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AXIAL ANOMALY AND BOUNDARY TERMS FOR GENERAL SPINOR FIELDS

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A B S T R A C T

Index and G index for general spinor fields are evaluated by topological methods. This leads to a simple determination of the axial anomaly for general spinor fields and of the non-local boundary term, which allows for fractional winding numbers in spite of the integrality of the index. The results are relevant for gravitational vacuum tunneling of higher spin fields.

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In this note we shall compute the indices and G-indices of elliptic operators between very general spinor bundles. This solves the problem of determining axial anomalies for arbitrary spins, yields information about the behaviour of fermionic zero modes under the action of the isometry group of the underlying space-time manifold and leads to a determination of a non-local boundary correction term, which helps to render the index an integer also for compact manifolds with boundary and for certain non-compact manifolds. The simplicity of the calculations will illustrate the power of the topological methods employed.

To fix our notions and notations let us start out with the following situation. Consider an $N = 4\ell$ -dimensional compact oriented Riemannian manifold M (Euclidean space-time), which is furthermore assumed to have a spin structure in order to assure that spinor fields can be defined on M . So, one can construct vector bundles $\Delta_+(M)$ and $\Delta_-(M)$ of fundamental spinor fields of positive and negative γ_5 -parity. We have $\dim \Delta_{\pm}(M) = 2^{2\ell-1}$, and for $N = 4$ the sections of these bundles just correspond to spinor fields of spin $s = \frac{1}{2}$ with definite chirality. It is well known that the Dirac operator $D = \gamma^i \nabla_i$ is an elliptic operator

$$D : \Delta_+(M) \rightarrow \Delta_-(M) \quad (1)$$

It is further known^{1,2)} that the index density of this operator, expressed in terms of the Riemannian curvature of M is just twice the gravitational anomaly of the $s = \frac{1}{2}$ axial current.

We now consider the general spinor bundles

$$\Delta^{mn}(M) = S^m \Delta_+(M) \otimes S^n \Delta_-(M) \quad (2)$$

where $S^r \Delta_{\pm}(M)$ is the r -fold symmetric tensor product of $\Delta_{\pm}(M)$. For $N = 4$ these are the most general spinor bundles, which correspond to finite dimensional irreducible representations of $\text{Spin}(4)$, the two-fold covering group of $\text{SO}(4)$. The bundles $\Delta^{mn}(M)$ in general do not describe physical particles of definite spin, but a superposition of spins between $\frac{1}{2}|m - n|$ and $\frac{1}{2}|m + n|$. Particles of definite higher spin are normally not given by any finite dimensional bundles over M but by formal linear combinations of bundles $\Delta^{mn}(M)$ with integer and not always positive coefficients. The contributions with negative coefficients correspond to ghost fields, which cancel unphysical degrees of freedom. The appropriate mathematical notion for these objects is the K -module³⁾ $K(M)$ over M , and fields of definite spin are sections of virtual bundles in $K(M)$.

It is easy to see that there exist universally interpretable⁴⁾ elliptic operators

$$D_{mn} : \Delta^{mn}(M) \rightarrow \Delta^{nm}(M) \quad (3)$$

Our first task is to evaluate

$$I_{mn}[M] = \text{ind } D_{mn} = \dim \text{kernel } D_{mn} - \dim \text{kernel } D_{mn}^+ \quad (4)$$

The index theorem^{4,5)} tells us

$$\text{ind } D_{mn} = (-1)^{\frac{N(N+1)}{2}} \frac{\text{ch } \Delta^{mn}(M) - \text{ch } \Delta^{nm}(M)}{e(TM)} \text{td}(TM \otimes \mathbb{C})[M] \quad (5)$$

independent of D_{mn} and the spin structure. Here TM is the tangent bundle of the space-time manifold M , $TM \otimes \mathbb{C}$ its complexification, td , e and ch are the Todd class, the Euler class and the Chern character⁴⁾, respectively. Altogether the right-hand side is a non-homogeneous cohomology class, expressible as a closed differential form, which in turn is a polynomial in the Riemannian curvature tensor. The symbol $[M]$ means that one has to extract the part of degree $N = \dim M$ and evaluate it on the compact manifold M .

According to the splitting principle⁴⁾ one can treat all the occurring bundles as if they were direct sums of one-dimensional vector bundles. Moreover, for the complexified tangent bundle there exists a formal splitting

$$TM \otimes \mathbb{C} = \bigoplus_{i=1}^{2\ell} (T_i \oplus \bar{T}_i) \quad (6)$$

such that the first Chern classes $c_1(T_i)$, $c_1(\bar{T}_i) \in H^2(M)$ fulfil

$$c_1(T_i) = -c_1(\bar{T}_i) = x_i \quad (7)$$

$$e(TM) = \prod_{i=1}^{2\ell} x_i \quad (8)$$

For details on the calculation of the index and the splitting principle see Ref. 4. In terms of the quantities x_i the Todd class is given by

$$\begin{aligned} \text{td}(TM \otimes \mathbb{C}) &= \prod_{i=1}^{2\ell} \frac{-x_i^2}{(1-e^{-x_i})(1-e^{x_i})} = 1 - \frac{1}{12} \sum x_i^2 + \dots \\ &= 1 - \frac{1}{12} p_1 + \dots \end{aligned} \quad (9)$$

where p_1 is the first Pontryagin class of TM , which, expressed by the Riemannian curvature, is

$$p_1 = \frac{1}{8\pi^2} \text{tr } R \wedge R \quad (10)$$

Finally $\text{ch } \Delta^{\text{mn}}$ has to be determined and expressed by the quantities x_i . Generally, for one-dimensional complex vector bundles E_i with $c_1(E_i) = y_i$, $\text{ch } E_i$ is defined by

$$\text{ch } E_i = e^{y_i} \quad (11a)$$

and for splittable bundles $E = \oplus E_i$ one has

$$\text{ch } E = \sum e^{y_i} \quad (11b)$$

This leads to an additivity property of ch :

$$\text{ch}(E \oplus F) = \text{ch } E + \text{ch } F \quad (11c)$$

Furthermore, because of $c_1(E_i \otimes E_j) = y_i + y_j$, multiplicativity also holds :

$$\text{ch}(E \otimes F) = \text{ch } E \text{ ch } F \quad (11d)$$

The spinor bundles $\Delta_{\pm}(M)$ have formal splittings

$$\Delta_{\pm}(M) = \bigoplus_{i=1}^{2^{\ell-1}} \Delta_{\pm}^i(M) \quad (12a)$$

$$\text{ch } \Delta_{\pm}(M) = \sum e^{y_i^{\pm}} \quad (12b)$$

The quantities $y_i^{\pm}$ are found in terms of the quantities $x_i^{\pm}$ by expressing the characters of the fundamental spinor representation of $\text{Spin}(N)$ in terms of its natural N -dimensional representation.

Introducing the $2^{2\ell}$ quantities

$$\frac{1}{2} (\pm x_1 \pm x_2 \pm \dots \pm x_{2\ell}) \quad (13)$$

one shows^{4,5)} that $\{y_i^+\}$ and $\{y_i^-\}$ are given by the combinations with an even and an odd number of plus signs, respectively. This checks with $\dim \Delta_{\pm}(M) = 2^{2\ell-1}$.

In particular for $N = 4$

$$\begin{aligned} y_1^+ &= \frac{1}{2}(x_1 + x_2) = -y_2^+ := y^+ \\ y_1^- &= \frac{1}{2}(x_1 - x_2) = -y_2^- := y^- \end{aligned} \quad (14)$$

The remaining purely algebraic problem is to calculate for a splittable bundle V the Chern character of the symmetric tensorial products $S^n V$. The n^{th} exterior power $\Lambda^n V$ and the n^{th} symmetric power $S^n V$ fulfil the following evident identities

$$\Lambda^n (E \oplus F) = \bigoplus_{k=0}^n \Lambda^k E \otimes \Lambda^{n-k} F \quad (15a)$$

$$S^n (E \oplus F) = \bigoplus_{k=0}^n S^k E \otimes S^{n-k} F, \quad (15b)$$

which, introducing the formal power series

$$\Lambda_t V = \bigoplus t^n \Lambda^n V; \quad S_t V = \bigoplus t^n S^n V \quad (16)$$

can be written as

$$\Lambda_t (E \oplus F) = \Lambda_t E \otimes \Lambda_t F \quad (17a)$$

$$S_t (E \oplus F) = S_t E \otimes S_t F \quad (17b)$$

Now, for one-dimensional bundles one finds at once

$$\text{ch } \Lambda^n E_i = \begin{cases} 1 & \text{for } n=0 \\ e^{y_i} & \text{for } n=1 \\ 0 & \text{for } n>1 \end{cases}, \text{ hence } \text{ch } \Lambda_t E_i = 1 + t e^{y_i} \quad (18a)$$

$$\text{ch } S^n E_i = e^{n y_i}, \text{ hence } \text{ch } S_t E_i = \sum t^n e^{n y_i} = \frac{1}{1 - t e^{y_i}} \quad (18b)$$

Using Eqs. (11) and (17) we finally obtain

$$\text{ch } \Lambda_t E = \prod_i (1 + t e^{y_i}) \quad (19)$$

$$\text{ch } S_t E = \prod_i \frac{1}{1 - t e^{y_i}},$$

from which $\text{ch } S^{\mathbf{n}}E$ and $\text{ch } \Lambda^{\mathbf{n}}E$ are obtained by taking the coefficient of $t^{\mathbf{n}}$. Notice in particular

$$\text{ch } S_+ E \text{ ch } \Lambda_- E = 1 \quad (20)$$

Now the evaluation of $I_{\mathbf{mn}}[M] = \text{ind } D_{\mathbf{mn}}$ is straightforward. We restrict ourselves to the case $N = \dim M = 4$, which is of principal physical interest (higher dimensions may, for instance, be relevant for extended supergravity theories).

Using Eq. (14) one gets

$$\begin{aligned} \text{ch } S_{\pm} \Delta_{\pm}(M) &= \frac{1}{(1 - t e^{y^{\pm}})(1 - t e^{-y^{\pm}})} \\ &= \sum_n t^n \sum_{k=0}^n e^{(2k-n)y^{\pm}} \\ &= \sum_n t^n \frac{\sinh(n+1)y^{\pm}}{\sinh y^{\pm}} \end{aligned} \quad (21)$$

Hence

$$\text{ch } S^{\mathbf{n}} \Delta_{\pm}(M) = \sum_{k=0}^n e^{(2k-n)y^{\pm}} = \frac{\sinh(n+1)y^{\pm}}{\sinh y^{\pm}} \quad (22)$$

Equations (5) to (9) tell us that we have to expand $\text{ch } S^{\mathbf{n}}\Delta_{\pm}$ up to terms of order four in $y^{\pm}$:

$$\text{ch } S^{\mathbf{n}} \Delta_{\pm}(M) = A_n + 2B_n(y^{\pm})^2 + \frac{2}{3} C_n (y^{\pm})^4 + \dots \quad (23a)$$

with

$$A_n = n+1 \quad ; \quad B_n = \sum_{k=0}^n (k - \frac{n}{2})^2 \quad ; \quad C_n = \sum_{k=0}^n (k - \frac{n}{2})^4 \quad (23b)$$

This gives with Eq. (14)

$$\begin{aligned} \text{ch } \Delta^{\mathbf{mn}}(M) - \text{ch } \Delta^{\mathbf{nm}}(M) &= \text{ch } S^{\mathbf{m}} \Delta_+(M) \text{ch } S^{\mathbf{n}} \Delta_-(M) - \text{ch } S^{\mathbf{n}} \Delta_+(M) \text{ch } S^{\mathbf{m}} \Delta_-(M) \\ &= 2(A_n B_m - A_m B_n) x_1 x_2 \\ &\quad + \frac{1}{3} (A_n C_m - A_m C_n) x_1 x_2 (x_1^2 + x_2^2) + \dots \end{aligned} \quad (24)$$

Inserting into Eq. (5) together with Eqs. (8) and (9) finally yields

$$\begin{aligned} I_{mn}[M] = \text{ind } D_{mn} &= \left\{ 2(A_n B_m - A_m B_n) + \frac{1}{3}(A_n C_m - A_m C_n) p_1 + \dots \right\} \left\{ 1 - \frac{1}{12} p_1 + \dots \right\} [M] \\ &= \frac{1}{6} \left\{ A_n (2C_m - B_m) - A_m (2C_n - B_n) \right\} p_1 [M] \end{aligned} \quad (25)$$

Using standard formulae for the arithmetic series B_r and C_r we obtain our final formula for the index

$$I_{mn}[M] = -\frac{(m+1)(n+1)}{720} \left\{ n(n+2)(3n^2+6n-14) - m(m+2)(3m^2+6m-14) \right\} p_1 [M] \quad (26)$$

in complete agreement with the result announced in Ref. 6. Very recently, related results have been derived⁷⁾ by heat transport methods. In particular

$$I_{1,0} = \text{ind } D_{1,0} = -\frac{1}{24} p_1 [M] \quad (27)$$

the standard result for the index of the Dirac operator. One can in fact convince oneself that

$$I_{mn}[M] = r I_{1,0}[M] \quad (28)$$

where r is always an integer. This means that the higher bundles Δ^{mn} do not give stronger vanishing theorems than the Dirac operator. Another special case is the massless field of definite spin $\frac{3}{2}$ (6-8). The result determines the axial anomaly of supergravity. One obtains

$$\begin{aligned} I_{3/2}[M] &= I_{2,1}[M] - 2I_{1,0}[M] = \left(\frac{19}{24} + \frac{2}{24} \right) p_1 [M] = \frac{21}{24} p_1 [M] \\ &= -21 I_{1,0}[M] \end{aligned} \quad (29)$$

So, we see again⁶⁻⁸⁾ that the spin $\frac{3}{2}$ axial anomaly is -21 times the spin $\frac{1}{2}$ result.

Furthermore, one finds

$$I_{2,0}[M] = \frac{1}{3} p_1 [M] = \tau \quad (30)$$

the signature of M . This has to be so, because spinor fields in $S^2 \Delta_{\pm}$ can be identified with self dual and anti-self dual two-forms for $N = 4$. As another very useful quantity we now evaluate the G-index^{4,5)} for our spinor bundles.

We consider a compact group G acting on M by orientation-preserving isometries. Given a spin structure on M one can construct canonical actions on the bundles $\Delta^{mn}(M)$ such that for every $m \in M$ and every $g \in G$ the fibre over m is linearly mapped onto the fibre over gm . Furthermore, we can choose the operator D_{mn} such that it commutes with the action of G . Then kernel D_{mn} and kernel D_{mn}^* are stable under M and provide us with finite dimensional representation spaces of G . Taking traces of $g \in G$ in these spaces one can define

$$I_{mn}^g[M] = \text{ind}_g D_{mn} = \text{tr} g \Big|_{\text{kernel } D_{mn}} - \text{tr} g \Big|_{\text{kernel } D_{mn}^*}.$$

The G-index theorem⁵⁾ then states

$$I_{mn}^g[M] = \varepsilon^{m+n} (-1)^{\frac{N}{2}} \frac{\text{ch}_g j^* \Delta^{mn} - \text{ch}_g j^* \Delta^{nm}}{e(TM^g) \text{ch}_g \wedge_{-1}(N^g \otimes \mathbb{C})} \text{Id}(TM^g \otimes \mathbb{C})[M^g], \quad (31)$$

where M^g is the fixed point set of g acting on M , TM^g the tangent bundle of M^g , N^g the normal bundle of M^g in TM . ch_g denotes the equivariant Chern character⁵⁾, which can be computed by a generalization of the splitting principle once the action of G on the bundle is known, j^* means restriction to M^g , and $\varepsilon = \pm 1$ depending on the spin structure. We cannot go into details here⁹⁾ and restrict ourselves to the following special case. We take $N = \dim M = 4$ and assume that M^g consists of a single fixed point. More general cases like fixed surfaces are analogous^{4,9)}. Then the action of g on the four-dimensional vector space $N^g \otimes \mathbb{C}$ can be block-diagonalized and is given by two rotation angles $\theta_1(g)$ and $\theta_2(g)$.

In this case one has

$$\text{ch}_g \wedge_{-1}(N^g \otimes \mathbb{C}) = 16 \sin^2 \frac{\theta_1}{2} \sin^2 \frac{\theta_2}{2} \quad (32)$$

$$\begin{aligned} \text{ch}_g j^* S^m \Delta_{\pm}(M) &= \sum_{k=0}^m e^{i(k - \frac{m}{2})(\theta_1 \pm \theta_2)} \\ &= \frac{\sin \frac{m+1}{2} (\theta_1 \pm \theta_2)}{\sin \frac{1}{2} (\theta_1 \pm \theta_2)} \end{aligned} \quad (33)$$

Altogether

$$\begin{aligned}
 I_{mn}^g[M] &= \varepsilon^{m+n} \frac{\sin \frac{m+n}{2}(\theta_1 + \theta_2) \sin \frac{n+n}{2}(\theta_1 - \theta_2) - \sin \frac{n+n}{2}(\theta_1 + \theta_2) \sin \frac{m+n}{2}(\theta_1 - \theta_2)}{16 \sin \frac{1}{2}(\theta_1 + \theta_2) \sin \frac{1}{2}(\theta_1 - \theta_2) \sin^2 \frac{\theta_1}{2} \sin^2 \frac{\theta_2}{2}} \\
 &= \varepsilon^{m+n} \frac{(n+1) \sin(n+1)\theta - (m+1) \sin(m+1)\theta}{16 \sin \theta \sin^4 \frac{\theta}{2}} \quad \text{for } \theta_1 = \theta_2 = \theta \\
 &= \varepsilon^{m+n} \frac{1}{16} (m+1)(n+1) [(-1)^m - (-1)^n] \quad \text{for } \theta = \pi
 \end{aligned} \tag{34}$$

This result is, for instance, important for asymptotically locally Euclidean gravitational instantons¹⁰⁾. These are non-compact Euclidean solutions of Einstein's equations with an (anti-) self dual Riemannian curvature tensor which tends to zero at infinity and can be conceived as having at infinity a boundary of the form S^3/G . Here S^3 is the three-dimensional sphere, equipped with its standard metric of constant curvature, and G is a finite group of isometries, which acts freely on S^3 . For these non-compact gravitational instanton spaces, the index theorem has to be supplemented by the non-local boundary term of Atiyah, Patodi and Singer¹¹⁾ and for the operator D_{mn} assumes the form

$$I_{mn}[M] = \overset{\circ}{I}_{mn}[M] - \xi_{mn}[\partial M] \tag{35}$$

Here $\overset{\circ}{I}_{mn}$ is the expression of Eq. (26) with $p_1[M]$ replaced by the convergent but not necessarily integer-valued integral

$$\overset{\circ}{p}_1[M] = \frac{1}{8\pi^2} \int_M \text{tr} R \wedge R \tag{36}$$

and $\xi_{mn}[\partial M]$ is the non-local boundary term, which is given by^{9,11,12)}

$$\xi_{mn}[\partial M] \equiv \frac{1}{|G|} \sum_{g \neq 1} I_{mn}^g[M] \pmod{\mathbb{Z}} \tag{37}$$

As the simplest example¹³⁾ we can treat the Eguchi-Hanson gravitational instanton¹⁴⁾. Here we have $\overset{\circ}{p}_1[M] = -3$, $G = \mathbb{Z}_2$, the cyclic group of order two and $S^3/G = \mathbb{RP}(3)$, the three-dimensional real projective space. With $\theta_1 = \theta_2 = \pi$ we get

$$\xi_{mn} [\partial M] = -\frac{1}{32} (m+1)(n+1) [(-1)^m - (-1)^n] , \quad (38)$$

generalizing the result of Ref. 12. [Notice that $\xi_{mn} [\partial M] = 0$ for $m+n$ even, corresponding to a univalued representation of $SO(4)$.]

For example, one rediscovers $\text{ind } D_{1/2} = 0$, $\text{ind } D_{3/2} = -2^{12,15}$ for spinor fields of definite spin $1/2$ and $3/2$.

Indeed, one can see that

$$\begin{aligned} I_{mn} [M] &= \frac{(m+1)(n+1)}{240} \{ n(n+2)(3n^2+6n-14) - m(m+2)(3m^2+6m-14) \} \\ &\quad + \frac{1}{32} (m+1)(n+1) [(-1)^m - (-1)^n] \end{aligned}$$

is always an integer.

Some immediate generalizations of our results are possible. First, one can add a gauge degree of freedom with a local gauge group H . This corresponds to tensorizing with a vector bundle V_H over M with structural group H . In this case e.g. for $N=4$, $I_{mn} [M]$ in Eq. (26) has to be replaced by

$$\begin{aligned} I_{mn}^H [M] &= \dim V_H \cdot I_{mn} [M] + 2 (A_n B_m - A_m B_n) \text{ch}_2 V_H [M] \\ &= \dim V_H \cdot I_{mn} [M] + \frac{(m+1)(n+1)}{6} [m(m+2) - n(n+2)] \text{ch}_2 V_H [M] . \end{aligned} \quad (39)$$

Here $\text{ch}_2 V_H$ is a multiple of the well-known winding number density, a quadratic polynomial in the Yang-Mills field strength. The easily computable coefficient only depends on the representation of H , which defines V_H .

Secondly, one can consider the slightly more general case

$$D_{mn}^{(k)} : \Delta^{mn} (M) \rightarrow \Delta^{m-k, n+k} (M) \quad (40a)$$

This can be reduced to the case already treated by writing

$$D_{mn}^{(k)} = 1 \otimes D_{k,0} : \Delta^{m-k, n} (M) \otimes \Delta^{k,0} (M) \rightarrow \Delta^{m-k, n} (M) \otimes \Delta^{0,k} (M) \quad (40b)$$

One only has to multiply the index density of $D_{k,0}$ by $\text{ch} \Delta^{m-k, n}$. The operators, which are treated in Ref. 7 by heat transport methods, are closely related to $D_{mn}^{(1)}$.

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