are for S-waves. We will simplify the derivation, generalize to higher partial waves and recall why duality between resonance averages and free particle behaviour emerges in such an approach.

The Born approximation for (1) gives the cross section

$$\sigma = \frac{4\pi\alpha^2}{3s} 3e_q^2 \frac{v(3-v^2)}{2} \theta(s-4m^2) \quad (2)$$

where e_q is the quark charge (in units of e), m is the quark mass and $(1-v^2) = 4m^2/s$. On the other hand in the narrow resonance approximation, the cross section can be written as the sum

$$\sigma = \sum_j 12\pi^2 \delta(s-M_j^2) M_j^{-1} \Gamma_j(e^+e^-) \quad (3)$$

where M_j is the mass of resonance j and Γ_j is its partial width to decay into e^+e^- . Using the Weisskopf-Van Royen formula⁷⁾

$$\Gamma_j(e^+e^-) = 16\pi\alpha^2 e_q^2 |\psi_j(0)|^2 M_j^{-2} \quad (4)$$

duality between (3) and (2) requires

$$\frac{| \psi_j(0) |^2}{\Delta M_j} = \frac{1}{16\pi^2} v_j M_j^2 (1 - v_j^2/3) \quad (5)$$

where ΔM_j is the spacing of S wave ($J = 1$) $c\bar{c}$ resonances in the neighborhood of j and

$$1 - v_j^2 = 4m^2/M_j^2. \quad (6)$$

With a hypothetical "scalar photon" which couples with strength g to e^+e^- and gg_Q with $c\bar{c}$ we would have instead of (2)

$$\sigma = 4\pi \left(\frac{g^2}{4\pi} \right)^2 3g_Q^2 \frac{v^3}{4s} \quad (7)$$

$$1 - v^2 = 4m^2/s$$

and instead of (3) and (4)

$$\sigma = \sum_j 4\pi^2 \delta(s - M_j^2) M_j^{-1} \Gamma_j \quad (8)$$

and

$$\Gamma_j = 72 \left(\frac{g^2}{4\pi} \right)^2 3g_Q^2 \frac{|R'_j(0)|^2}{M_j^4} \quad (9)$$

where the radial wave function $R_j(r)$ of the j^{th} 3P_0 state satisfies the normalization

$$\int r^2 |R_j(r)|^2 dr = 1.$$

Demanding duality between (7) and (8) leads to the requirement

$$\frac{|R'_j(0)|^2}{\Delta M_j} = \frac{1}{144\pi} v_j^3 M_j^4. \quad (10)$$

where ΔM_j is the spacing of P wave (even parity) $J = 0$ $c\bar{c}$ resonances.

Krammer and Leal-Ferreira⁵⁾ and Quigg and Rosner⁶⁾ showed that Eq. (5) emerges directly from a non-relativistic ($v^2 \ll 1$) potential model in the WKB approximation. We will use instead^{*)} the closely related Thomas-Fermi approximation for the density of a Fermi gas with one particle per level in a given potential well $V(r)$ which we shall assume is non-singular at the origin and set $V(0) = 0$. If all levels up to an energy E are filled we have for the density

$$\sum_{E_n \leq E} |\psi_n(r)|^2 = \int_{|\vec{k}| \leq k_F(r)} \frac{d^3 \vec{k}}{(2\pi)^3} \quad (11)$$

$$= k_F^3(r) / 6\pi^2. \quad (12)$$

with

$$E - V(r) = k_F^2(r) / 2\mu \quad (13)$$

where $\mu = m/2$ is the reduced mass appearing in Schrödinger equation. Differentiating Eq. (12) with respect to E and recalling that only S-waves contribute at the origin, we get using $E_j = mv_j^2$

$$\frac{|\psi_j(0)|^2}{\Delta M_j} = \frac{m^2 v_j}{4\pi^2} \quad (14)$$

which coincides with Eq. (5) [apart from the relativistic correction factor $(1 - v^2/3)$] since in the non-relativistic approximation $M_j \approx 2m$.

To get an expression for the p wave function, consider the expression for the kinetic energy of a Fermi gas

*) Our formulae follow also from an improved WKB approximation for partial waves based on spherical Bessel functions.

$$\sum_{E_n \leq E} |\vec{\nabla} \psi_n(0)|^2 = \int_{|\vec{k}| \leq k_F(0)} k^2 \frac{d^3 \vec{k}}{(2\pi)^3} \quad (15)$$

$$= \frac{3}{5} k_F^2(0) \frac{k_F^3(0)}{6\pi^2} \quad (16)$$

so that we get

$$\frac{|R'_j(0)|^2}{\Delta M_j} = \frac{1}{9\pi} m^4 v_j^3 \quad (17)$$

in agreement with Eq. (10).

Clearly such relations can be further generalized to higher partial waves. The result expressed in terms of the normalized radial wave function and the reduced mass $\mu (= m/2)$ is

$$\frac{\left| d^l R_{n,l} / dr^l \right|_{r=0}^2}{\Delta E_{n,l}} = \frac{1}{\pi} \left(\frac{l!}{(2l+1)!!} \right)^2 (2\mu E_{n,l})^{l+1/2} \quad (18)$$

where $\Delta E_{n,l}$ is the spacing of energy levels with a given l in the neighborhood of $E_{n,l}$. To illustrate how this approximation works for a hadronic well we show in Table I the ratio of the correct values of the left-hand side of Eq. (18) to the right-hand side Thomas-Fermi values. The spacing ΔE in this case is constant.

Now it is not just a happy accident that the Thomas Fermi formulae exhibit the duality in question. The relevance of the Born approximation for (1) depends on the hypothesis that ψ and $\bar{\psi}$ move freely for a short time, and that the subsequent influence of the confining potential introduces only fine structure in energy dependence. Let us recall how the Thomas Fermi formulae also emerge directly from these ideas.

Consider for the potential well the function

$$G(\vec{r}, \vec{r}', t) = \sum_n \psi_n(\vec{r}) \psi_n(\vec{r}') e^{-iE_n t} \quad (19)$$

where the sum is over the complete set of energy eigenstates. It reduces at $t = 0$ to $\delta(\vec{r} - \vec{r}')$ and satisfies the Schrödinger equation for all t . Plausibly it is given for small t by the free space value

$$G(\vec{r}, \vec{r}', t) \approx \int \frac{d^3 \vec{k}}{(2\pi)^3} e^{i\vec{k} \cdot (\vec{r} - \vec{r}')} e^{-i(\frac{k^2}{2m} + V(r))t} \quad (20)$$

in which the potential (at $r \approx r'$) enters only as origin of energy. Multiplying by a damping factor

$$e^{-\frac{1}{2} \Delta^2 t^2} \quad (21)$$

to eliminate large times, and Fourier transforming with respect to t , we have from (19) and (20)

$$\begin{aligned} & \sum_n \psi_n(\vec{r}) \psi_n(\vec{r}') e^{-\frac{1}{2} (E_n - \omega)^2 / \Delta^2} \\ & \approx \int \frac{d^3 \vec{k}}{(2\pi)^3} e^{i\vec{k} \cdot (\vec{r} - \vec{r}')} e^{-\frac{1}{2} [\frac{k^2}{2m} + V(r) - \omega]^2 / \Delta^2} \quad (22) \end{aligned}$$

That is to say that an average over products of wave function in the neighborhood of a given energy ω is approximated by the corresponding integral over plane waves. Integrating over ω to a given "Fermi level"

gives what we have called the Thomas Fermi approximation, with the Fermi surface fuzzed by an amount Δ . Differentiating with respect to $\vec{r}$ and $\vec{r}'$ (treating V as constant), before setting $\vec{r} = \vec{r}' = 0$, gives the extra factors $\vec{k}$ in formulae like (15).

So far we considered the potential V to be constant near the origin. But it is commonly supposed that the quark-antiquark potential contains also a singular part due to gluon exchange. Then explicit formulae like (18) no longer hold. But the idea of duality remains plausible. For short times, or coarse graining in energy, it should be sufficient to pay attention to the singular part of the potential. The plane waves of (22), and therefore the Born approximation (1), should be modified to allow for it⁸⁾.

TABLE: The correct value of

$$\left| \frac{d^l R_{nl}(\gamma)}{d\gamma^l} \right|_{\gamma=0}^2$$

to its Thomas Fermi value Eq. (18) for the potential $V(r) = \frac{1}{2}m\omega^2 r^2$.

	n = 1	n = 2	n = 3	n = 4
l = 0	1.023	1.005	1.002	1.001
l = 1	0.951	0.985	0.993	0.996
l = 2	0.820	0.927	0.961	0.976

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