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HIGHER ORDER EFFECTS IN ASYMPTOTICALLY FREE
GAUGE THEORIES : II. FLAVOUR SINGLET WILSON
OPERATORS AND COEFFICIENT FUNCTIONS

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A B S T R A C T

We complete the calculation of all parameters needed to discuss deep inelastic scattering in QCD to subleading order by calculating the anomalous dimensions of the twist two flavour singlet operators in the Wilson expansion to two loops and the coefficient functions to order $\bar{g}^2$. The calculation is performed in the dimensional regularization scheme with the minimal subtraction renormalization prescription. The application of the results to deep inelastic scattering is discussed.

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1. INTRODUCTION

Deep inelastic lepton-nucleon scattering seems to be the cleanest place to test the only serious candidate as a field theory of strong interactions which we have so far, Quantum Chromodynamics^{1),2)}.

The successful confrontations of QCD with the deep inelastic lepton-nucleon scattering data have been followed by positive tests of QCD in other hard scattering processes $e^+e^- \rightarrow \text{hadrons}$, lepton-pair production in pp collisions and large transverse momentum inclusive production in pp scattering^{*)}.

In two previous papers^{7),8)} we analyzed the scaling violation data for the flavour non-singlet part of the electroproduction structure functions including higher order corrections. The aim was to see how good an expansion scheme is provided by the renormalization group, even for relatively large values of the running coupling constant ($\bar{g}^2/4\pi \sim 0.2-0.5$). Since then new experimental data have been presented and a new analysis has been done by Bardeen et al.⁹⁾ for the non-singlet part of the structure functions and it was found that the higher order corrections are indeed significant.

In this article we present the details and results of the full next to leading order corrections for the singlet part of the structure functions. From the phenomenological point of view the singlet part of the structure functions is concentrated towards the low end of the Bjorken scaling variable $0 < x < 0.3$. Most of the neutrino-nucleon and muon-nucleon deep inelastic data also lie in this region.

In order to carry out the analysis for the singlet part we need to calculate the anomalous dimensions of the singlet Wilson operators to order g^4 and the coefficient functions to order $\bar{g}^2$ in the same renormalization scheme. So in section 2 we discuss the operator mixing problem in two loops and develop the formalism for the moments of the structure functions including all subleading order corrections; in section 3 we describe the method of evaluating the anomalous dimensions of the singlet operators in two loops; in section 4 we discuss the calculation of the coefficient functions to order $\bar{g}^2$. Because of the enormous complexity of the general answer for the anomalous dimensions (we consider about one hundred diagrams) in section 5 we present the necessary information for calculating corrections to the moments up to the 20th moment and for 3 to 6 flavours, in the form of numerical tables. In the Appendix we give

*) There are far too many papers on this subject to make a full list of references possible here. A sample set is presented in references 3) to 6).

the simple and double pole parts of the 14 basic two-loop integrals which are needed for the calculations of the anomalous dimensions (these are calculated in $4-\epsilon$ dimensions).

2. THE OPERATOR MIXING PROBLEM

From the Wilson operator product expansion one can obtain the following relation for the N^{th} moment of one of the structure functions [e.g. $F_2(x, q^2)$] for lepton hadron scattering

$$\int_0^1 dx x^{N-2} F_2(x, q^2) = [C_{2,NS}^N(q^2) \langle p | O_{NS}^N | p \rangle + C_{2,f}^N(q^2) \langle p | O_f^N | p \rangle + C_{2,g}^N(q^2) \langle p | O_g^N | p \rangle] \quad (2.1)$$

where $\langle p | O_{NS}^N | p \rangle = p^{\mu_1} \dots p^{\mu_N} = \langle p | O_{NS}^{\mu_1 \dots \mu_N} | p \rangle$

$O_{NS}^{\mu_1 \dots \mu_N}$ being the spin N , twist 2, flavour non-singlet operator. Similarly O_f^N and O_g^N refer to the fermion and gluon components of the spin N , twist 2, flavour singlet operators. The q^2 dependence of the coefficient functions is obtained from the renormalization group equation which involves the anomalous dimension of the operator. For the non-singlet operator this is a single equation and the anomalous dimensions up to order g^4 have been discussed in ref. 7). The two components of the singlet operators (with the same spin) have identical quantum numbers so that they mix in higher orders; i.e., one can generate $O_f^{\mu_1 \dots \mu_N}$ from the higher order corrections to $O_g^{\mu_1 \dots \mu_N}$ and vice versa. This mixing is infinite so that the anomalous dimension (for each N) is a 2×2 matrix. Thus the renormalization group equation is a pair of coupled linear differential equations.

$$\left(\left(-2 \frac{\partial}{\partial \ln q^2} + \beta(g) \frac{\partial}{\partial g} \right) \delta_{ab} - \gamma_{ba}^N \right) C_b^N(q^2) = 0 \quad (2.2)$$

where we have suppressed the suffix of C_b^N which indicates which structure function is under consideration; a and b run over f and g . The matrix γ_{ab}^N may be written (to order g^4)

$$\gamma_{ab}^N = \frac{g^2}{16\pi^2} \gamma_{(a)ab}^N + \frac{g^4}{(16\pi^2)^2} \gamma_{(4)ab}^N \quad (2.3)$$

One can define a matrix U_{ab}^N which diagonalizes $\gamma_{(0)ab}^N$

$$U_{ia}^{N-1} \gamma_{\omega ab}^N U_{bj}^N = \begin{pmatrix} \delta_+^N & 0 \\ 0 & \delta_-^N \end{pmatrix} \quad (2.4)$$

(i,j run over + -).

The matrix U^N is independent of g and therefore if we work to leading order only we could diagonalize Eq.(2.2) giving us two uncoupled equations. However, if we try to diagonalize the whole of γ_{ab}^N of Eq.(2.3) we would obtain a diagonalizing matrix which contains the coupling constant, g , and would therefore not commute with $\frac{\partial}{\partial g}$, so that Eq.(2.2) cannot be diagonalized in this way. Gross and Wilczek ¹⁰⁾ have given a general solution to Eq.(2.2)

$$C_a^N(q^2, g) = \sum_b \left\{ T \exp \left[- \int_0^t \gamma^N(g) dg \right] \right\}_{ba} C_b^N(q^2 = \mu^2, \bar{g}(q^2)) \quad (2.5)$$

where T indicates path ordering of the integrals in the expansion of the exponential and $t = \ln q^2 / \mu^2$. In practice Eq. (2.5) is rather difficult to handle and we prefer to write Eq.(2.2) as an inhomogeneous differential equation, using Eq.(2.3)

$$\begin{aligned} & \left(\left(-2 \frac{\partial}{\partial \ln q^2} + \beta(g) \frac{\partial}{\partial g} \right) \delta_{ab} - \frac{g^2}{16\pi^2} \gamma_{\omega ba}^N \right) C_b^N(q^2) \\ & = \frac{g^4}{(16\pi^2)^2} \gamma_{\omega ba}^N C_b^N(q^2) \end{aligned} \quad (2.6)$$

Now the LHS of Eq.(2.6) can be diagonalized using the matrix U^N , and furthermore since we are working to order g^4 , we may replace $C_b^N(g^2)$ on the RHS by the solution to the lowest order equation. Thus we have the two uncoupled, but inhomogeneous equations

$$\begin{aligned} & \left(-2 \frac{\partial}{\partial \ln q^2} + \beta(g) \frac{\partial}{\partial g} - \frac{g^2 \delta_+^N}{16\pi^2} \right) C_+^N(q^2, g) = \frac{g^4}{(16\pi^2)^2} \cdot \\ & \cdot \left[\gamma_{++}^N C_+^N(q^2 = \mu^2, 0) \left(\frac{\bar{g}^2(q^2)}{g^2} \right)^{\delta_+^N/2\beta_0} + \gamma_{+-}^N C_-^N(q^2 = \mu^2, 0) \left(\frac{\bar{g}^2(q^2)}{g^2} \right)^{\delta_-^N/2\beta_0} \right] \end{aligned} \quad (2.7)$$

with a similar equation for $+ \rightleftharpoons -$, where C_+^N and C_-^N are obtained from

$$C_+^N(q^2, g) = U_{+f}^N C_f^N(q^2) + U_{+g}^N C_g^N(q^2, g) \quad (2.8)$$

and similarly for C_-^N , and $\gamma_{++}^N, \gamma_{+-}^N$, etc., are obtained from

$$U_{ia}^{N-1} \gamma_{ab}^N U_{bj}^N = \begin{pmatrix} \gamma_{++}^N & \gamma_{+-}^N \\ \gamma_{-+}^N & \gamma_{--}^N \end{pmatrix} \quad (2.9)$$

If we write the perturbative expansion for $C_F^N(q^2 = \mu^2, g)$ and $C_g^N(q^2 = \mu^2, g)$ up to one loop as

$$C_F^N(q^2 = \mu^2, g) = C_F^N(q^2 = \mu^2, 0) \left[1 + \epsilon_F^N \frac{g^2}{16\pi^2} + \dots \right] \quad (2.10)$$

$$C_g^N(q^2 = \mu^2, g) = \epsilon_g^N \frac{g^2}{16\pi^2} \quad (2.11)$$

[Note that the perturbative expansion for the gluon operator coefficient function begins in one loop], then we can define $\epsilon_{\pm}^N$ by

$$\epsilon_{\pm}^N = U_{\pm+}^N \epsilon_F^N + U_{\pm-}^N \epsilon_g^N \quad (2.12)$$

and the solution to Eq. (2.7) can be written

$$\begin{aligned} C_+^N(q^2) = C_+^N(q_0^2) \left\{ 1 + \frac{(\bar{g}^2(q^2) - \bar{g}^2(q_0^2))}{16\pi^2} \left[\epsilon_+^N + \frac{\gamma_{++}^N}{2\beta_0} - \frac{\gamma_{+-}^N \beta_1}{2\beta_0^2} \right] \right\} \left[\frac{\bar{g}^2(q^2)}{\bar{g}^2(q_0^2)} \right]^{\delta_+^N/2\beta_0} \\ + C_-^N(q_0^2) \frac{\gamma_{-+}^N}{(\gamma_-^N - \gamma_+^N - 2\beta_0)} \left\{ \left[\frac{\bar{g}^2(q^2)}{\bar{g}^2(q_0^2)} \right]^{\delta_-^N/2\beta_0} \frac{\bar{g}^2(q_0^2)}{16\pi^2} - \left[\frac{\bar{g}^2(q^2)}{\bar{g}^2(q_0^2)} \right]^{\delta_-^N/2\beta_0} \frac{\bar{g}^2(q^2)}{16\pi^2} \right\} \end{aligned} \quad (2.13)$$

with a similar equation for $C_-^N(q^2)$ with all $+$ and $-$ indices interchanged. Here β_0 and β_1 are obtained from the expansion of β

$$\beta(g) = - \frac{g^3}{16\pi^2} \beta_0 - \frac{g^5}{(16\pi^2)^2} \beta_1 + \dots \quad (2.14)$$

That this is a solution of Eq.(2.7) may be seen by inserting it into the LHS of Eq.(2.7) and using the fact that if we write $\bar{g}(q^2)$ as a function of q^2 and $\bar{g}(q_0^2)$ we have the relation

$$\left[-2 \frac{\partial}{\partial \ln q^2} - \beta(\bar{g}(q_0^2)) \frac{\partial}{\partial \bar{g}(q_0^2)} \right] \bar{g}(q^2) = 0 \quad (2.15)$$

Eq.(2.13) is still not in a form which can be directly compared with experiment. However it is a simple matter to use the matrix U^N to express $C_{(-)}^N$ in terms of C_f^N and C_g^N . We write the structure function $F_2(x, q^2)$ as

$$F_2(x, q^2) = x \left(\frac{4}{9} u(x, q^2) + \frac{4}{9} \bar{u}(x, q^2) + \frac{1}{9} d(x, q^2) + \frac{1}{9} \bar{d}(x, q^2) + \frac{1}{9} s(x, q^2) + \frac{1}{9} \bar{s}(x, q^2) + \dots \right) \quad (2.16)$$

where u, d, s ($\bar{u}, \bar{d}, \bar{s}$) represent up, down and strange quark (antiquark) distribution functions. Now the flavour non-singlet part of F_2 has a q^2 dependence which is given by the anomalous dimensions of the non-singlet operators, but the total quark distribution

$$Q(x, q^2) \equiv u(x, q^2) + \bar{u}(x, q^2) + d(x, q^2) + \bar{d}(x, q^2) + s(x, q^2) + \bar{s}(x, q^2) + \dots \quad (2.17)$$

as well as the gluon distribution, $G(x, q^2)$ (which is not directly observable in lepton production), have q^2 dependences given by the singlet operators. From Eqs.(2.1) and (2.13) we have for the N^{th} moment $Q^N(q^2)$

$$Q^N(q^2) = A^N(q^2, q_0^2) Q^N(q_0^2) + B^N(q^2, q_0^2) G^N(q_0^2) \quad (2.18)$$

where

$$\begin{aligned} A^N(q^2, q_0^2) &= U_{f+}^{N-1} U_{f+}^N \left[1 + \frac{\bar{g}^2(q^2) - \bar{g}^2(q_0^2)}{16\pi^2} \left(\epsilon_+^N + \frac{\delta_{++}^N}{2\beta_0} - \frac{\delta_{+-}^N \beta_1}{2\beta_0^2} \right) \right] \left[\frac{\bar{g}^2(q^2)}{\bar{g}^2(q_0^2)} \right]^{\delta_{++}^N/2\beta_0} \\ &+ U_{f-}^{N-1} U_{f-}^N \left[1 + \frac{\bar{g}^2(q^2) - \bar{g}^2(q_0^2)}{16\pi^2} \left(\epsilon_-^N + \frac{\delta_{--}^N}{2\beta_0} - \frac{\delta_{-+}^N \beta_1}{2\beta_0^2} \right) \right] \left[\frac{\bar{g}^2(q^2)}{\bar{g}^2(q_0^2)} \right]^{\delta_{--}^N/2\beta_0} \\ &+ U_{f+}^{N-1} U_{f-}^N \frac{\delta_{+-}^N}{(\delta_-^N - \delta_+^N - 2\beta_0)} \left[\left(\frac{\bar{g}^2(q^2)}{\bar{g}^2(q_0^2)} \right)^{\delta_{+-}^N/2\beta_0} \frac{\bar{g}^2(q_0^2)}{16\pi^2} - \left(\frac{\bar{g}^2(q^2)}{\bar{g}^2(q_0^2)} \right)^{\delta_{+-}^N/2\beta_0} \frac{\bar{g}^2(q^2)}{16\pi^2} \right] \end{aligned} \quad (2.19a)$$

$$\begin{aligned}
 & + u_{f-}^{N-1} u_{+f}^N \frac{\delta_{-+}^N}{(\delta_+^N - \delta_-^N - 2\beta_0)} \left[\left(\frac{\bar{g}^2(q^2)}{\bar{g}^2(q_0^2)} \right)^{\delta_{++}^N/2\beta_0} \frac{\bar{g}^2(q_0^2)}{16\pi^2} - \left(\frac{\bar{g}^2(q^2)}{\bar{g}^2(q_0^2)} \right)^{\delta_{--}^N/2\beta_0} \frac{\bar{g}^2(q^2)}{16\pi^2} \right] \\
 & + u_{f+}^{N-1} u_{+g}^N \left[\left(\frac{\bar{g}^2(q^2)}{\bar{g}^2(q_0^2)} \right)^{\delta_{++}^N/2\beta_0} - \left(\frac{\bar{g}^2(q^2)}{\bar{g}^2(q_0^2)} \right)^{\delta_{--}^N/2\beta_0} \right] \frac{\bar{g}^2(q_0^2)}{16\pi^2} \varepsilon_g^N
 \end{aligned}
 \tag{2.19a}$$

cont.

$$B^N(q^2, q_0^2) =$$

$$\begin{aligned}
 & u_{g+}^{N-1} u_{+f}^N \left[1 + \frac{\bar{g}^2(q^2) - \bar{g}^2(q_0^2)}{16\pi^2} \left(\varepsilon_+^N + \frac{\delta_{++}^N}{2\beta_0} - \frac{\delta_+^N \beta_1}{2\beta_0^2} \right) \right] \left[\frac{\bar{g}^2(q^2)}{\bar{g}^2(q_0^2)} \right]^{\delta_{++}^N/2\beta_0} + \\
 & u_{g-}^{N-1} u_{-f}^N \left[1 + \frac{\bar{g}^2(q^2) - \bar{g}^2(q_0^2)}{16\pi^2} \left(\varepsilon_-^N + \frac{\delta_{--}^N}{2\beta_0} - \frac{\delta_-^N \beta_1}{2\beta_0^2} \right) \right] \left[\frac{\bar{g}^2(q^2)}{\bar{g}^2(q_0^2)} \right]^{\delta_{--}^N/2\beta_0} + \\
 & u_{g+}^{N-1} u_{-f}^N \frac{\delta_{+-}^N}{(\delta_+^N - \delta_-^N - 2\beta_0)} \left[\left(\frac{\bar{g}^2(q^2)}{\bar{g}^2(q_0^2)} \right)^{\delta_{++}^N/2\beta_0} \frac{\bar{g}^2(q_0^2)}{16\pi^2} - \left(\frac{\bar{g}^2(q^2)}{\bar{g}^2(q_0^2)} \right)^{\delta_{--}^N/2\beta_0} \frac{\bar{g}^2(q^2)}{16\pi^2} \right] + \\
 & u_{g-}^{N-1} u_{+f}^N \frac{\delta_{-+}^N}{(\delta_+^N - \delta_-^N - 2\beta_0)} \left[\left(\frac{\bar{g}^2(q^2)}{\bar{g}^2(q_0^2)} \right)^{\delta_{++}^N/2\beta_0} \frac{\bar{g}^2(q_0^2)}{16\pi^2} - \left(\frac{\bar{g}^2(q^2)}{\bar{g}^2(q_0^2)} \right)^{\delta_{--}^N/2\beta_0} \frac{\bar{g}^2(q^2)}{16\pi^2} \right] + \\
 & (u_{g+}^{N-1} u_{+g}^N - u_{f+}^{N-1} u_{+f}^N) \left\{ \left(\frac{\bar{g}^2(q^2)}{\bar{g}^2(q_0^2)} \right)^{\delta_{++}^N/2\beta_0} - \left(\frac{\bar{g}^2(q^2)}{\bar{g}^2(q_0^2)} \right)^{\delta_{--}^N/2\beta_0} \right\} \frac{\varepsilon_g}{16\pi^2} \bar{g}^2(q_0^2)
 \end{aligned}
 \tag{2.19b}$$

Since the gluon distribution is not directly measurable its exact definition is in general arbitrary. However Eq.(2.18) provides its own definition of the gluon distribution, since it can be inverted to give

$$S^N(q^2) = - \frac{A^N(q_0^2, q^2) B^N(q_0^2, q^2)}{B^N(q_0^2, q^2)} S^N(q_0^2) + \frac{(1 - A^N(q_0^2, q_0^2) A^N(q_0^2, q^2))}{B^N(q_0^2, q^2)} \gamma^N(q_0^2) \quad (2.20)$$

To leading order this definition coincides with the definition obtained by considering the matrix elements of the gluon operators with subtraction point $\mu^2 = q^2$.

In the next sections we discuss the calculations of γ_{ff}^N , γ_{fg}^N , γ_{gf}^N and γ_{gg}^N as well as e_g^N and e_f^N so that the quantities in Eq.(2.19) may be calculated.

We note that if we are working consistently to subleading order then the quantities $\bar{g}^2(q^2)$ and $\bar{g}_0^2(q_0^2)$ must be obtained from the β function taken to two loops, i.e.

$$\frac{\bar{g}^2(q^2)}{16\pi^2} = \frac{1}{\beta_0 \log(q^2/\Lambda^2)} - \frac{\beta_1}{\beta_0^2} \frac{\log \log(q^2/\Lambda^2)}{\log^2(q^2/\Lambda^2)} \quad (2.21)$$

3. THE ANOMALOUS DIMENSIONS OF TWIST TWO SINGLET WILSON OPERATORS

It is known that in the Wilson expansion for the structure functions [Eq. (2.1)], the gauge invariant twist two flavour singlet operators appear. These operators are

$$O_f^{\mu_1 \dots \mu_N} = \frac{i^{N-1}}{N!} [\bar{\psi} \gamma^{\mu_1} D^{\mu_2} \dots D^{\mu_N} \psi - \text{traces} + \text{permutations}] \quad (3.1a)$$

$$O_g^{\mu_1 \dots \mu_N} = \frac{i^{N-2}}{2N!} [G^{\alpha\mu_1} D^{\mu_2} \dots D^{\mu_{N-1}} G^{\alpha\mu_N} - \text{traces} + \text{permutations}] \quad (3.1b)$$

It is also known that in the renormalization of these operators (i.e. the calculation of their matrix elements in higher orders) they mix not only with each other but also with other operators. Let us call the operators of Eq.(3.1) type I operators and the other operators, which mix with them in higher orders, type II operators (which are in general not gauge invariant).

Operators of type I plus the operators of type II form a closed set under renormalization. Now the following facts have been proved¹¹⁾ for the renormalization matrix Z_{ij} defined by

$$O_{i,B}^{\mu_1 \dots \mu_N} = Z_{ij}^N O_{j,R}^{\mu_1 \dots \mu_N} \quad (3.2)$$

where i, j run over all the operators of types I and II:

1) The operators of type II renormalize into each other, i.e. Z_{ij} is of the form

$$Z = \begin{pmatrix} Z_{I,I} & Z_{I,II} \\ 0 & Z_{II,II} \end{pmatrix} \quad (3.3)$$

2) The matrix elements between quark states of the operators of type I have counterterms obtained from the matrix elements of type I operators only.

3) The matrix elements of type I operators between gluon states have counterterms whose $g_{\mu\nu}$ coefficient (μ and ν being the polarizations indices of the external gluons) are obtained from the matrix elements of type I operators only.

This gives us a unique way of isolating the 2×2 submatrix

$$Z_{I,I} = \begin{pmatrix} Z_{gg} & Z_{gf} \\ Z_{fg} & Z_{ff} \end{pmatrix} \quad (3.4)$$

from which we calculate the anomalous dimensions of the physically interesting operators (i.e. type I).

We use the method of dimensional regularization¹²⁾ and the minimal subtraction renormalization prescription, i.e. removing the pole at $\epsilon = 0$ in $4 - \epsilon$ dimensions. We work in the Feynman gauge (although in the minimal subtraction prescription the anomalous dimensions are gauge invariant). If we denote by $O_{a,b}^N$ the matrix element of $O_a^{\mu_1 \dots \mu_N}$ between spacelike off mass-shell b states we have

$$O_{a,c|R}^N = \sum_b Z_c (Z^N)^{-1}_{ab} O_{b,c|B}^N \quad (3.5)$$

where a, b, c refer to quark or gluon and Z_c is the wave function renormalization of the c field. From Eq.(3.5) we obtain the renormalization group equations ^{*)}.

$$\left[\left(\mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} - 2\alpha \gamma_a \frac{\partial}{\partial \alpha} - 2\gamma_c \right) \delta_{ab} + \gamma_{ab}^N \right] O_{bc}^N = 0 \quad (3.6)$$

where

$$\gamma_{ab}^N = \left(Z^{N^{-1}} \mu \frac{dZ^N}{d\mu} \right)_{ab} \quad (3.7a)$$

$$\gamma_c = \frac{1}{2Z_c} \mu \frac{dZ_c}{d\mu} \quad (3.7b)$$

and

$$\beta(g) = \mu \frac{dg_R}{d\mu} \quad (3.7c)$$

It is easy to show that in the minimal subtraction renormalization prescription, γ_{ab}^N and β are independent of the gauge parameter α . It has also been shown in ref. 7) that γ_{ab}^N is simply related to the residue of the pole at $\epsilon = 0$ in the expansion of Z_{ab}^N as a power series in $1/\epsilon$

$$Z_{ab}^N = \sum_{k=1}^{\infty} \frac{a_{k,ab}^N}{\epsilon^k} \quad (3.8a)$$

$$\gamma_{ab}^N = -g_R^2 \frac{\partial a_{1,ab}^N}{\partial g_R^2} \quad (3.8b)$$

Thus in order to determine the elements of the anomalous dimension matrix we have to calculate the coefficient of $1/\epsilon$ for the matrix elements of

^{*)} When the matrix elements of gauge invariant operators are taken between physical states the term $\partial/\partial\alpha$ vanishes. Thus we have a renormalization group equations for the coefficient functions in which this term does not appear.

the operators $O_f^{\mu_1 \dots \mu_n}$ and $O_g^{\mu_1 \dots \mu_n}$ between off mass-shell quark and gluon states up to order g^4 . We also need the coefficient of $1/\epsilon$ for the fermion and gluon propagators calculated to the same order.

The lowest order effective vertices for $O_f^{\mu_1 \dots \mu_n}$ are the same as that for the non-singlet operators and have been discussed in ref. 7). In fig. 1 we show the 2,3 and 4 gluon effective vertices for the gluon operator. Using a source $\Delta_{\mu_1} \Delta_{\mu_2} \dots \Delta_{\mu_n}$ with $\Delta^2 = 0$ (this automatically projects out the traceless symmetric part) we have for the two gluon effective vertex (fig. 1a) the Feynman rule,

$$\delta_{ab} \frac{[1 + (-1)^N]}{2} (\Delta \cdot p)^{N-2} [g^{\mu\nu} (\Delta \cdot p)^2 - \Delta \cdot p (\Delta^\mu p^\nu + \Delta^\nu p^\mu) + \Delta^\mu \Delta^\nu p^2] \quad (3.9a)$$

for the three-gluon effective vertex (fig. 1b),

$$\begin{aligned} & -\frac{ig}{3!} f^{abc} \left\{ \Delta_\nu (\Delta_\mu r_\mu (\Delta \cdot p) + p_\mu \Delta_\mu (\Delta \cdot r) - g_{\mu\nu} (\Delta \cdot p) (\Delta \cdot r) \right. \\ & - \Delta_\mu \Delta_\nu p \cdot r) \sum_{j=1}^{N-2} (-1)^{j-1} (\Delta \cdot p)^{j-1} (\Delta \cdot r)^{N-j-2} + [(g_{\mu\nu} \Delta_\nu - \Delta_\mu g_{\nu\mu}) \cdot \\ & \cdot (\Delta \cdot r) + \Delta_\nu (\Delta_\mu r_\nu - \Delta_\nu r_\mu)] (\Delta \cdot r)^{N-2} \Big\} + \text{permutations} \end{aligned} \quad (3.9b)$$

and for the four gluon effective vertex (fig. 1c)

$$\begin{aligned} & -\frac{g^2}{4!} f^{abc} f^{cde} \left\{ (g_{\mu\sigma} (\Delta \cdot p) (\Delta \cdot s) - p_\sigma \Delta_\mu (\Delta \cdot s) - \Delta_\sigma s_\mu (\Delta \cdot p) + \right. \\ & \Delta_\mu \Delta_\sigma (p \cdot s)) \sum_{j=1}^{N-3} \sum_{i=1}^j (\Delta \cdot s)^{N-j-3} (\Delta \cdot (r+s))^{j-2} (\Delta \cdot p)^{i-1} - \Delta_\nu (g_{\mu\sigma} \Delta_\nu (\Delta \cdot p) + \\ & \Delta_\mu \Delta_\sigma s_\nu) \sum_{j=1}^{N-2} (\Delta \cdot s)^{N-j-2} (\Delta \cdot (p+q))^{j-1} + \\ & \Delta_\nu (g_{\mu\sigma} \Delta_\nu (\Delta \cdot p) + \Delta_\mu \Delta_\sigma p \cdot r) \sum_{j=1}^{N-2} (\Delta \cdot p)^{N-j-2} (\Delta \cdot (r+s))^{j-1} \\ & \left. - g_{\mu\sigma} \Delta_\mu \Delta_\sigma (\Delta \cdot (r+s))^{N-2} \right\} + \text{permutations} \end{aligned} \quad (3.9c)$$

The diagrams contributing to γ_{gg} are shown in figs. 2,3 and 4, those contributing to γ_{gf} in figs. 5,6,7 and those contributing to γ_{fg} in figs. 8,9,10. Because of the complexity of these diagrams it has been necessary to calculate them using SCHOONSCHIP, a program which performs algebraic manipulations and which was written by Veltman in 1967¹³⁾. Because we work in $4 - \epsilon$ dimensions certain modifications to the program had to be done. Firstly it was necessary to define a function $G(MU,NU)$ which had the properties of the (built in) Kronecker delta functions $D(MU,NU)$ with the exception that $G(MU,MU)$ was replaced by $4 - \epsilon$ instead of just 4. Secondly the algorithms in SCHOONSCHIP which calculate the traces of Dirac γ matrices are valid in four dimensions only. This causes ambiguities in $4 - \epsilon$ dimensions when we try to take the trace of a product of more than 8 such γ - matrices. It was therefore necessary to insert substitutions which reduced all products of γ matrices before using the standard SCHOONSCHIP trace routine.

The first step in the calculation of any of the diagrams in figs. 2-10 is to reduce tensor or spinor forms into scalars, which are much easier to handle. For diagrams with external quark lines this is achieved by averaging over the spins of the external lines i.e. by multiplying by $\gamma \cdot p/4$ and taking the trace (p being the momentum of the external line). For external gluon states, as has been explained above, we need to project out the coefficient of $g_{\mu\nu}$. Using the fact that $\Delta^2 = 0$ this can be achieved by contracting the external gluon polarization indices μ, ν with the projection operator

$$P_{\mu\nu} = \frac{1}{(2-\epsilon)} \left[g_{\mu\nu} - \frac{\Delta_\mu p_\nu + \Delta_\nu p_\mu}{\Delta \cdot p} + p^2 \frac{\Delta_\mu \Delta_\nu}{(\Delta \cdot p)^2} \right] \quad (3.10)$$

We have divided the diagrams into three types. In figs. 2,5, and 8 we have the diagrams which consist of the one loop diagrams with a self energy insertion on one of the internal lines. These are the easiest to handle. The renormalized propagators are of the form

$$\frac{\delta \cdot k}{k^2} \left(\Gamma\left(\frac{\epsilon}{2}\right) (A+B\epsilon) (k^2)^{-\frac{\epsilon}{2}} - \frac{2A}{\epsilon} \right) \quad (3.11a)$$

for fermions and

$$- \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) \frac{1}{k^2} \left(\Gamma\left(\frac{\epsilon}{2}\right) (A + B\epsilon) (k^2)^{-\epsilon/2} - \frac{2A}{\epsilon} \right) \quad (3.11b)$$

for gluons, where k is the momentum of the internal line in which we have the self energy insertions [note that in Eq.(3.11) the term $-2A/\epsilon$ is the one-loop self energy counterterm which renormalizes the propagators]. The self energy insertion diagrams are thus calculated by calculating the one loop diagram with one of the internal propagators by Eq.(3.11).

In figs. 3,6 and 9 we have all the other diagrams which do not contain overlapping divergences. The only subdivergence of these diagrams corresponds to the one loop renormalization of the operators. These diagrams are handled by first of all calculating the full tensor or spinor form of the finite parts of the operator matrix elements to one loop (i.e. we do not take traces for fermion lines or project out the $g_{\mu\nu}$ coefficient for gluon lines) and inserting this expression into the second loop.

The diagrams which contain overlapping divergences are shown in figs. 4, 7 and 10. For these diagrams SCHOONSCHIP is first used to calculate the numerators and to expand scalar products of momenta as linear combinations of propagators appearing in the denominator, e.g. we have

$$k_1 \cdot k_2 = \frac{1}{2} [k_1^2 + k_2^2 - (k_1 - k_2)^2] \quad (3.12)$$

where k_1 and k_2 are the loop momenta to be integrated. This gives us 18 different types of integrals, which after a suitable change of variable may be reduced to 14 basic integrals whose simple and double pole parts are given in the Appendix. It is found that each integral is divergent either in k_1 or k_2 but not both, so that this method separates out a diagram with an overlapping divergence into a number of integrals without overlapping divergences, and whose simple pole parts may be directly substituted (again using SCHOONSCHIP). The double pole parts are needed because the numerator contains terms proportional to ϵ and these terms multiplied by the double pole of the integrals contribute to the simple pole part of the diagram. As has been fully described in ref. 7), there is a complication with these terms since the double pole parts of the integrals have been calculated without taking into account the counterterms for the

subdivergences. This is correct if the term proportional to ϵ arose in the calculation of the loop which gives rise to the subdivergence, but if it arose in the external loop we must subtract off the double pole part of the contribution from the counterterm to the subdivergence. In practice this only happens when the subdivergence is of the form of the one loop corrections to an ordinary three point vertex (i.e. not an effective operator vertex). Thus it was necessary to subtract off the terms proportional to ϵ in the numerator of the one loop correction to the operator matrix elements multiplied by the counterterms associated with the renormalization of the ordinary three point couplings. A detailed example of the calculation of such an overlapping divergence diagram is given in ref. 7).

The diagrams contributing to γ_{ff} are identical to those given in ref. 7) for the non-singlet plus the two diagrams of fig. 11 (these cannot contribute to the non-singlet anomalous dimensions since the intermediate two gluon state must be a flavour singlet). These two diagrams are of the type which do not contain overlapping divergences. Their contribution when added to the anomalous dimensions of the non-singlet operators give us γ_{ff} .

We also had to calculate the gluon propagator to two loops. The diagrams contributing are shown in fig. 12. These may also be divided into those consisting of self energy insertions into one loop diagrams and diagrams with overlapping divergences. We also found it convenient to use SCHOONSCHIP to calculate these. We find in the Feynman gauge for γ_g (defined by Eq. 3.7b)

$$\gamma_g = \frac{g^2}{16\pi^2} \left(\frac{5}{3} C_A - \frac{4}{3} T_R \right) + \frac{g^4}{(16\pi^2)^2} \left(\frac{23}{2} C_A^2 - 5 C_A T_R - 4 C_F T_R \right) \quad (3.13)$$

where C_A , T_R and C_F are Casimir operators of the gauge group defined by

$$\delta_{cd} C_A = f^{abc} f^{abd} \quad (3.14a)$$

$$\delta_{ij} C_F = T_{ik}^a T_{kj}^a \quad (3.14b)$$

FIG. 10

FIG. 11

and $e_{\mu\nu}$ are defined by

$$e_{\mu\nu} = g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \quad (4.2)$$

and

$$d_{\mu\nu} = - \frac{p_\mu p_\nu}{p^2} q^2 + \frac{p_\mu q_\nu + p_\nu q_\mu}{p \cdot q} - g_{\mu\nu} \quad (4.3)$$

Since the operator product expansion is an operator statement, (4.1) is true for all states p , and indeed we will choose the states $|p\rangle$ to be quark states or gluon states depending on which coefficient function we are extracting. We work in the Feynman gauge, the coefficient functions are however gauge invariant in the renormalization scheme which we use.

(a) The Quark Coefficient Functions

In order to calculate the quark coefficient functions $C_{i,f}^N$, we choose $|p\rangle$ to be an off-shell quark state with momentum p . We now have to calculate the one loop diagrams for $T_{\mu\nu}$ (i.e. those of Fig. 13) and for $\langle p | O_P^N | p \rangle$ (i.e. those of Fig. 14). Defining

$$S_N = \sum_{j=1}^N \frac{1}{j} \quad (4.4)$$

we find ^{*)}

$$T_L = \frac{g^2 C_F}{4\pi^2} \sum_{\substack{N=2 \\ \text{even } N}}^{\infty} \left(\frac{1}{x}\right)^N \frac{1}{N+1} \quad (4.5)$$

$$T_2 = 2 \sum_{\substack{N=2 \\ \text{even } N}}^{\infty} \left(\frac{1}{x}\right)^N \left[1 + \frac{g^2 C_F}{16\pi^2} \left\{ \log \frac{q^2}{p^2} \left[3 + \frac{2}{N(N+1)} - 4S_N \right] + 3S_N \right. \right. \\ \left. \left. - 2 - 8 \sum_{j=1}^N \frac{1}{j^2} - \frac{1}{N} + \frac{4}{N^2} - \frac{4}{(N+1)^2} + \frac{6}{(N+1)} \right\} \right] \quad (4.6)$$

^{*)} We adopt the consistent procedure of tracing over the quark spins and dividing by 4 in (4.5) to (4.8), which normalizes the lowest order operator matrix element to 1.

$$T_3^{\bar{v}-v} = \frac{g^2 C_F}{16\pi^2} \sum_{\substack{N=2 \\ \text{even } N}}^{\infty} \left(\frac{1}{2}\right)^N \left\{ \log \frac{q^2}{\mu^2} \left[3 + \frac{2}{N(N+1)} - 4S_N \right] \right. \\ \left. + 3S_N - 2 - 8 \sum_{j=1}^N \frac{1}{j^2} - \frac{3}{N} + \frac{4}{N^2} - \frac{4}{(N+1)^2} + \frac{4}{(N+1)} \right\} \quad (4.7)$$

For $T_3^{v+\bar{v}}$ the expression is identical to (4.7) except that the sum runs over odd N , starting at $N = 1$. Finally

$$\langle p | O_F^N | p \rangle = 1 - \frac{C_F g^2}{16\pi^2} \left\{ \left(\log \frac{p^2}{\mu^2} - \log 4\pi + \gamma \right) \left(3 + \frac{2}{N(N+1)} - 4S_N \right) \right. \\ \left. + 2 \frac{1-S_N}{N(N+1)} - 7 + 4 \sum_{j=1}^N \left(\frac{S_j}{j} + \frac{1}{j^2} \right) + \frac{4}{(N+1)^2} - \frac{2}{N^2} + \frac{2}{N} \right\} \quad (4.8)$$

where γ is Euler's constant. Substituting (4.5) to (4.7) on the left hand side of (4.1) and (4.8) on the right hand side, and comparing the coefficients of g^2 one readily obtains;

$$C_{L,f}^N(q^2=\mu^2, \bar{g}^2(q^2)) = \frac{\bar{g}^2(q^2) C_F}{4\pi^2} \cdot \frac{1}{N+1} \quad (4.9)$$

$$C_{2,f}^N(q^2=\mu^2, \bar{g}^2(q^2)) = 2 \left\{ 1 + \frac{\bar{g}^2(q^2) C_F}{16\pi^2} \left[3S_N - 4 \sum_{j=1}^N \frac{1}{j^2} \right. \right. \\ \left. - \frac{2}{N(N+1)} S_N + 4 \sum_{j=1}^N \frac{S_j}{j} + \frac{3}{N} + \frac{4}{N+1} + \frac{2}{N^2} - 9 + \right. \\ \left. \left. (\log 4\pi - \gamma) \left(4S_N - 3 - \frac{2}{N(N+1)} \right) \right] \right\} \quad (4.10)$$

and

$$C_{3,f}^N(q^2=\mu^2, \bar{g}^2(q^2)) = 1 + \frac{\bar{g}^2(q^2)C_F}{16\pi^2} \left[3S_N - 4 \sum_{j=1}^N \frac{1}{j^2} - \frac{2}{N(N+1)} S_N \right. \\ \left. + 4 \sum_{j=1}^N \frac{S_j}{j} + \frac{1}{N} + \frac{2}{N+1} + \frac{2}{N^2} - 9 + (\log \mu \bar{u} - 8) \cdot \right. \\ \left. \cdot \left(4S_N - 3 - \frac{2}{N(N+1)} \right) \right] \quad (4.11)$$

where for $\bar{v} = v$ ($\bar{v} + v$), N is even (odd). These results, (4.9) - (4.11), agree with those of Bardeen et al.⁹⁾.

b) The Gluon Coefficient Functions :

In order to extract the gluon coefficient functions we choose $|p\rangle$ to be an off-shell gluon state with momentum p . Now $T_3^v = T_3^{\bar{v}}$, and since (by charge conjugation) gluons can only contribute to terms even in x whereas $xT_3^v + xT_3^{\bar{v}}$ is odd in x we conclude that $C_{3,g}^N \equiv 0$. To extract the $C_{1,g}^N$ and $C_{2,g}^N$ functions we must calculate the one loop diagrams contributing to $T^{\mu\nu}$ (i.e. those of Fig. 15) and to $\langle p | O_g^N | p \rangle$ (i.e. those of Fig. 16). As has already been explained in detail in the previous section, in order to avoid mixing with operators which are not gauge invariant it is necessary to calculate the coefficients of $g^{\lambda\rho}$ in $T^{\mu\nu}$ and $\langle p | O_g^N | p \rangle$ where λ and ρ are the polarization indices of the gluons, and the results quoted below are for these coefficients. We find^{*)}

$$T_L = - \frac{g^2 T_R}{\pi^2} \sum_{\substack{N=2 \\ \text{even } N}}^{\infty} \left(\frac{1}{x} \right)^N \frac{1}{(N+1)(N+2)} \quad (4.12)$$

$$T_2 = - \frac{g^2 T_R}{2\pi^2} \sum_{\substack{N=2 \\ \text{even } N}}^{\infty} \left(\frac{1}{x} \right)^N \left\{ \frac{N^2+N+2}{N(N+1)(N+2)} \log \frac{q^2}{p^2} - \frac{2}{N} + \frac{2}{N^2} \right. \\ \left. - \frac{4}{(N+1)^2} + \frac{4}{(N+2)^2} + \frac{8}{N+1} - \frac{8}{N+2} \right\} \quad (4.13)$$

^{*)} In order to normalize the lowest order operator matrix element to 1, we have included a factor of 1/2 in (4.12) to (4.14).

and

$$\begin{aligned} \langle p | O_g^N | p \rangle = & 1 + \frac{g^2 T_R}{4\pi^2} \left\{ \frac{1}{N} - \frac{1}{N^2} + \frac{4}{(N+1)^2} - \frac{4}{(N+2)^2} - \frac{2}{N+1} \right. \\ & \left. + \frac{2}{(N+2)} - \frac{N^2+N+2}{N(N+1)(N+2)} \left(\log 4\pi \frac{\mu^2}{p^2} + S_N - \gamma \right) \right\} \end{aligned} \quad (4.14)$$

Substituting (4.12) and (4.13) on the left hand side of (4.1) and (4.14) on the right hand side we find

$$C_{L,g}^N(q^2=\mu^2, \bar{g}^2(q^2)) = - \frac{\bar{g}^2(q^2) T_R}{\pi^2} \frac{1}{(N+1)(N+2)} \quad (4.15)$$

and ^{*})

$$\begin{aligned} C_{2,g}^N(q^2=\mu^2, \bar{g}^2(q^2)) = & \frac{\bar{g}^2(q^2) T_R}{2\pi^2} \left\{ \frac{1}{N} - \frac{1}{N^2} - \frac{6}{N+1} + \frac{6}{N+2} \right. \\ & \left. + \frac{N^2+N+2}{N(N+1)(N+2)} \left(\log 4\pi + S_N - \gamma \right) \right\} \end{aligned} \quad (4.16)$$

The quantities ϵ_F^N , ϵ_g^N defined in Eqs. (2.10) and (2.11) can be obtained from the expression for the coefficient functions, Eqs. (4.9), (4.10), (4.11), (4.15) and (4.16).

5. RESULTS

This section contains the results of our calculation of the anomalous dimension matrix in two loops, and related quantities required for the formulae of section 2.

^{*}) There is an error in ref. 9) in the equation for $C_{2,g}^N$ (in equations (6.25) and (A.22) of ref. 9). The authors of ref. 9) now agree with (4.16) and this error will be corrected in the published version of their paper ¹⁵⁾.

We start by presenting the results for the anomalous dimension matrix. The analytic forms for the results are far too complicated to present here (there are a total of 692 terms in the four entries of the matrix), and so we present the results in numerical form.

We can write

$$\gamma_{(1)gg}^N \equiv C_A^2 A^{gg} + C_F T_R B^{gg} + C_A T_R C^{gg} \quad (5.1)$$

and then the coefficients A^{gg} , B^{gg} , C^{gg} are tabulated in Table 1 for low values of N .

For $\gamma_{(1)gf}^N$ we write

$$\gamma_{(1)gf}^N \equiv C_F T_R B^{gf} + C_F^2 D^{gf} + C_F C_A E^{gf} \quad (5.2)$$

and list the coefficients of B^{gf} , D^{gf} and E^{gf} in Table 2.

For the other off-diagonal entry $\gamma_{(1)fg}^N$ we write

$$\gamma_{(1)fg}^N \equiv C_F T_R B^{fg} + C_A T_R C^{fg} \quad (5.3)$$

and list the coefficients B^{fg} and C^{fg} in Table 3. Finally we define B^{ff} , D^{ff} and E^{ff} by

$$\gamma_{(1)ff}^N \equiv C_F T_R B^{ff} + C_F^2 D^{ff} + C_F C_A E^{ff} \quad (5.4)$$

and tabulate their numerical values for low values of N in Table 4.

We would like to point out two features of these results. Firstly the values B^{ff} in Table 4 are different from those of the non-singlet operator ⁷⁾, due to the extra diagrams of Fig. 11. Secondly for $N = 2$, the entries in Table 1 are identical to those in Table 3, and those in Table 2 are identical to those in Table 4. This is a consequence of the fact that the minimal subtraction renormalization scheme which we are using preserves

the Slavnov-Taylor identities and in particular the energy momentum tensor is not renormalized. The combination $\gamma_{(1)gg}^{N=2} - \gamma_{(1)fg}^{N=2} (\gamma_{(1)ff}^{N=2} - \gamma_{(1)gf}^{N=2})$ is proportional to the $\frac{1}{\epsilon}$ term in the matrix element of the energy momentum tensor taken between gluon (quark) states, and must therefore be zero (note that because of our definition of the operators the signs of the off-diagonal elements are opposite to those of ref. 10).

For phenomenological applications it is more useful to have the anomalous dimension matrix in a form which is diagonal in order g^2 (see section 2). In Table 5 we tabulate the lowest order contributions to the eigenvalues of the anomalous dimension matrix (γ_-^N and γ_+^N) for 3 to 6 flavours, and for low values of n . The colour group has been taken to be $SU(3)$. The entries in the diagonalizing matrix U for 3 to 6 flavours and for low values of μ are tabulated in Table 6. Finally in Table 7 we list the values of γ_{--} , γ_{-+} , γ_{+-} and γ_{++} for 3 to 6 flavours and for low values of N .

If we insert these numbers into Eq.(2.18) we find for the second moment of the singlet part of $F_2(x, q^2)$, for four flavours

$$\begin{aligned} Q^{N=2}(q^2) &= \int_0^1 F_2(x, q^2) dx \Big|_{\text{singlet}} = \\ &\left\{ \left[0.57 + 6.89 \frac{\bar{g}^2(q^2)}{16\pi^2} - 3.13 \frac{\bar{g}^2(q_0^2)}{16\pi^2} \right] \left[\frac{\bar{g}^2(q^2)}{\bar{g}^2(q_0^2)} \right]^{0.75} + \left[0.43 + 0.57 \frac{\bar{g}^2(q^2)}{16\pi^2} - 4.33 \frac{\bar{g}^2(q_0^2)}{16\pi^2} \right] \right\} \cdot \\ &\cdot Q^{N=1}(q_0^2) + \left\{ \left[0.43 + 5.17 \frac{\bar{g}^2(q^2)}{16\pi^2} - 6.46 \frac{\bar{g}^2(q_0^2)}{16\pi^2} \right] \left[\frac{\bar{g}^2(q^2)}{\bar{g}^2(q_0^2)} \right]^{0.75} - 0.43 - 0.57 \frac{\bar{g}^2(q^2)}{16\pi^2} + 1.86 \frac{\bar{g}^2(q_0^2)}{16\pi^2} \right\} Q^{N=1}(q_0^2) \end{aligned} \quad (5.5)$$

A full investigation of the effects of these higher order corrections in deep inelastic scattering is in progress, and a preliminary investigation has shown that for the moments higher than $N = 2$, the largest contributions to the correction comes from the one loop correction to the coefficient functions, discussed in Section 3.

In conclusion we comment on the behaviour of the anomalous dimensions for large values of N (the operator spin). In Ref. 7) we stated that the anomalous dimension of the non singlet operators had an asymptotic form (as $N \rightarrow \infty$) of $\frac{2}{3} g^4 (16\pi^2)^2 C_F C_A \ln^3 N$, which was in agreement with a statement by Gross ¹⁷⁾. Since then Yndurain and collaborators ¹⁸⁾ have

found that the expression for the non-singlet anomalous dimensions can be greatly simplified and that they grow no faster than $\ln N$ (i.e. at the same rate as the one loop anomalous dimension). We now agree with Yndurain et al. and disagree with Gross. We have found, moreover, that the diagonal element of the singlet anomalous dimension γ_{gg} grows no faster than $\ln^2 N$ as $N \rightarrow \infty$, whereas the off diagonal elements fall as $\ln^3 N/N$.

CONCLUSIONS

The results presented in this paper, together with those in Ref. 7) provide all the parameters needed to discuss deep inelastic scattering up to subleading order.

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Table I

Results for $\gamma_{(1)gg}^N \equiv C_A^2 A^{gg} + C_{F^T R} B^{gg} + C_{A^T R} C^{gg}$

N	A^{gg}	B^{gg}	C^{gg}
2	0	10.96	5.185
4	28.98	9.312	-22.34
6	40.63	8.666	-31.14
8	48.42	8.400	-36.67
10	54.45	8.267	-40.77
12	59.44	8.191	-44.06
14	63.71	8.143	-46.81
16	67.46	8.111	-49.18
18	70.81	8.089	-51.27
20	73.84	8.073	-53.13

Table II

Results for $\gamma_{(1)gf}^N \equiv C_{F^T R} B^{gf} + C_F^2 D^{gf} + C_F C_A E^{gf}$

N	B^{gf}	D^{gf}	E^{gf}
2	-15.41	-8.296	27.85
4	-1.855	-3.119	9.474
6	-0.2665	-2.058	5.663
8	0.2119	-1.658	4.116
10	0.4047	-1.462	3.298
12	0.4919	-1.348	2.798
14	0.5319	-1.274	2.462
16	0.5484	-1.221	2.220
18	0.5524	-1.180	2.037
20	0.5494	-1.147	1.894

Table III

Results for $\gamma_{(1)fg}^N \equiv C_{FR}^T B^{fg} + C_{AR}^T C^{fg}$

N	B^{fg}	C^{fg}
2	10.96	5.185
4	12.51	-7.283
6	12.79	-9.365
8	12.51	-9.887
10	12.07	-9.906
12	11.58	-9.732
14	11.10	-9.481
16	10.65	-9.203
18	10.23	-8.921
20	9.841	-8.645

Table IV

Results for $\gamma_{(1)ff}^N \equiv C_{FR}^T B^{ff} + C_{FD}^{2ff} + C_{FA}^T C_{E}^{ff}$

N	B^{ff}	D^{ff}	E^{ff}
2	-15.41	-8.296	27.85
4	-19.98	-10.64	47.87
6	-25.95	-12.34	59.49
8	-30.46	-13.52	67.90
10	-34.05	-14.39	74.53
12	-37.03	-15.07	80.03
14	-39.57	-15.60	84.74
16	-41.80	-16.05	88.85
18	-43.78	-16.41	92.51
20	-45.56	-16.73	95.79

Table V

Values of γ_-^N, γ_+^N for 3 to 6 flavours and for low values of N . f denotes the number of flavours.

N \ f	3		4		5		6	
	γ_-^N	γ_+^N	γ_-^N	γ_+^N	γ_-^N	γ_+^N	γ_-^N	γ_+^N
2	0	11.11	0	12.44	0	13.78	0	15.11
4	13.68	29.48	13.62	30.87	13.56	32.26	13.52	33.64
6	17.93	39.65	17.90	41.01	17.89	42.36	17.87	43.71
8	20.88	46.57	20.87	47.91	20.86	49.26	20.85	50.60
10	23.17	51.87	23.17	53.21	23.16	54.55	23.16	55.88
12	25.05	56.18	25.05	57.51	25.05	58.85	25.04	60.19
14	26.65	59.82	26.65	61.15	26.64	62.49	26.64	63.82
16	28.03	62.97	28.03	64.30	28.03	65.64	28.03	66.97
18	29.26	65.75	29.26	67.08	29.26	68.42	29.26	69.75
20	30.36	68.23	30.36	69.57	30.36	70.90	30.36	72.24

Table VI

The matrix $\begin{pmatrix} U_{f-} & U_{f+} \\ U_{g-} & U_{g+} \end{pmatrix}$ for 3 to 6 flavours and for low values of N .

$f \backslash N$	3		4		5		6	
2	-0.4903	0.7071	-0.6000	0.7071	-0.6839	0.7071	-0.7474	0.7071
	0.8716	0.7071	0.8000	0.7071	0.7295	0.7071	0.6644	0.7071
4	0.9922	0.1403	0.9934	0.1708	0.9943	0.1964	0.9951	0.2182
	-0.1250	0.9901	-0.1148	0.9853	-0.1062	0.9805	-0.09887	0.9759
6	0.9987	0.07241	0.9988	0.09072	0.9989	0.1069	0.9991	0.1214
	-0.05156	0.9974	-0.04853	0.9959	-0.04584	0.9943	-0.04344	0.9926
8	0.9995	0.04802	0.9996	0.06080	0.9996	0.07234	0.9997	0.08282
	-0.03051	0.9988	-0.02899	0.9982	-0.02762	0.9974	-0.02637	0.9966
10	0.9998	0.03549	0.9998	0.04519	0.9998	0.05405	0.9998	0.06217
	-0.02104	0.9994	-0.02010	0.9990	-0.01924	0.9985	-0.01845	0.9981
12	0.9999	0.02789	0.9999	0.03565	0.9999	0.04279	0.9999	0.04938
	-0.01578	0.9996	-0.01513	0.9994	-0.01453	0.9991	-0.01398	0.9988
14	0.9999	0.02293	0.9999	0.02925	0.9999	0.03520	0.9999	0.04071
	-0.01249	0.9997	-0.01201	0.9996	-0.01156	0.9994	-0.01114	0.9992
16	0.9999	0.01926	1.000	0.02469	1.000	0.02976	1.000	0.03448
	-0.01026	0.9998	-0.009877	0.9997	-0.009527	0.9996	-0.009200	0.9994
18	1.000	0.01654	1.000	0.02127	1.000	0.02568	1.000	0.02980
	-0.008650	0.9999	-0.008344	0.9998	-0.008060	0.9997	-0.007794	0.9996
20	1.000	0.01447	1.000	0.01884	1.000	0.02353	1.000	0.02617
	-0.007447	0.9999	-0.007194	0.9998	-0.006957	0.9997	-0.006735	0.9997

Table VII

The matrix $\begin{pmatrix} y_{--} & y_{-+} \\ y_{+-} & y_{++} \end{pmatrix}$ for 3 to 6 flavours and for low values of N.

$\begin{matrix} f \\ N \end{matrix}$	3		4		5		6	
2	0 10.13	0 111.1	0 21.12	0 115.9	0 34.01	0 120.7	0 48.03	0 125.5
4	130.4 22.57	-14.60 181.1	116.4 23.53	-16.30 154.5	102.6 24.18	-17.04 127.7	88.79 24.59	-17.01 100.9
6	163.9 14.40	-22.26 243.1	146.5 15.63	-27.06 202.3	129.0 16.71	-30.90 161.5	111.5 17.67	-33.95 120.8
8	186.7 10.87	-24.30 287.5	166.4 12.02	-30.29 238.1	146.0 13.07	-35.48 188.8	125.6 14.04	-39.96 139.5
10	204.6 8.911	-24.64 323.0	181.8 9.959	-31.10 267.4	159.1 10.94	-36.87 211.8	136.4 11.87	-42.04 156.2
12	219.4 7.671	-24.36 353.0	194.5 8.632	-30.98 292.3	170.0 9.544	-37.01 231.7	145.3 10.41	-42.50 171.1
14	232.2 6.814	-23.81 378.9	205.8 7.702	-30.45 314.1	179.4 8.551	-36.55 249.4	153.0 9.365	-42.19 184.6
16	243.4 6.179	-23.16 402.0	215.5 7.006	-29.73 333.6	187.6 7.800	-35.82 265.2	159.7 8.565	-41.49 196.9
18	253.4 5.690	-22.48 422.7	224.2 6.465	-28.94 351.2	195.0 7.211	-34.97 279.6	165.8 7.933	-40.62 208.1
20	262.4 5.301	-21.81 441.6	232.0 6.029	-28.15 367.2	201.6 6.734	-34.09 292.9	171.2 7.417	-39.67 218.6

APPENDIX

In this Appendix we give the simple and double pole parts of the 14 basic two loop integrals which were needed to calculate the diagrams containing overlapping divergences. These were calculated in $n = 4 - \epsilon$ dimensions. The double pole parts are without counterterms (the effect of considering the counterterms is to reverse the signs of these terms), whereas for the simple pole parts we have subtracted off the counterterms corresponding to the divergent subintegrals ^{*}).

The answers are given in terms of the function $S(N)$ where

$$S(N) = \sum_{j=1}^N \frac{1}{j} \quad (A.1)$$

$$\begin{aligned} I_1 &\equiv \int \frac{d^n k_1}{(2\pi)^n} \frac{d^n k_2}{(2\pi)^n} \frac{(\Delta \cdot k_1)^M (\Delta \cdot k_2)^N}{(k_1 - p)^2 (k_1 - k_2)^2 k_2^2 (k_2 - p)^2} \\ &= \frac{1}{(4\pi^2)^2} \frac{1}{(M+1)} \left\{ \frac{1}{\epsilon^2} [2S(N) - 2S(M+N+1)] + \frac{1}{\epsilon} \left[\sum_{j=1}^M \left(\frac{S(j) + S(M-j)}{j + N + 1} + \frac{1}{(j + N + 1)^2} \right) \right. \right. \\ &\quad \left. \left. - 2S(M+1)S(M+N+1) + 2S(M+1)S(N) \right] \right\} \quad (A.2) \end{aligned}$$

$$\begin{aligned} I_2 &\equiv \int \frac{d^n k_1}{(2\pi)^n} \frac{d^n k_2}{(2\pi)^n} \frac{(\Delta \cdot k_1)^M (\Delta \cdot k_2)^N}{k_1^2 (k_1 - k_2)^2 k_2^2 (k_2 - p)^2} \\ &= \frac{1}{(4\pi^2)^2} \frac{1}{(M+1)(M+N+1)} \left\{ -\frac{2}{\epsilon^2} + \frac{1}{\epsilon} \left[S(M+N+1) - S(M) - \frac{2}{M+1} \right] \right\} \quad (A.3) \end{aligned}$$

$$\begin{aligned} I_3 &\equiv \int \frac{d^n k_1}{(2\pi)^n} \frac{d^n k_2}{(2\pi)^n} \frac{(\Delta \cdot k_1)^M (\Delta \cdot k_2)^N}{k_1^4 (k_1 - k_2)^2 (k_2 - p)^2} \\ &= -\frac{1}{(4\pi^2)^2} \frac{1}{\epsilon} \frac{1}{M(M+N+1)} \quad (A.4) \end{aligned}$$

^{*}) In (A.2) to (A.15) we have set $\Delta \cdot p = 1$.

$$I_4 \equiv \int \frac{d^n k_1}{(2\pi)^n} \frac{d^n k_2}{(2\pi)^n} \frac{(\Delta \cdot k_1)^M k_2 \cdot p}{k_1^4 (k_1 - p)^2 (k_1 - k_2)^2 k_2^2}$$

$$= \frac{1}{(16\pi^2)^2} \frac{1}{2(M+1)} \left[-\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon} \left(\frac{1}{2} S(M+1) - 1 \right) \right] \quad (A.5)$$

$$I_5 \equiv \int \frac{d^n k_1}{(2\pi)^n} \frac{d^n k_2}{(2\pi)^n} \frac{(\Delta \cdot k_1)^M (\Delta \cdot k_2) k_2 \cdot p}{k_1^4 (k_1 - p)^2 (k_1 - k_2)^2 k_2^2}$$

$$= \frac{1}{(16\pi^2)^2} \left\{ \frac{1}{\varepsilon^2} \left(\frac{1}{6(M+1)} - \frac{1}{3(M+2)} \right) + \frac{1}{\varepsilon} \left(\frac{2}{9(M+1)} - \frac{S(M+1)}{12(M+1)} - \frac{13}{36(M+2)} + \frac{S(M+2)}{6(M+2)} \right) \right\} \quad (A.6)$$

$$I_6 \equiv \int \frac{d^n k_1}{(2\pi)^n} \frac{d^n k_2}{(2\pi)^n} \frac{(\Delta \cdot k_1)^M (k_2 \cdot p)}{k_1^4 (k_1 - p)^2 (k_1 - k_2)^2 (k_2 - p)^2}$$

$$= \frac{1}{(16\pi^2)^2} \left\{ -\frac{1}{\varepsilon^2} \frac{1}{2(M+1)} + \frac{1}{\varepsilon} \left(\frac{1}{4(M+1)^2} - \frac{1}{2(M+1)} - \frac{1}{4M(M+1)} \right) \right\} \quad (A.7)$$

$$I_7 \equiv \int \frac{d^n k_1}{(2\pi)^n} \frac{d^n k_2}{(2\pi)^n} \frac{(\Delta \cdot k_1)^M (\Delta \cdot k_2) k_2 \cdot p}{k_1^4 (k_1 - p)^2 (k_1 - k_2)^2 (k_2 - p)^2}$$

$$= \frac{1}{(16\pi^2)^2} \left\{ \frac{1}{\varepsilon^2} \left(\frac{1}{6(M+1)} + \frac{1}{3(M+2)} \right) + \frac{1}{\varepsilon} \left(\frac{1}{6M} - \frac{5}{36(M+1)} - \frac{19}{36(M+2)} + \frac{1}{6(M+2)^2} + \frac{1}{12(M+1)^2} \right) \right\} \quad (A.8)$$

$$I_8 \equiv \int \frac{d^n k_1}{(2\pi)^n} \frac{d^n k_2}{(2\pi)^n} \frac{(\Delta \cdot k_1)^M (\Delta \cdot k_2)^N k_1 \cdot p}{k_1^4 (k_1 - k_2)^2 k_2^2 (k_2 - p)^2}$$

$$= \frac{1}{(16\pi^2)^2} \frac{1}{(M+1)(M+N)} \left\{ -\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon} \left(\frac{S(M+N)}{2} - \frac{S(M)}{2} - \frac{1}{(M+N+1)} - \frac{1}{2M} - \frac{1}{2(M+1)} \right) \right\} \quad (A.9)$$

$$I_9 \equiv \int \frac{d^n k_1}{(2\pi)^n} \frac{d^n k_2}{(2\pi)^n} \frac{(\Delta \cdot k_1)^M (\Delta \cdot k_2)^N}{k_1^2 (k_1 - p)^2 k_2^2 (k_2 - p)^2}$$

$$= - \frac{4}{(16\pi^4)^2} \frac{1}{\varepsilon^2} \frac{1}{(M+1)(N+1)} \quad (A.10)$$

$$I_{10} \equiv \int \frac{d^n k_1}{(2\pi)^n} \frac{d^n k_2}{(2\pi)^n} \frac{(\Delta \cdot k_1)^M (\Delta \cdot k_2)^N k_1 \cdot k_2}{k_1^2 (k_1 - p)^2 k_2^2 (k_2 - p)^2}$$

$$= - \frac{2}{(16\pi^4)^2} \frac{1}{\varepsilon^2} \frac{1}{(M+1)(N+2)} \quad (A.11)$$

$$I_{11} \equiv \int \frac{d^n k_1}{(2\pi)^n} \frac{d^n k_2}{(2\pi)^n} \frac{(\Delta \cdot k_1)^M (\Delta \cdot k_2)^N}{(k_1 - k_2 + p)^2 (k_1 - k_2)^2 k_2^2 (k_2 - p)^2}$$

$$= \frac{1}{(16\pi^4)^2} \frac{(-1)}{\varepsilon^2} \left\{ \frac{4}{(M+1)(M+N+1)} + \frac{(-1)^M 4 M! N!}{(M+1)(M+N+1)!} \right\} \quad (A.12)$$

$$I_{12} \equiv \int \frac{d^n k_1}{(2\pi)^n} \frac{d^n k_2}{(2\pi)^n} \frac{(\Delta \cdot k_1)^M (\Delta \cdot k_2)^N}{(k_1 - k_2 + p)^2 k_1^2 (k_1 - k_2)^2 k_2^2}$$

$$= \frac{1}{(16\pi^4)^2} \frac{(-1)^M M! N!}{(M+N+1)(M+N+1)!} \left\{ -\frac{2}{\varepsilon^2} + \frac{1}{\varepsilon} [S(M) + S(N) - S(M+N+1)] \right\} \quad (A.13)$$

$$I_{13} \equiv \int \frac{d^n k_1}{(2\pi)^n} \frac{d^n k_2}{(2\pi)^n} \frac{(\Delta \cdot k_1)^M (\Delta \cdot k_2)^N}{(k_1 - k_2 + p)^2 k_1^2 (k_1 - k_2)^2 (k_2 - p)^2} = \frac{1}{(16\pi^4)^2} \sum_{j=1}^N \frac{N! (j+M)!}{j! (N+M+1)! (M+j+1)}$$

$$\cdot \left\{ -\frac{2}{\varepsilon^2} + \frac{1}{\varepsilon} [S(M+N+1) - S(M+j) - S(N-j) - \frac{2}{j+M+1}] \right\} \quad (A.14)$$

$$I_{14} \equiv \int \frac{d^n k_1}{(2\pi)^n} \frac{d^n k_2}{(2\pi)^n} \frac{(\Delta \cdot k_1)^M (\Delta \cdot k_2)^N}{(k_1 - k_2 + p)^2 k_1^2 k_2^2 (k_2 - p)^2}$$

$$= \frac{-1}{(16\pi^4)^2} \frac{(-1)^M M! N!}{(M+1)(M+N+1)!} \left\{ \frac{2}{\varepsilon^2} + \frac{1}{\varepsilon} [S(M) + S(N) + \frac{2}{(M+1)} - S(M+N+1)] \right\} \quad (A.15)$$

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FIGURE CAPTIONS

- Figure 1 Effective vertices for the gluon operator $O_g^{\mu_1 \dots \mu_n}$.
- Figure 2 Two-loop diagrams contributing to γ_{gg}^N which involve a self-energy insertion into a one loop diagram. The wiggly, solid, broken lines represent gluon, fermion and ghost propagators, respectively.
- Figure 3 Two-loop diagrams contributing to γ_{gg}^N which do not have overlapping divergences (and which were not included in Fig. 2).
- Figure 4 Two-loop diagrams contributing to γ_{gg}^N which have overlapping divergences.
- Figure 5 Two-loop diagrams contributing to γ_{gf}^N which involve a self-energy insertion into a one loop diagram.
- Figure 6 Two-loop diagrams contributing to γ_{gf}^N which do not have overlapping divergences (and which were not included in Fig. 5).
- Figure 7 Two-loop diagrams contributing to γ_{gf}^N which have overlapping divergences.
- Figure 8 Two-loop diagrams contributing to γ_{fg}^N which involve a self-energy insertion into a one loop diagram.
- Figure 9 Two-loop diagrams contributing to γ_{fg}^N which do not have overlapping divergences (and which were not included in Fig. 8).
- Figure 10 Two-loop diagrams contributing to γ_{fg}^N which have overlapping divergences.
- Figure 11 The two-loop diagrams which contribute to γ_{ff}^N but which do not contribute to the anomalous dimension of the non-singlet operators.
- Figure 12 Two loop diagrams contributing to the gluons' self energy.
- Figure 13 Diagrams contributing to the forward elastic amplitude for a quark scattering on a weak or electromagnetic current.

Figure 14 Diagrams contributing to $\langle p | O_f^N | p \rangle$ where $|p\rangle$ is a quark state.

Figure 15 Diagrams contributing to the forward elastic amplitude for a gluon scattering on a weak or electromagnetic current.

Figure 16 Diagrams contributing to $\langle p | O_f^N | p \rangle$ where $|p\rangle$ is a gluon state.

FIG.1

FIG. 2

FIG. 3

FIG. 4

FIG. 5

FIG. 6

FIG. 7

FIG.8

FIG.9

FIG.10

FIG.11

FIG. 12

FIG.13

FIG.14

FIG.15

FIG.16