Ref.TH.2612-CERN

SOME USES OF THE HOLONOMY GROUP

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A B S T R A C T

Holonomy group considerations are a useful tool in studying the problem of embedding in non-Abelian gauge theories defined on S^4 . The (ir-)reducibility of a gauge potential is characterized by the local behaviour of the gauge field and its covariant derivatives. The semi-continuity property of the holonomy group leads to the conclusion that (anti)-self-dual orbits form open sets in the orbit space of irreducible connections.

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Ref.TH.2612-CERN
10 December 1978

It is well known ^{1),2)} that, in pure Yang-Mills gauge theory in R^4 , the requirement of the finiteness of the action ($\|F_A\|^2 < \infty$) leads to the construction of principal bundles over S^4 . Here S^4 is the one-point compactification of R^4 .

It is of interest to study the set of connections on any one of such bundles, P . We seek a global characterisation of these connections and, therefore, we turn our attention to their holonomy groups. The following global considerations are, now, in order :

- (a) S^4 is connected ; hence the holonomy groups $\Phi(u)$ of a connection $\forall u \in P$ are all isomorphic. So we just refer to the holonomy group Φ of the connection under study ;
- (b) S^4 is also simply connected ; consequently Φ is a connected group for any connection in P ;
- (c) suppose the holonomy group of a connection were Abelian ; then we could have a reduced bundle with structure group an Abelian $U(1)$ group ; however, the isomorphism classes of $U(1)$ bundles over S^4 are in 1-1 correspondence with $H^2(S^4, Z)$; the vanishing of the latter shows that there are no non-trivial $U(1)$ bundles over S^4 and consequently there are no non-trivial $U(1)$ connections.

So, if we assume that the gauge group of the Y-M theory is some compact Lie group, then from (b) and (c) we could conclude that the holonomy group of any connection on P is necessarily semi-simple.

Now, it is shown by Singer ³⁾ that the irreducible connections form a dense set in the space of all connections. Hence we could concentrate our attention on irreducible connections in P . For such connections the holonomy group coincides with the structure group of the bundle. Thus, we are led to the study of irreducible connections in $P(S^4, G)$ with $G = G_1 \times G_2 \times \dots$, where the factors G_i are simple (non-Abelian) Lie groups. However, any such irreducible connection gives rise to a gauge field which can be decoupled into a number of non-interacting fields ⁴⁾. Equivalently the original G bundle splits into a product of bundles with structure groups the simple factors G_i .

At this stage we can make use of a further topological property of S^4 , namely $H^2(S^4, Z_n) = 0$, for all n (Z_n is the group of integers modulo n) and lift any connection in $P(S^4, G_1)$ to a connection in a G_1 -

bundle where $\bar{G}_i$ is the covering group of G_i . To see this we recall the following exact sequence ⁵⁾

$$\longrightarrow H^1(S^4, Z_n) \longrightarrow B_{\bar{G}_i}(S^4) \longrightarrow B_{G_i}(S^4) \longrightarrow H^2(S^4, Z_n) \longrightarrow$$

where $G_i = \bar{G}_i / Z_n$, for some n , and $B_{\bar{G}_i}$ (resp. B_{G_i}) are the isomorphism classes of $\bar{G}_i$ (resp. G_i) bundles. Since $H^1(S^4, Z_n) = H^2(S^4, Z_n) = 0$ we get the desired result.

Conclusion

The general problem of studying irreducible connections arising from Y-M theory on S^4 with gauge group a compact Lie group is equivalent to the study of irreducible connections in G -bundles for all compact simply-connected simple Lie groups G .

Specializing the above statement to (anti)-self-dual connections we see that, if we know all irreducible (anti)-self-dual G -connections, where G is any one of the following groups $SU(n)$, $Spin(n)$ ($n \geq 7$), $Sp(n)$, G_2 , F_4 , E_8 , $\bar{E}_6$, $\bar{E}_7$, then we can construct any irreducible (anti)-self-dual gauge potential with compact Lie group as a gauge group. A procedure for getting all irreducible (anti)-self-dual G -connections is given in Refs. 6) and 7).

The problem of embedding

In trying to construct irreducible G -connections one encounters the following situation. A number of G -connections result from embeddings of subgroups $H \subset G$ in G . These are the reducible connections. Of course, it is not always easy to recognize such reducible connections, because in general an H -connection can look very complicated after a number of gauge transformations taking values in G .

A way of checking the (ir-)reducibility of a given connection is provided by the following theorem ⁸⁾.

Reduction theorem

Let $P(M, G)$ be a principal G -bundle with a connection Γ , where M is connected and paracompact. Let u_0 be an arbitrary point in P . Denote by P_{u_0} the set of points in P which can be joined to u_0 by a horizontal curve. Then (i) P_{u_0} is a reduced bundle with structure group the holonomy group $\mathfrak{H}(u_0)$; (ii) the connection Γ is reducible to a connection in P_{u_0} .

According to this theorem all we have to do is to check the holonomy group Φ , of the given G connection. If Φ coincides with G then our connection is irreducible. Otherwise it is reducible to a Φ -connection.

Now the holonomy groups of a connection in $P(S^4, G)$ are all isomorphic. So we can choose a point $x_0 \in S^4$ and examine the holonomy group in an open neighbourhood of x_0 . From Ref. 4) we know that the Lie algebra of the holonomy group is generated by $F_{\mu\nu}, \nabla_\mu F_{k\lambda}, \nabla_\mu \nabla_\nu F_{k\lambda}, \dots$ evaluated at x_0 . Thus, if the gauge field and its covariant derivatives generate a proper subalgebra of $\mathfrak{g}$, then the connection is an embedding of a subgroup of G .

The above conclusion is stated as a theorem in Ref. 9), but it is now seen to be a consequence of the reduction theorem and the analysis of Loos. Because of this it is not necessary to introduce an open covering of S^4 and check the consistency of the reduction on the overlap. The consistency is automatic, because the reduction theorem is a global statement concerning the bundle $P(S^4, G)$. Introducing a covering of S^4 together with transition functions gives only rise to a co-ordinate bundle over S^4 . Any statement, however, about the bundle P is also true about any co-ordinate bundle, since P is the equivalence class of such co-ordinate bundles. This is another indication of how powerful the language of fibre bundles is, when we are dealing with global problems.

Semi-continuity of Φ

We would like to end this short note with the following application of the semi-continuity property of the holonomy group.

In Ref. 10) (using deformation theory), it is shown that in the vicinity of an irreducible self-dual orbit there exist only self-dual orbits. [This is a result "half-way" towards proving the non-existence of non-(anti)-self-dual connections on S^4]. In order to conclude from this result that self-dual orbits form open sets in the orbit space of irreducible connections we have to show the irreducibility of the deformed connection.

Now, the irreducibility of the deformed connection follows immediately from the semi-continuity property¹¹⁾ of the holonomy group. Let Φ_0 be the holonomy group of Γ_0 and Φ_t the holonomy group of Γ_t obtained from Γ_0 via local deformations. Then

$$\dim \Phi_t \geq \dim \Phi_0$$

Since Γ_0 is assumed irreducible $\dim \Phi_0 = \dim \mathbb{G}$. Thus, $\dim \Phi_t \geq \dim \mathbb{G}$. However, from general considerations we know that the holonomy group of any connection in a $\mathbb{G}$ -bundle is a subgroup of $\mathbb{G}$. Hence $\Phi_t \subset \mathbb{G}$. Consequently $\dim \Phi_t = \dim \mathbb{G}$ and Γ_t is irreducible. Thus, we reach the following conclusion.

Self-dual orbits form open sets in the orbit space of irreducible connections on S^4 .

I thank W. Nahm for reading the manuscript and for helpful comments.

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