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MASS CORRECTIONS TO SOME INSTANTON EFFECTS

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A B S T R A C T

We develop an approximation scheme which allows us to determine the massive scalar propagator in a background instanton field to second order in a supposedly small mass parameter m , i.e., including terms of order $m^2 \ln m$ and m^2 , and the massive spinor propagator to the same order. The approximate scalar propagator is used to study the mass corrections to the short-distance behaviour of a two-point scalar loop. Applying this as a toy model for $\sigma(e^+e^- \rightarrow \text{anything})$ we find that the large momentum behaviour of the leading mass corrections is at most of the same order as the massless instanton effects.

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1. INTRODUCTION

Most predictions of quantum chromodynamics have so far been based on perturbative calculations performed at large momenta where non-perturbative effects are supposed to be unimportant or small. Only recently first attempts¹⁻³⁾ have been made at trying to understand whether these hopes are justified.

The new understanding of the QCD vacuum structure implied by the existence of instantons⁴⁻⁶⁾ has led to the introduction of a calculational scheme based on the dilute instanton gas approximation which allows an estimate of non-perturbative effects on some processes. This scheme is not devoid of difficulties -- in most cases it gives divergent integrals which as yet cannot be regularized in a convincing manner. Although one may not obtain numerical estimates, the scheme allows one to make statements about the large momentum behaviour of some non-perturbative effects.

In all calculations performed so far only one-loop diagrams were considered and all internal lines were taken to be massless. The main reasons for these limitations are simplicity and the fact that the exact analytical form of a massive propagator in an external instanton field is not known.

The massless propagators for scalar, spinor, and vector particles have been studied extensively⁷⁻¹³⁾. This is also the case for the relation between the massive scalar propagator and the massive spinor and vector propagators¹⁴⁾ as well as for the mechanism of mass generation¹⁵⁾. In this paper, we present an approximation of the massive scalar propagator in terms of the small mass parameter m . We find the coefficient of the $m^2 \ln m$ term exactly and the coefficient of the m^2 term to be given by one-dimensional integrals related to Spence functions, which are exhibited to lowest order in a short-distance variable. We also present some results on the massive spin $\frac{1}{2}$ propagator with a brief indication of its use in a spinor loop.

The approximate scalar propagator is applied to calculate the mass-dependent instanton corrections to a two-point scalar loop - used as a simple model for estimating instanton corrections to $e^+e^- \rightarrow \text{anything}$. We find that the m^0 and m^2 instanton contributions exhibit a less singular short-distance behaviour than the corresponding instanton free terms. The instanton contributions to the real part of the scalar loop decrease relatively, for large momentum Q^2 , as Q^{-2} . The numerical coefficient of this term is unfortunately not calculable as it is given by a divergent integral (where the divergence comes from both the integration over size and position of the instanton). However the cross-section for $e^+e^- \rightarrow \text{anything}$ is given by an imaginary part of the amplitude (appropriately continued to Minkowski space). It turns out that, owing to the behaviour mentioned above, the imaginary part of the m^0 instanton contribution behaves like $Q^{-1}(\ln Q)^6$ (for

zero quark flavours) and its coefficient is calculable. We are also able to calculate exactly the $m^2 \ln m$ contribution - it behaves like $Q^{-13} (\ln Q)^6$ (with the coefficient again being calculable). For the m^2 terms we are only able to show that they behave at most like $Q^{-11} (\ln Q)^6$. Thus we see that to order m^2 all instanton corrections are very strongly suppressed at large momenta.

2. THE SCALAR PROPAGATOR IN AN EXTERNAL INSTANTON FIELD

The propagator of a spin 0 particle with mass m in an external instanton field $A_\mu^a(x)$ is determined by

$$(-D^2 + m^2) \Delta_m(x, y) = \delta(x - y) \quad (2.1)$$

where

$$D_\mu = \partial_\mu - i \tau_a A_\mu^a(x) \quad (2.2)$$

and the non-Abelian gauge field in the singular gauge is given by

$$A_\mu^{a\pm}(x) = -\frac{1}{2} \eta_{\mu\nu a}^\pm \partial_\nu \ln \Pi(x), \quad a = 1, 2, 3, \quad (2.3)$$

where $\pm$ refer to instanton and anti-instanton, respectively. We may for simplicity choose

$$\Pi(x) = 1 + \frac{\rho^2}{x^2} \quad (2.4)$$

so as to represent a single (anti-) instanton of size ρ located at the origin.

In what follows we consider a $SU(2)$ colour gauge group. Then τ_a are the Pauli matrices satisfying

$$\tau_a \tau_b = \delta_{ab} + i \varepsilon_{abc} \tau_c \quad (2.5)$$

It is convenient to introduce

$$\tau_\mu = (\vec{\tau}, i), \quad \tau_\mu^\dagger = (\vec{\tau}, -i) \quad (2.6)$$

in terms of which the $\eta_{\mu\nu a}^\pm$ tensors in (2.3) are given by the relations

$$\begin{aligned} \tau_\mu^\dagger \tau_\nu &= \delta_{\mu\nu} + i \eta_{\mu\nu a}^+ \tau_a \\ \tau_\mu \tau_\nu^\dagger &= \delta_{\mu\nu} + i \eta_{\mu\nu a}^- \tau_a \end{aligned} \quad (2.7)$$

We also introduce operators $\Delta = (-D^2)^{-1}$ and $\Delta_m = (-D^2 + m^2)^{-1}$, whose configuration space matrix elements reproduce the propagator solutions of (2.1) for zero and non-zero mass so that, for example

$$\langle x | \Delta | y \rangle = \Delta(x, y) \quad (2.8)$$

fulfils $-D^2 \Delta(x, y) = \delta(x-y)$.

The corresponding operators for the free (no instanton) propagators are defined as

$$\Delta^{(0)} = (-\partial^2)^{-1}, \quad \Delta_m^{(0)} = (-\partial^2 + m^2)^{-1} \quad (2.9)$$

with

$$\begin{aligned} \langle x | \Delta^{(0)} | y \rangle &= \Delta^{(0)}(x, y) = [4\pi^2(x-y)^2]^{-1} \\ \langle x | \Delta_m^{(0)} | y \rangle &= \Delta_m^{(0)}(x, y) = \frac{m}{4\pi^2|x-y|} K_1(m|x-y|) \end{aligned} \quad (2.10)$$

It is remarkable that a simple expression exists for $\Delta(x, y)$ ⁸⁾. The most direct way to obtain it proceeds by noticing that for an instanton

$$D^2 = \Pi^{1/2} \tau \partial \Pi^{-1} \tau^\dagger \partial \Pi^{1/2} \quad (2.11)$$

Then using pure operator manipulations one finds

$$\begin{aligned} \Delta &= -(\partial^2)^{-1} = -\Pi^{-1/2} (\tau^\dagger \partial) \Pi (\tau \partial)^{-1} \Pi^{-1/2} = -\Pi^{-1/2} (\tau^\dagger \partial)^{-1} (1 + \frac{\rho^2}{x^2}) (\tau \partial)^{-1} \Pi^{-1/2} \\ &= -\Pi^{-1/2} (\partial^2)^{-1} \Pi^{-1/2} - \rho^2 \Pi^{-1/2} (\tau^\dagger \partial)^{-1} \frac{1}{x^2} (\tau \partial)^{-1} \Pi^{-1/2} \\ &= \Pi^{-1/2} \Delta^{(0)} \Pi^{-1/2} - \rho^2 \Pi^{-1/2} \frac{\tau x}{x^2} (\partial^2)^{-1} \frac{\tau^\dagger x}{x^2} \Pi^{-1/2} = \Pi^{-1/2} (\Delta^{(0)} + \rho^2 \frac{\tau x}{x^2} \Delta^{(0)} \frac{\tau^\dagger x}{x^2}) \Pi^{-1/2} \end{aligned} \quad (2.12)$$

where we have used that

$$\begin{aligned} [(\tau^\dagger \partial)^{-1} \frac{1}{x^2} (\tau \partial)^{-1}]^{-1} &= (\tau \partial) x^2 (\tau^\dagger \partial) = x^2 \partial^2 + 2\tau x \tau^\dagger \partial \\ &= \tau x \partial^2 \tau^\dagger x = [(\tau^\dagger x)^{-1} (\partial^2)^{-1} (\tau x)^{-1}]^{-1} = [\frac{\tau x}{x^2} (\partial^2)^{-1} \frac{\tau^\dagger x}{x^2}]^{-1} \end{aligned}$$

This derivation also goes through for the n-instanton 't Hooft solution. To write the result of (2.12) in a more compact form one may introduce the column and row vectors N, N^+ (in an operator sense)

$$N(x) = \Pi^{-1/2}(x) \begin{pmatrix} 1 \\ \rho \frac{\tau^\dagger x}{x^2} \end{pmatrix}, \quad N^+(x) = \Pi^{-1/2}(x) \left(1, \rho \frac{\tau x}{x^2} \right) \quad (2.13)$$

such that $N^+ N = 1$; then

$$\Delta = N^+ \Delta^{(0)} N, \quad (2.14)$$

which is also the form of the massless propagator in a general n-instanton field^{12,13)} when N is constructed appropriately. Although we for simplicity

treat the case with N given by (2.13), the following discussion of the massive propagator will hold in the more general case.

We shall be primarily interested in the scalar propagator Δ_m , because the propagators for the spin $\frac{1}{2}$ and spin 1 cases can be derived in a straightforward way¹⁴⁾ once the scalar propagator is known. At the end we will discuss the spin $\frac{1}{2}$ propagator explicitly.

Let us now introduce the projection matrix operator

$$P(x) = NN^+ = [g^2 + x^2]^{-1} \begin{pmatrix} x^2 & g^2 \tau x \\ g^2 \tau x & g^2 \end{pmatrix} \quad (2.15)$$

Then the massive propagator Δ_m is given by

$$\begin{aligned} \Delta_m &= \Delta (1 + m^2 \Delta)^{-1} = N^+ \Delta^{(0)} N [1 + m^2 N^+ \Delta^{(0)} N] \\ &= N^+ \Delta^{(0)} [1 + m^2 P \Delta^{(0)}]^{-1} N, \end{aligned} \quad (2.16)$$

where we have used that $N[1 + N^+ \Delta^{(0)} N] = [1 + m^2 P \Delta^{(0)}] N$. As it seems difficult to find an exact expression for $\Delta^{(0)} [1 + m^2 P \Delta^{(0)}]^{-1}$ [basically because P does not commute with $\Delta^{(0)}$], we shall develop an approximation for small m , which will correspond to the expansion of the free propagator

$$\Delta_m^{(0)}(x, y) = \frac{1}{4\pi^2(x-y)^2} + \frac{m^2}{8\pi^2} \ln \frac{m|x-y|}{2} + \frac{m^2}{8\pi^2} \left(\gamma - \frac{1}{2}\right) + \dots \quad (2.17)$$

where γ is Euler's constant.

It is easy to observe that a naive expansion of (2.16) in powers of m^2 would lead to divergences [as exemplified by the appearance of the $\ln m$ term in (2.17)]. We will proceed differently by expanding around the (constant) matrix.

$$E = \lim_{\substack{|x| \rightarrow \infty \\ g \text{ fixed}}} P(x) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad (2.18)$$

Using the identity

$$(1 + m^2 P \Delta^{(0)})^{-1} = (1 + m^2 E \Delta^{(0)})^{-1} + m^2 (1 + m^2 E \Delta^{(0)})^{-1} (E - P) \Delta^{(0)} (1 + m^2 P \Delta^{(0)})^{-1}, \quad (2.19)$$

we obtain

$$\begin{aligned} \Delta^{(0)} (1 + m^2 P \Delta^{(0)})^{-1} &= \Delta^{(0)} (1 + m^2 E \Delta^{(0)})^{-1} + m^2 \Delta^{(0)} (1 + m^2 E \Delta^{(0)})^{-1} (E - P) \Delta^{(0)} (1 + m^2 E \Delta^{(0)})^{-1} \\ &\quad + m^4 \Delta^{(0)} (1 + m^2 E \Delta^{(0)})^{-1} (E - P) \Delta^{(0)} (1 + m^2 E \Delta^{(0)})^{-1} (E - P) \Delta^{(0)} (1 + m^2 P \Delta^{(0)})^{-1} \end{aligned} \quad (2.20)$$

Owing to $[E, \Delta^{(0)}] = 0$ it is easy to invert $[1 + m^2 E \Delta^{(0)}]$ and so

$$\Delta^{(0)} [1 + m^2 E \Delta^{(0)}]^{-1} = \Delta^{(0)} (1 - E + [1 + m^2 \Delta^{(0)}]^{-1} \Delta^{(0)} E) = \Delta^{(0)} (1 - E) + \Delta_m^{(0)} E = \begin{pmatrix} \Delta_m^{(0)} & 0 \\ 0 & \Delta^{(0)} \end{pmatrix}. \quad (2.21)$$

We note that

$$(E - P)(x) = \frac{1}{\rho^2 + x^2} \begin{pmatrix} \rho^2 & -\rho \tau x \\ -\rho \tau^\dagger x & -\rho^2 \end{pmatrix} \quad (2.22)$$

for any finite ρ goes to zero as $|x|^{-1}$ or faster when $|x| \rightarrow \infty$ and thus improves the convergence properties of the terms in (2.20) when viewed as integrals. Each term of the right-hand side of (2.20) is well defined, because, when evaluated in configuration space, all integrals are convergent owing to the damping factors $\Delta^{(0)} [1 + m^2 E \Delta^{(0)}]^{-1}$. In the last term of (2.20) the least convergent contribution comes from terms in which the non-diagonal elements of $E - P$ are multiplied by each other corresponding to, for example, $\Delta^{(0)} (\tau^\dagger x / \rho^2 + x^2) \Delta_m^{(0)} (\tau x / \rho^2 + x^2) \Delta^{(0)}$. When we let $m \rightarrow 0$ in the m^4 term of (2.20) we thus encounter at most logarithmically divergent integrals and this term turns out to be of order $m^4 \ln m$ or higher. In the second term of the right-hand side of (2.20) the coefficient of m^2 is well defined even for $m^2 \rightarrow 0$. We may thus evaluate it at $m = 0$ and using $\Delta_m^{(0)} = \Delta^{(0)} - m^2 \Delta_m^{(0)} \Delta^{(0)}$ it can be seen that one thereby neglects terms of order $m^4 \ln^2 m$ or higher.

To order m^2 (neglecting terms of at most order $m^4 \ln^2 m$) we therefore have an approximate propagator

$$\begin{aligned} \Delta_m(x, y) &\simeq \langle x | N^+ [\Delta_m^{(0)} E + \Delta^{(0)} (1 - E) + m^2 \Delta^{(0)} (E - P) \Delta^{(0)}] N | y \rangle \\ &= \Pi_{(x)}^{-1/2} \Delta_m^{(0)}(x, y) \Pi_{(y)}^{-1/2} + \rho^2 \Pi_{(x)}^{-1/2} \frac{\tau x}{x^2} \Delta^{(0)}(x, y) \frac{\tau^\dagger y}{y^2} \Pi_{(y)}^{-1/2} \\ &\quad + m^2 N^\dagger(x) \int d^4 z \Delta^{(0)}(x, z) (E - P)(z) \Delta^{(0)}(z, y) N(y), \end{aligned} \quad (2.23)$$

where of course we may expand $\Delta_m^{(0)}$ to the required order as given in (2.17). The problem is therefore reduced to the calculation of

$$\int d^4 z \Delta^{(0)}(x, z) (E - P)(z) \Delta^{(0)}(z, y) = \begin{pmatrix} \rho^2 I_1(x, y) & -\rho \tau \cdot I_2(x, y) \\ -\rho \tau^\dagger \cdot I_2(x, y) & -\rho^2 I_1(x, y) \end{pmatrix} \quad (2.24)$$

where

$$I_1(x, y) = \int d^4 z \frac{1}{(x - z)^2} \frac{1}{z^2 + \rho^2} \frac{1}{(y - z)^2} \quad (2.25)$$

and

$$I_2^M(x, y) = \int d^4 z \frac{1}{(x - z)^2} \frac{z^\mu}{z^2 + \rho^2} \frac{1}{(y - z)^2} \quad (2.26)$$

In terms of these integrals the propagator is

$$\begin{aligned} \Delta_m(x,y) = & \Delta(x,y) + \frac{1}{4\pi^2} \Pi^{-1/2}(x) \Pi^{-1/2}(y) \\ & \cdot \left\{ \frac{m^2}{2} \ln \frac{m|x-y|}{2} + \frac{m^2}{2} \left(\gamma - \frac{1}{2} \right) \right. \\ & + \frac{m^2 \gamma^2}{4\pi^2} \left[\left(1 - \gamma^2 \frac{\tau x}{x^2} \frac{\tau y}{y^2} \right) I_1(x,y) \right. \\ & \quad \left. \left. - \tau \cdot I_2(x,y) \frac{\tau y}{y^2} - \frac{\tau x}{x^2} \tau \cdot I_2(x,y) \right] \right\} \\ & + O(m^4 \ln^2 m). \end{aligned} \quad (2.27)$$

As the integral for I_1 resembles a momentum space expression for a triangle contribution to the vertex function we may, in principle, use the results of Källén and Wightman to write I_1 in terms of dilogarithms or the related Spence functions. Since this is not particularly illuminating we may simply give the answer in terms of a one-dimensional Feynman type parameter integral. Putting $x = y + a$ and

$$\begin{aligned} Z(p) &= a^2 + 2ya p + p^2 y^2 \\ R(p) &= a^2 + p(p^2 + y^2 + 2ya) \end{aligned} \quad (2.28)$$

we find

$$\begin{aligned} I_1 &= -\pi^2 \int_0^1 dp \frac{1}{Z(p)} \ln \left(1 - \frac{Z(p)}{R(p)} \right) \\ I_2 &= \pi^2 x_\mu \int_0^1 dp \left\{ \frac{1}{Z(p)} \left[1 - \ln \left(1 - \frac{Z(p)}{R(p)} \right) \right] + \frac{R(p)}{[Z(p)]^2} \ln \left(1 - \frac{Z(p)}{R(p)} \right) \right\} \\ &\quad - \pi^2 y_\mu \int_0^1 dp (1-p) \left\{ \frac{1}{Z(p)} \left(1 + \frac{R(p)}{Z(p)} \right) \ln \left(1 - \frac{Z(p)}{R(p)} \right) \right\} \end{aligned} \quad (2.29)$$

Since $|Z(p)| \leq R(p)$ we can make a series expansion of the logarithm where the integral of each term can be calculated. To order $O(a \ln |a|)$ we obtain

$$\begin{aligned} I_1(x,y) &= \pi^2 \left\{ \frac{1}{\gamma^2 + y^2} \ln \frac{(\gamma^2 + y^2)^2}{\gamma^4 a^2} + \frac{1}{\gamma^2 + y^2} - \frac{1}{y^2} \ln \frac{\gamma^2 + y^2}{\gamma^2} \right\} \\ I_2^\mu(x,y) &= \pi^2 \frac{x^\mu + y^\mu}{2} \left\{ \frac{1}{\pi^2} I_1(x,y) - \frac{1}{2(\gamma^2 + y^2)} - \frac{1}{y^2} + \frac{\gamma^2}{y^4} \ln \frac{\gamma^2 + y^2}{\gamma^2} \right\} \end{aligned} \quad (2.30)$$

In closing this section we may point out that the integrals I_1 and I_2 can be evaluated exactly in the case where the instanton sits at one of the end points of the propagator.

3. THE SPIN $\frac{1}{2}$ PROPAGATOR

We define operators for the massive and massless spinor propagator in an instanton background as

$$\begin{aligned} S_m &= (\gamma D + im)^{-1} \\ S &= (\gamma D')^{-1}, \end{aligned} \quad (3.1)$$

where the prime denotes that the operator D' acts only in the space spanned by the non-zero energy eigenmodes. Similarly,

$$\begin{aligned} S_m^{(0)} &= (\gamma \partial + im)^{-1} \\ S^{(0)} &= (\gamma \partial)^{-1} \end{aligned} \quad (3.2)$$

for the no-instanton case. The matrix elements are defined in analogy with the scalar case. Contrary to the scalar case, an explicit expression for the spinor propagator to lowest order in the mass already exists, albeit not precisely in a form similar to the one discussed in Section 2. Carlitz and Creamer¹⁵⁾ used the expansion

$$\begin{aligned} S_m - S_m^{(0)} &= \frac{P_0}{im} - \frac{1}{1+imS^{(0)}} (S^{(0)} - S) \frac{1}{1+imS^{(0)}} \\ &\quad - \frac{1}{1+imS^{(0)}} (S^{(0)} - S) \frac{im}{1+imS^{(0)}} (S^{(0)} - S) \frac{1}{1+imS^{(0)}} + \dots \end{aligned} \quad (3.3)$$

to derive

$$S_m = S + S_m^{(0)} - S^{(0)} + \frac{P_0}{im} (1 - (mp)^2 \ln(mp)) + \dots \quad (3.4)$$

P_0 is here the projection operator for the zero modes

$$\langle x | P_0 | y \rangle = \psi_0(x) \psi_0^\dagger(y) \quad (3.5)$$

While their derivation does not show it explicitly, one can use the exact relation

$$S - S^{(0)} = \frac{1}{\gamma D} (1 - P_0) - \frac{1}{\gamma \partial} \quad (3.6)$$

(which in Ref. 15 was presented in a version with $|x-y| \rightarrow \infty$) to show rigorously that (3.4) describes the limit $p \rightarrow 0$ with other variables fixed and, somewhat counter-intuitively, that the neglected correction terms in the first non-zero mode term in (3.3) are of higher order than $m \ln m$. Taken at its face value the propagator (3.4) is of dubious value in calculations of short-range effects, since such calculations involve integration over instanton size.

However, massive as well as massless spinor propagators are uniquely given by their scalar counterparts¹⁴⁾ as

$$S_m = (\gamma D - im) \Delta_m \frac{1+\gamma_5}{2} + \Delta_m \gamma D \frac{1-\gamma_5}{2} + \frac{1}{im} (1 - \gamma D \Delta_m \gamma D) \frac{1-\gamma_5}{2} \quad (3.7)$$

(with a sign flip of the γ_5 matrices for an anti-instanton). The order to which S_m can be found thus depends entirely on our knowledge of Δ_m . Using the approximate form of Δ_m found in Section 2 it is thus a simple matter to find an expression for S_m valid to order m^2 . For instance the coefficient of the $m \ln m$ term turns out to have the same ϕ -dependence as was exhibited in Ref. 15:

$$\frac{i}{m \ln m} (S_m - S)(x, y) = \frac{-2g^4}{\pi^2 (y^2 + x^2)^{3/2} (y^2 + y^2)^{3/2}} \frac{\tau x}{|x|} \frac{\tau^\dagger \gamma \tau \gamma}{16} \frac{1-\gamma_5}{2} \frac{\tau^\dagger y}{|y|} + \dots \quad (3.8)$$

Accordingly (3.4) is also useful in the short-range limit with ϕ constant and arbitrary.

Instead of giving S_m explicitly to order m^2 it will be more transparent to display the simpler structure which results after performing the γ -matrix algebra in actual applications of the spinor propagator.

As an example, we take the case of a two-point spinor loop, which enters the calculation of the total e^+e^- cross-section at high energies (and which at the other end of the energy spectrum goes into expressions involving the GIM mechanism and gluon corrections to non-leptonic decays).

Defining

$$\Pi_m^{\mu\nu}(x, y) = - \text{Tr} (\gamma^\mu S_m(x, y) \gamma^\nu S_m(y, x)) , \quad (3.9)$$

one finds

$$\begin{aligned} \Pi_m^{\mu\nu}(x, y) = & -4 \text{Tr} \left\{ \vec{D}_\mu \Delta_m(x, y) \vec{D}_\nu \Delta_m(y, x) \right. \\ & + \Delta_m(x, y) \overleftarrow{D}_\mu \Delta_m(y, x) \overleftarrow{D}_\nu \\ & + 2(\delta(x-y) + \vec{D}_\mu \Delta_m(x, y) \overleftarrow{D}_\mu) \Delta_m(y, x) \\ & \left. + 2\Delta_m(x, y) (\delta(x-y) + \vec{D}_\nu \Delta_m(y, x) \overleftarrow{D}_\nu) \right\} \quad (3.10) \end{aligned}$$

where the remaining trace operation refers to the τ matrices in the scalar propagator; the arrows indicate the direction in which the derivatives act, a leftward arrow implying as usual that the (Euclidean) spatial derivative has a negative sign. This expression has the advantage of including the zero-mode contributions in a natural way, as well as avoiding the ill-defined expansion in mass of Refs. 1 and 2.

As a simpler calculational laboratory we will, however, in the next section treat the case of a massive scalar loop. While it is much simpler to calculate, the lowest-order instanton contribution to the imaginary part exhibits the same large momentum behaviour as the spinor loop and we also expect the mass corrections in both cases to have essentially the same qualitative behaviour.

4. MASS CORRECTIONS TO A SCALAR LOOP

In this section we will consider a simple model of $\sigma(e^+e^- \rightarrow \text{anything})$ along the lines of Ref. 2 and try to estimate the large-momentum behaviour of the mass corrections to the instanton effects on a two-point scalar loop Π in the dilute gas approximation. Thus we have to calculate the Fourier transform of the instanton correction term ($\Pi = \Pi_0 + \Pi_I$):

$$\Pi_I(x, y) = \int d^4z \int d\rho D(\rho) \text{Tr} \left\{ \Delta_m(x, y, z, \rho) \Delta_m(y, x, z, \rho) - \Delta_m^{(0)}(x, y) \Delta_m^{(0)}(y, x) \right\} \quad (4.1)$$

where $D(\rho)$ is the instanton density function⁵⁾ and z is the instanton position. As we have determined $\Delta_m(x, y, z, \rho)$ only modulo $O(m^4 \ln^2 m)$ terms, this will also be the case for $\Pi_I(x, y)$.

If we formally write

$$\Delta_m = A + B m^2 \ln m + C m^2 + \dots, \quad (4.2)$$

choosing for simplicity a unit mass scale in the $\ln m$ term, we then have

$$\begin{aligned} \text{Tr}(\Delta_m \Delta_m) = & \text{Tr}[A(x, y) A(y, x)] + m^2 \ln m \text{Tr}[A(x, y) B(y, x) + A(y, x) B(x, y)] \\ & + m^2 \text{Tr}[A(x, y) C(y, x) + A(y, x) C(x, y)] + \dots \end{aligned} \quad (4.3)$$

Thus, when we sum over instanton and anti-instanton contributions, we obtain (with the instanton situated at the origin)

$$\begin{aligned} \text{Tr}(\Delta_m \Delta_m) = & \frac{4}{(4\pi^2)^2 (x-y)^4} \left[1 - \frac{\rho^2 (x-y)^2}{(x^2 + \rho^2)(y^2 + \rho^2)} \right] \\ & + \frac{2m^2 \ln m^2}{(4\pi^2)^2 (x-y)^2} \left\{ 1 - \frac{\rho^2}{(x^2 + \rho^2)(y^2 + \rho^2)} [(x-y)^2 + (\rho^2 + xy)] \right\} \\ & + \frac{4m^2}{(4\pi^2)^2 (x-y)^2} \left\{ \ln \frac{|x-y|}{2} + \gamma - \frac{1}{2} - \frac{\rho^2}{(x^2 + \rho^2)(y^2 + \rho^2)} \left[\left(\ln \frac{|x-y|}{2} + \gamma - \frac{1}{2} \right) \right. \right. \\ & \cdot ((x-y)^2 + \rho^2 + xy) + \frac{1}{2\pi^2} (\rho^4 - x^2 y^2) I_1(x, y, \rho) \\ & \left. \left. + \frac{1}{2\pi^2} (y^2 + \rho^2)(x \cdot I_2(x, y, \rho) + \frac{1}{2}(x^2 + \rho^2)y \cdot I_2(x, y, \rho)) \right] \right\} + O(m^4 \ln^2 m). \end{aligned} \quad (4.4)$$

The splitting of the $\ln(m/2)|x-y|$ is justified as calculation step because I_1 and I_2 contain $\ln \rho$ and $\ln |x-y|$ terms, which can be combined with $\ln m$. Any final expression should, however, exhibit that the real expansion parameter is dimensionless, see Ref. 15.

The leading short-distance behaviour of the m^0 term in the no-instanton contribution is given by $1/(x-y)^4$ and the instanton contribution is relatively suppressed by a factor $(x-y)^2$. This relative suppression does not hold for the $m^2 \ln m$ terms. Both the no-instanton contribution and the instanton correction exhibit the short-distance behaviour $1/(x-y)^2$, i.e. the same as the instanton contribution to the m^0 term.

To understand the behaviour of the m^2 terms we use the approximate form of the I_1 and I_2 integrals as displayed in (2.30). Thus, to order $(a \ln |a|)$, the expression in the curly bracket of (4.4) is found to be

$$\begin{aligned}
 & \ln |a| + \gamma - \frac{1}{2} - \frac{p^2}{(y^2 + p^2)} \left(\ln |a| + \gamma - \frac{1}{2} \right) \\
 & - \frac{p^2}{(y^2 + p^2)^2} \left[\frac{1}{2} (p^2 + y^2) \ln \frac{p^2 + y^2}{a^2} - \frac{p^2}{2y^2} (p^2 + y^2) \ln \frac{p^2}{p^2 + y^2} - \frac{p^2 + 2y^2}{2} \right] \\
 & = \ln |a| + \gamma - \frac{1}{2} - \frac{p^2}{p^2 + y^2} \left(\ln \frac{p}{2} + \gamma - \frac{1}{2} \right) - \frac{p^2}{2y^2} \ln \frac{p^2 + y^2}{p^2} \\
 & \quad + \frac{p^2 (p^2 + 2y^2)}{2(y^2 + p^2)^2} \quad . \quad (4.5)
 \end{aligned}$$

As the leading short-distance behaviour of the m^2 term of the no-instanton contribution is given by $2/(4\pi^2)^2 a^2 (\ln a + \gamma - \frac{1}{2})$ we see that the leading instanton contribution is proportional to $1/a^2$ and that the next higher corrections behave like $\ln |a|$ and a constant. The leading short-distance behaviour of the no-instanton contribution ($\sim 1/a^2 \ln |a|$) is thus again suppressed in Π_1 .

In the discussion so far we have considered the leading short-distance behaviour of the full expression for the scalar loop. For physical applications, however, it is usually only of relevance to know its momentum space discontinuity. This is the case if we consider the scalar loop as a simple model for $\sigma(e^+e^- \rightarrow \text{anything})$. Therefore we need just Fourier-transform our expressions and, having continued the results to Minkowski space (which we assume to be possible), take the imaginary part. At the same time we have to perform the integration over instanton positions and sizes as indicated in (4.1).

Using translational invariance we may set one vertex of the loop at 0, the other one at a_μ and the instanton at y_μ . Then the m^0 terms in Π are

$$\Pi_0^{(0)} = \frac{2}{(4\pi^2)^2} \int d^4 a \frac{e^{iQ \cdot a}}{a^4} \sim -\ln Q^2 (+\text{infinity}) \quad (4.6)$$

and

$$\Pi_I^{(0)} = -\frac{2}{(4\pi^2)^2} \int d^4 a \frac{e^{iQ \cdot a}}{a^2} \int d^4 y \int d^4 p D(p) \frac{p^2}{(y^2 + p^2)((y+a)^2 + p^2)} \quad (4.7)$$

for the no-instanton and the instanton pieces, respectively. Continuing to Minkowski space $Q^2 \rightarrow -Q^2$ and noting that, for $Q^2 > 0$, $\text{Im} \ln(-Q^2) = \text{const.}$, we recover the well-known scaling result that $\sigma(e^+e^- \rightarrow \text{anything}) \sim \text{const.}$ for $Q^2 \rightarrow \infty$. As $\int d^4 a (e^{iQ \cdot a}/a^2)^2 \sim Q^{-2}$ has no imaginary part when continued to Minkowski space we see that the instanton correction to the imaginary part of Π is of higher order than the correction to the real part.

At first sight one might think that $\Pi_I^{(0)}$ has no imaginary part for $Q^2 > 0$ whatsoever, since we could expand $[(y+a)^2 + p^2]^{-1}$ in powers of a^μ and observe that

$$\int d^4 a \frac{e^{iQ \cdot a}}{a^2} a^n = (2\pi)^4 (-\Delta)^{n-1} \delta(Q) = 0 \quad \text{for } Q^2 > 0 \quad (4.8)$$

This argument is, however, incorrect. When the integration over instanton position y_μ is performed (with a cut-off if necessary to avoid divergences) the expansion in powers of a_μ becomes an expansion in powers of a^2/ρ^2 , but above a certain power the integral over instanton sizes begins to diverge at the lower end of integration. To see that this corresponds to a generation of an imaginary part it is convenient to perform the integrations over y_μ and a_μ in (4.7) explicitly and then consider the resultant integral over instanton sizes. This indeed was the method used in Ref. 2 in the calculation of the spinor loop.

Thus we obtain, using a convenient cut-off on the y integration,

$$\begin{aligned} \frac{1}{a^2} \int d^4 y \frac{1}{(y^2 + p^2)((y+a)^2 + p^2)} &= \frac{1}{a^2} \int_0^1 d\varphi \int d^4 y \frac{1}{[y^2 + p^2 + \varphi(1-\varphi)a^2]^2} \\ &= \frac{\pi^2}{a^2} \ln \frac{\Lambda'^2}{p^2} - \frac{\pi^2}{a^2} - \pi^2 \int_0^1 d\varphi \int_0^\infty dv \frac{1}{(v+p^2)(a^2 + \frac{v+p^2}{\varphi(1-\varphi)})} \end{aligned} \quad (4.9)$$

as the cut-off $\Lambda' \rightarrow \infty$.

The first two terms, when Fourier-transformed, contribute only to the real part and therefore we have to concentrate on the study of the last term. As

$$\int d^4 a e^{iQ \cdot a} \frac{1}{a^2 + \frac{v+p^2}{\varphi(1-\varphi)}} = 4\pi^2 \sqrt{\frac{v+p^2}{Q^2 \varphi(1-\varphi)}} K_1\left(\sqrt{\frac{Q^2(v+p^2)}{\varphi(1-\varphi)}}\right) \quad (4.10)$$

we are left with

$$\int_0^1 d\varphi \int_0^\infty dv \sqrt{\frac{1}{(v+p^2)\varphi(1-\varphi)Q^2}} K_1\left(\sqrt{\frac{Q^2(v+p^2)}{\varphi(1-\varphi)}}\right).$$

Next we perform the v -integration and obtain

$$\frac{2}{Q^2} \int_0^1 d\varphi K_0\left(\frac{pQ}{\sqrt{\varphi(1-\varphi)}}\right).$$

Thus the total contribution to the imaginary part for $Q^2 > 0$ can be obtained from

$$\begin{aligned} & \frac{1}{Q^2} \int d\varphi D(\varphi) \varphi^2 \int_0^1 d\varphi K_0\left(\frac{pQ}{\sqrt{\varphi(1-\varphi)}}\right) \\ &= 0.1 \left(\frac{Q}{\Lambda}\right)^{-11} \int_0^1 d\varphi \int_0^\infty du u^8 K_0\left(\frac{u}{\sqrt{\varphi(1-\varphi)}}\right) \left(11 \ln \frac{\Lambda u}{Q}\right)^6, \end{aligned} \quad (4.11)$$

where we have taken the instanton density function calculated by 't Hooft⁵⁾ in a Pauli-Villars regularization scheme, changed the variable of integration $\rho = u/Q$, and set the number of quark fields to zero.

For large Q^2 , $[\ln(\Lambda u/Q)]^6 \sim (\ln Q)^6$ and we can perform the remaining integrations obtaining

$$0.1 \left(\frac{Q}{\Lambda}\right)^{-11} (11 \ln Q)^6 2^7 \frac{(\Gamma(\frac{9}{2}) \Gamma(\frac{11}{2}))^2}{\Gamma(11)} \quad (4.12)$$

Continuing this expression to Minkowski space $Q^2 \rightarrow -Q^2$ we obtain an imaginary part from the continuation of $(Q/\Lambda)^{11}$ and see that the instanton contribution, for large Q^2 , is given by $\sim (Q/\Lambda)^{-11} (\ln Q)^6$. This behaviour was also found by Baulieu et al.²⁾ working with the physically more relevant spinor-loop model.

We now proceed to look at the $m^2 \ln m^2$ and m^2 contributions to the scalar loop reckoning that the corresponding contributions to the spinor loop will be of a similar nature. For the $m^2 \ln m$ term the no-instanton contribution is

$$\Pi_0^{(1)}(Q) = \frac{m^2 \ln m^2}{(4\pi^2)^2} \int d^4a e^{iQ \cdot a} \frac{1}{a^2} = \frac{m^2 \ln m}{4\pi^2} \frac{1}{Q^2} \quad (4.13)$$

and thus gives no contribution to the imaginary part. The instanton correction can be rewritten as

$$\begin{aligned} \Pi_I^{(1)}(Q) &= \frac{m^2 \ln m^2}{(4\pi^2)^2} \int d\varphi D(\varphi) \int d^4a e^{iQ \cdot a} \int d^4y \\ &\quad \cdot \frac{1}{2} \left\{ \frac{p^2}{[p^2 + (y - \frac{a}{2})^2][p^2 + (y + \frac{a}{2})^2]} + \frac{p^2}{a^2[p^2 + (y - \frac{a}{2})^2]} + \frac{p^2}{a^2[p^2 + (y + \frac{a}{2})^2]} \right\} \end{aligned} \quad (4.14)$$

The last two contributions do not give any imaginary part as

$$\int d^4y \frac{1}{p^2 + (y \pm a)^2} = \int d^4y \frac{1}{p^2 + y^2} = \pi^2 (\Lambda^2 - p^2 \ln(\Lambda^2 + p^2) + p^2 \ln p^2), \quad (4.15)$$

where Λ' is the necessary position integration cut-off. Then the Fourier transform of these terms is proportional to $\int d^4a e^{iQ \cdot a} (1/a^2) = 4\pi^2 (1/Q^2)$ and with the cut-off as introduced above we see that they do not give any imaginary part. The remaining term can be rewritten as

$$\frac{m^2 \ln m^2}{2 (4\pi^2)^2} \Delta_Q \int d^4p \mathcal{D}(p) \int d^4y \int d^4a \frac{p^2 e^{iQ \cdot a}}{a^2 [p^2 + (y - \frac{a}{2})^2] [p^2 + (y + \frac{a}{2})^2]} \quad (4.16)$$

where Δ_Q denotes the Laplacian in Q -space. The expression to the right of the Laplace operator is exactly the expression we have calculated earlier -- thus we see that for large Q^2 the imaginary part of $\Pi_I^{(1)}(Q^2)$ is given by

$$\begin{aligned} & -\frac{1}{2} m^2 \ln m^2 \Delta_Q \sim 0.1 \left(\frac{Q}{\Lambda}\right)^{-11} (11 \ln Q)^6 \frac{(\Gamma(\frac{9}{2}) \Gamma(\frac{11}{2}))^2}{\Gamma(11)} \\ & \sim m^2 \ln m^2 \left(\frac{Q}{\Lambda}\right)^{-13} (\ln Q)^6 \end{aligned} \quad (4.17)$$

The contribution of the $m^2 \ln m$ term is therefore, for large Q^2 , suppressed by a factor Q^{-2} relative to the contribution of the m^0 term to $\Pi(Q)$.

Next we look at the m^2 terms. The no-instanton contribution is given by

$$\frac{2m^2}{(4\pi^2)^2} \int d^4a e^{iQ \cdot a} \frac{1}{a^2} [\ln a + \gamma - \frac{1}{2}] = -\frac{2m^2}{4\pi^2} \frac{1}{Q^2} \left(\frac{1}{2} + \ln \frac{Q}{2}\right) \quad (4.18)$$

and the imaginary part is proportional to m^2/Q^2 . Comparing with (4.13) we find further justification for using the splitting $m^2 \ln m (a/2) = m^2 \ln m + m^2 \ln (a/2)$, as the γ factor from the Fourier transform of $m^2 \ln (a/2)$ cancels with the similar explicit factor in the coefficient of m^2 in (4.4).

To calculate the imaginary part of the instanton correction exactly we would have to know the I_1 and I_2 functions in (2.25) and (2.26) exactly and not only in the limit $a \rightarrow 0$ as given in (2.30).

We may nevertheless argue along the line of reasoning for the m^0 term and find that the imaginary part will decrease at least as $Q^{-11} (\ln Q)^6$ for large Q^2 . A more precise idea on the leading behaviour of the m^2 term may be found by using a better approximation of I_1 and I_2 ; however, we will not pursue this problem further in this paper.

We can thus state that to order m^0 , $m^2 \ln m$, and m^2 the instanton corrections to $\sigma(e^+e^- \rightarrow \text{anything})$, calculated in the scalar loop model with the dilute gas approximation, are for large Q^2 suppressed as $Q^{-11} (\ln Q)^6$ relative to the leading scaling piece.

One may of course ask what happens with the higher mass corrections to the imaginary part of $\Pi(Q)$. To try to answer this question we may note that, of all the short-distance terms a^{2n} , $n = -2, -1, 0, 1, \dots$ appearing in $\Pi(x,y)$, only the first term with $n = -2$ has a Fourier transform which contributes to the imaginary part of $\Pi(Q)$. Similarly, for the $a^{2n} \ln |a|$ terms, $n = -1, 0, 1, \dots$, only the first term gives an imaginary part. In calculating instanton corrections we found that these two leading terms were absent and therefore the leading imaginary part came from the scaling of the ρ -integration in (4.11).

In the no-instanton contribution we also have terms like $a^{2n} \ln^2 |a|$, $n = 0, 1, \dots$, which appear for the first time in the m^4 corrections. All of them give rise to an imaginary part. It would be extremely interesting to see whether such terms also appear in the instanton corrections as they might dominate the correction terms which we have been able to calculate. Such terms on the m^4 level of our approximation scheme would, if they are not cancelled somehow, give rise to instanton corrections of order m^4/Q^4 . Even if the leading singularity were cancelled, m^6 terms might still give an m^6/Q^6 contribution. Unfortunately, we are not yet able to determine what the corrections are like at the m^4 level, but we hope to report on this later on.

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