Ref.TH.2645-CERN

QUANTUM FLUCTUATIONS IN THE ONE INSTANTON SECTOR
OF THE CP^{n-1} MODEL

A.M. Din, P. Di Vecchia and W.J. Zakrzewski^{*)}
CERN--Geneva

ABSTRACT

Within the two-dimensional CP^{n-1} model we calculate the instanton density function in the dilute gas limit by studying the quantum fluctuations in the one instanton sector in the one loop approximation. The result disagrees with the $1/n$ expansion. Green's functions in the one instanton background are displayed.

^{*)} On leave of absence from the Department of Mathematics, University of Durham, Durham, England.

1. INTRODUCTION

Recently a new class of non-linear σ -models, the CP^{n-1} models, have been constructed¹⁾. Their interest stems from the fact that they are similar to four dimensional non-Abelian gauge theories in what concerns properties like conformal invariance, asymptotic freedom, dimensional transmutation and the topological structure. The low energy properties of these models have been extensively analyzed using the powerful technique of the $1/n$ expansion^{2),3)}. In particular it has been shown that topologically non-trivial field configurations are essential for providing the confinement of massive quarks and that there is no $U(1)$ problem in the chiral limit.

Due to the non-trivial topological structure which gives rise to instanton solutions, quite analogous to the Yang-Mills instantons, in addition to the $1/n$ expansion, one can also approximate the Green's functions of the theory with the minima of the classical action and then compare the two approximations. Although at first sight one would expect that these two approaches, at least in the dilute gas approximation, cannot match together because the $1/n$ expansion gives a power behaviour in n , while the instantons give an exponential decreasing in n ³⁾, one must, however, remember that in the dilute gas approximation one gets a divergent result due to the scale divergence for large scales. In order to obtain a finite expansion one must then cut-off the theory for large scales. The question is now the following: Can one fix the scale cut-off in such a way that there is no disagreement between the $1/n$ expansion and the dilute gas approximation?

In this paper we compute the quantum fluctuations around the one instanton in the CP^{n-1} model and we show that there is no choice of the scale cut-off which gives agreement between the $1/n$ expansion and the dilute gas approximation. This is, however, not due to the exponential behaviour in n given by the classical action, but it is produced by the quantum fluctuations.

The paper is organized as follows. In Section 2 we solve exactly the eigenvalue equation for small fluctuations in the one instanton sector. Section 3 is devoted to the computation of one loop quantum corrections in the one instanton sector. The Green's function in the one instanton background is evaluated in Section 4. In Section 5 we compare finally our result with the $1/n$ expansion.

2. EIGENVALUE EQUATION FOR THE FLUCTUATION AROUND AN INSTANTON

In this section we derive the eigenvalue equation for small fluctuations around an instanton and we solve it exactly in the case of an instanton with topological charge equal to ± 1 .

We start from the action for the CP^{n-1} model that can be written in the following form:

$$S = \int d^2x \, \overline{D_\mu z} D_\mu z \quad (2.1)$$

where z^α is a complex field transforming according to the fundamental representation of $SU(n)$ that satisfies the condition^{*)}

$$|z|^2 = 1/f \quad (2.2)$$

while the covariant derivative is defined by:

$$D_\mu = \partial_\mu - f \bar{z} \partial_\mu z \quad (2.3)$$

An instanton is a solution of the self-duality equation:

$$D_\mu z_0 = i \varepsilon_{\mu\nu} D_\nu z_0 \quad (2.4)$$

The most general solution of Eq. (2.4) with topological charge equal to 1 is given by:

$$z_0^\alpha = \frac{u^\alpha \lambda + v^\alpha (x_0 - a_0 - i(x_1 - a_1))}{\sqrt{\lambda^2 + (x - a)^2}} \frac{1}{\sqrt{f}} \quad (2.5)$$

where u^α and v^α are arbitrary vectors which satisfy the following conditions:

*) Notice that the coupling constant f used in this paper is connected to the coupling constant used in Ref. 2) by $f \rightarrow n/2f$.

$$|u|^2 = |v|^2 = 1, \quad \bar{u} \cdot v = 0. \quad (2.6)$$

It is easy to see that z_0^α depends on $4n - 2$ arbitrary real parameters, which corresponds to the number of zero modes.

We can now expand the action (2.1) around a classical solution $z_0^\alpha(x)$ of the self-duality Eq. (2.4) by writing:

$$z^r(x) = z_0^r \sqrt{1 - f|\phi|^2} e^{i\psi} + \phi^r. \quad (2.7)$$

Because of the gauge invariance of the action (2.1) the phase factor $e^{i\psi}$ can always be chosen such that

$$\bar{z}_0 \cdot \phi = 0 \quad (2.8)$$

Inserting now (2.7) into (2.1) and keeping only up to quadratic terms in the fluctuation ϕ^α one gets:

$$S = S_0 + \int d^2x \bar{\phi}^r M^{rs} \phi^s \quad (2.9)$$

with

$$M^{rs} = (-D_\mu D_\mu - f \overline{D_\mu z_0} D_\mu z_0) \delta^{rs} - 2f D_\mu z_0^r \overline{D_\mu z_0^s}. \quad (2.10)$$

S_0 is the classical action and the covariant derivative D_μ is computed with the classical solution $z_0^\alpha(x)$. In the case of the classical solution (2.5) with $\lambda = 1$ and $a_1 = 0$, M^{rs} is given by:

$$M^{\alpha\beta} = \left\{ -\square + \frac{2i \varepsilon_{\mu\nu} x_\nu \partial_\mu}{1+x^2} - \frac{2-x^2}{(1+x^2)^2} \right\} \delta^{\alpha\beta} - \frac{4}{(1+x^2)^2} p^\alpha \bar{p}^\beta \quad (2.11)$$

where

$$p^\alpha = \frac{v^\alpha - (x_0 + i x_i) u^\alpha}{\sqrt{1+x^2}} \quad (2.12)$$

is the vector lying in the space spanned by u^α and v^α that is orthogonal to $z_0^\alpha(x)$.

The most general expression for ϕ^α satisfying the constraint (2.8) is given by:

$$\phi^\alpha = w^\alpha G(x) + p^\alpha F(x) \quad (2.13)$$

where w^α is an x -independent unit vector orthogonal to both u^α and v^α :

$$\bar{w} \cdot w = \bar{v} \cdot w = 0 \quad (2.14)$$

Inserting (2.13) into (2.9) one gets finally:

$$S = S_0 + \int d^2x \left\{ \bar{G} \left[-\square + \frac{2i \varepsilon_{\mu\nu} x_\nu \partial_\mu}{1+x^2} + \frac{x^2-2}{(1+x^2)^2} \right] G + \bar{F} \left[-\square + \frac{4i \varepsilon_{\mu\nu} x_\nu \partial_\mu}{1+x^2} + 4 \frac{x^2-1}{(1+x^2)^2} \right] F \right\} \quad (2.15)$$

From (2.15) one can easily write down the eigenvalue equations for F and G . However, these equations are not easy to solve. As has been pointed out by 't Hooft⁴⁾ it is more convenient to solve the following related equations:

$$\left\{ -\square + \frac{2i\epsilon_{\mu\nu}x_\nu\partial_\mu}{1+x^2} + \frac{x^2-2}{(1+x^2)^2} \right\} G = \lambda_g \frac{4}{(1+x^2)^2} G \quad (2.16)$$

$$\left\{ -\square + \frac{4i\epsilon_{\mu\nu}x_\nu\partial_\mu}{1+x^2} + 4\frac{x^2-1}{(1+x^2)^2} \right\} F = \lambda_f \frac{4}{(1+x^2)^2} F \quad (2.17)$$

which are the equations resulting from writing the theory, defined by the action (2.1), on the unit sphere. To see this we observe that the integration volume $d\Omega$ on the unit sphere is related to d^2x by the formula:

$$d\Omega = d^2x \left(\frac{2}{1+x^2} \right)^2 \quad (2.18)$$

Equations (2.16) and (2.17) are now easy to solve if we write them in terms of the variable:

$$\cos \theta = \frac{1-x^2}{1+x^2} \quad (2.19)$$

They become simply:

$$\left\{ \frac{\partial^2}{\partial \theta^2} + \cot \theta \frac{\partial}{\partial \theta} - \frac{e^2 + e + \frac{1}{2} - (e + \frac{1}{2}) \cos \theta}{\sin^2 \theta} + \lambda_g + \frac{3}{4} \right\} G = 0 \quad (2.20)$$

$$\left\{ \frac{\partial^2}{\partial \theta^2} + \cot \theta \frac{\partial}{\partial \theta} - \frac{e^2 + 2e + 2 - (2e + 2) \cos \theta}{\sin^2 \theta} + \lambda_f + 2 \right\} F = 0 \quad (2.21)$$

where we have used the formula:

$$\frac{2 i \varepsilon_{\mu\nu} x_\nu \partial_\mu}{1+x^2} e^{i\varphi} = \frac{2e}{1+x^2} \quad (2.22)$$

where φ is defined by the expressions:

$$x_0 = |x| \cos \varphi, \quad x_1 = |x| \sin \varphi \quad (2.23)$$

Equations (2.20) and (2.21) are particular cases of the equation for the rotation functions $d_{mk}^n(\theta)$:

$$\left\{ \frac{\partial^2}{\partial \theta^2} + \cot \theta \frac{\partial}{\partial \theta} - \frac{m^2 + k^2 - 2mk \cos \theta}{\sin^2 \theta} + n(n+1) \right\} d_{mk}^n(\theta) = 0. \quad (2.24)$$

Comparing (2.20) and (2.21) with (2.24) we can immediately determine the eigenfunctions and the eigenvalues of Eqs. (2.20) and (2.21).

We get:

$$\begin{aligned} G &\sim d_{l+\frac{1}{2}, \frac{1}{2}}^{j+\frac{1}{2}}(\theta), \quad \lambda_g = j(j+2), \quad j = 0, 1, 2, \dots \\ F &\sim d_{l+1, 1}^{j+1}(\theta), \quad \lambda_f = j(j+3), \quad j = 0, 1, 2, \dots \end{aligned} \quad (2.25)$$

Recalling the decomposition (2.13), we see that the operator $M^{\alpha\beta}$ in (2.9) in the field of one instanton for the theory on the sphere has the following spectrum of eigenvalues:

$$\begin{aligned}\lambda &= j(j+2) \quad , \quad d_j = 2(j+1) \cdot 2(n-2) \quad , \quad j=0,1,2,\dots \\ \lambda &= j(j+3) \quad , \quad d_j = 2(2j+3) \quad , \quad j=0,1,2,\dots\end{aligned}\tag{2.26}$$

where each real mode is counted as one.

When $j = 0$ one gets $4n - 2$ real zero modes and this number coincides with the number obtained by considering the parameters of the classical solution (2.5).

We conclude this section by looking at the eigenvalue equation for the fluctuation around the vacuum solution which is given by $z_0^\alpha(x)$ equal to a constant vector. In this case one must solve the following eigenvalue equation:

$$N^{\alpha\beta} \phi^\beta = -\square \phi^\alpha = \frac{4}{(1+x^2)^2} \lambda \phi^\alpha \tag{2.27}$$

subject to the condition:

$$\bar{z}_0 \cdot \phi = 0 \tag{2.28}$$

Using the method described above we get the following branch of eigenvalues:

$$\lambda = j(j+1) \quad , \quad d_j = (2j+1)2(n-1) \quad , \quad j = 0,1,2,\dots \tag{2.29}$$

the eigenfunctions are just the spherical harmonics:

$$\phi^\alpha \sim Y_{je}(\theta, \varphi) \tag{2.30}$$

The number of zero modes is equal to $2(n-1)$ which agrees with the number of arbitrary parameters appearing in the classical $z_0^\alpha = \text{const}$ solution.

3. COMPUTATION OF THE QUANTUM FLUCTUATIONS AROUND A ONE INSTANTON CLASSICAL SOLUTION

The contribution to the partition function of the CP^{n-1} model of the quantum fluctuations around the one instanton solution, normalized with respect to the vacuum, and evaluated in the one loop approximation is given by:

$$\begin{aligned} Z^{(1)} &= e^{-\frac{2\pi}{f}} \frac{\int D\phi D\bar{\phi} e^{-\int d\Omega \bar{\phi} M' \phi}}{\int D\phi D\bar{\phi} e^{-\int d\Omega \bar{\phi} N \phi}} = \\ &= e^{-\frac{2\pi}{f}} \frac{\det N}{\det M'} \quad , \quad (3.1) \\ M' &= \frac{(1+x^2)^2}{4} M \end{aligned}$$

where the exponential term comes from the classical action, while the determinants of the operators M' and N defined in the previous section give the one loop correction.

If we compute the determinants (3.1) using the eigenvalues that we have found in the previous section we find that they are actually infinite and we must regularize them. As in Ref. 5) we used the ζ -function regularization procedure.

Proceeding in exactly the same way as in Ref. 5) we find the following expression for $Z^{(1)}$:

$$Z^{(1)} = e^{-\frac{2\pi}{f}} (\lambda\mu)^{\zeta(0) - \zeta_0(0) + \tilde{z} - \tilde{z}_0} \frac{1}{e} \frac{\zeta'(0) - \zeta'_0(0)}{I_0} \quad (3.2)$$

where:

$$\begin{aligned}\xi(s) &= 2 \sum_{j=1}^{\infty} (2j+3) [j(j+3)]^{-s} + 2(n-2) \sum_{j=1}^{\infty} (2j+2) [j(j+2)]^{-s} \\ \xi_0(s) &= 2(n-1) \sum_{j=1}^{\infty} (2j+1) [j(j+1)]^{-s}.\end{aligned}\quad (3.3)$$

$\tilde{z}[\tilde{z}_0]$ is the number of zero modes of the instanton [vacuum] solution and I and I_0 are determined by the contribution of the collective coordinates. Introduction of collective coordinates C_k gives the following formula for I :

$$I = \int dC_k \det \left[\frac{1}{2\pi} \int \bar{\phi}_i \phi_j d\Omega \right] \quad (3.4)$$

where ϕ_i are the zero mode eigenfunctions of the operator M . In order to compute explicitly $\zeta(0)$ and $\zeta'(0)$ it is convenient to introduce the following function:

$$f(s) = \sum_{n=1}^{\infty} (2n+a) [n(n+a)]^{-s} \quad (3.5)$$

By analytic continuation it is easy to prove that:

$$\begin{aligned}f(0) &= -\frac{1}{6} - \frac{a}{2} \\ f'(0) &= 4\zeta_R'(-1) - \frac{a^2}{2} + \sum_{n=1}^a (2n-a) \log n\end{aligned}\quad (3.6)$$

where $\zeta_R(s)$ is the ordinary Riemann ζ -function.

Using the previous formulae and the fact that $\tilde{z} - \tilde{z}_0 = 2n$ it is possible to compute the contributions of the non-zero modes to the determinants. One gets:

$$Z^{(1)} = e^{-\frac{2\pi}{f}} (\lambda\mu)^n e^{n[-\frac{1}{2} + 2\log 2] - 1 + 3\log 3 - 3\log 2} \frac{I}{I_0} \quad (3.7)$$

We must now introduce the following correction factor:

$$\begin{aligned} & e^{-f \frac{n}{2\pi} \int \overline{D_\mu z_0} D_\mu z_0 \log \frac{2\mu}{\mu_0(1+x^2)} d^2x} \\ & = e^{-n [\log \mu/\mu_0 - 1 + \log 2]} \end{aligned} \quad (3.8)$$

which allows the transition from a subtraction point μ which is x dependent to another subtraction point μ_0 that is x independent. The expression (3.7) takes therefore, the following form:

$$Z^{(1)} = e^{-\frac{2\pi}{f} + n \log(\lambda\mu_0) + n[-\frac{1}{2} + \log 2] - 1 + 3\log 3 - 3\log 2} \frac{I}{I_0} \quad (3.9)$$

We must now compute the contribution of the zero modes. We start by listing the eigenfunctions corresponding to the zero modes of the fluctuation in the instanton field. If we assume, for instance, that the vectors u^α and v^α which appear in the classical solution (2.5) span the two-dimensional subspace characterized by the vectors of the type (oo...a b)

where a and b are arbitrary numbers, then the zero mode eigenfunctions can be written as follows:

$$\phi_i^r = \frac{1}{\sqrt{f}} \delta_i^r \frac{1}{\sqrt{1+x^2}}, \quad i = 1, \dots, n-2 \quad (3.10)$$

$$\phi_j^r = \frac{1}{\sqrt{f}} \delta_j^r \frac{x_0 - ix_1}{\sqrt{1+x^2}}, \quad j = 1, \dots, n-2 \quad (3.11)$$

These are the eigenfunctions corresponding to the $4(n-2)$ real zero modes.

In addition, we have six further zero modes that lie in the space spanned by the vectors u^α and v^α . Their eigenfunctions are:

$$\begin{aligned} \phi_1^r &= \frac{i}{\sqrt{f}} \frac{1 - (x_0 - ix_1)}{1+x^2} p^r \\ \phi_2^r &= -\frac{1}{\sqrt{f}} \frac{1 + (x_0 - ix_1)^2}{1+x^2} p^r \\ \phi_3^r &= \frac{1}{\sqrt{f}} \frac{1}{1+x^2} p^r \\ \phi_4^r &= -\frac{i}{\sqrt{f}} \frac{1}{1+x^2} p^r \\ \phi_5^r &= -\frac{2i}{\sqrt{f}} \frac{x_0 - ix_1}{1+x^2} p^r \\ \phi_6^r &= \frac{1}{\sqrt{f}} \frac{x_0 - ix_1}{1+x^2} p^r \end{aligned} \quad (3.12)$$

The contribution of the zero modes (3.10) and (3.11) to the determinant in (3.4) is given by $(1/f)^{2(n-2)}$ while the zero modes (3.12) give the following contribution:

$$2^6 3^{-3} f^{-3} \quad (3.13)$$

Therefore I is given by the following expression:

$$I = \int d^2x \frac{d\lambda}{\lambda^3} \left(\frac{1}{f}\right)^{2n-1} 2^6 3^{-3} V \quad (3.14)$$

where V is the volume of $SU(n)/T(n)$ and where $T(n)$ is the stability group of the instanton. The stability group of the one instanton solution is the same as in the Yang-Mills case. We can therefore use the formula carefully computed in Ref. 6). In this way we obtain

$$I = \int d^2x \frac{d\lambda}{\lambda^3} \left(\frac{1}{f}\right)^{2n-1} 2^6 3^{-3} \frac{\pi^{2n-2}}{(n-1)!(n-2)!} \quad (3.15)$$

We must now compute the corresponding factor for the fluctuation around the vacuum. If we choose the classical vacuum solution of the type

$$z_0^a = (0 \ 0 \ \dots \ 1)$$

we can write the zero mode eigenfunctions as follows:

$$\phi_i^a = \frac{1}{\sqrt{f}} \delta_i^a, \quad i = 1, \dots, n-1 \quad (3.16)$$

The contribution of those eigenfunctions to the determinant in (3.4) is given by $(2/f)^{n-1}$. This number must then be multiplied by the volume of the stability group of the vacuum. Following the arguments described in Ref. 6) we find that this volume is equal to $\pi^{n-1}/(n-1)!$, thus giving

$$I_0 = \left(\frac{2}{f}\right)^{n-1} \frac{\pi^{n-1}}{(n-1)!} \quad (3.17)$$

Inserting (3.15) and (3.17) into (3.9) we obtain the final expression:

$$Z^{(1)} = \int d^2x \frac{d\lambda}{\lambda^2} e^{-\frac{2\pi}{f(\lambda)} - 1 - \frac{n}{2} + 2 \log 2} \left(\frac{\pi}{f}\right)^n \frac{4}{\pi} \frac{1}{(n-2)!} \quad (3.18)$$

where

$$\frac{1}{f(\lambda)} = \frac{1}{f(\mu_0)} + n \log(\lambda \mu_0) \quad (3.19)$$

In terms of the renormalization group invariant mass

$$m^2 = \mu_0^2 e^{-\frac{4\pi}{nf(\mu_0)}} \quad (3.20)$$

$f(\lambda)$ is given by the following expression:

$$f(\lambda) = \frac{2\pi}{n \log(1/\lambda m)} \quad (3.21)$$

Notice that the integral over λ is divergent for large values of λ as in the case of the Yang-Mills theory. For $n = 2$ there is also an ultra-violet divergence that has already been noticed in Refs. 7) where the quantum fluctuations in the one instanton sector of the $O(3)$ non-linear σ model were computed.

4. THE GREEN'S FUNCTION

We can use the results of Section 2 to evaluate the propagator for the z field in the one instanton background. These propagators are needed if one should want to go beyond the one loop approximation.

The object to evaluate is

$$\Delta'_{\alpha\beta}(x, y) = \langle \bar{z}_\alpha(x) z_\beta(y) \rangle_1 \quad (4.1)$$

where the index 1 refers to the one instanton background. We will only be interested in the non-zero mode part Δ of Δ' and decompose Δ in the following way:

$$\Delta_{\alpha\beta}(x,y) = \Delta^{(1)}(x,y) \frac{1}{f} \bar{w}_\alpha w_\beta + \Delta^{(n)}(x,y) \frac{1}{f} \bar{p}_\alpha(x) p_\beta(y) \quad (4.2)$$

where the vectors w and p are chosen as in (2.12) and (2.14).

The propagator $\Delta^{(1)}(x,y)$ can be determined from its eigenfunction expansion in terms of the eigenfunctions $\phi_{j\ell}$ and eigenvalues λ_j displayed in (2.25) and (2.26)

$$\phi_{je} = \sqrt{\frac{2j+2}{4\pi}} \mathcal{D}_{\ell+\frac{1}{2}, \frac{1}{2}}^{j+\frac{1}{2}}(\varphi, \theta, -\varphi) \quad (4.3)$$

$$\lambda_j = j(j+2)$$

where $\phi_{j\ell}$ is now normalized on the sphere. The rotation functions $D_{\ell\ell'}^j(\alpha, \beta, \gamma)$ are related to $d_{\ell\ell'}^j(\beta)$ in the usual way

$$\mathcal{D}_{\ell\ell'}^j(\alpha, \beta, \gamma) = e^{ie\alpha} d_{\ell\ell'}^j(\beta) e^{ie'\gamma} \quad (4.5)$$

we thus find

$$\Delta^{(1)}(x,y) = \sum_{\substack{j,\ell \\ j>0}} \frac{1}{\lambda_j} \phi_{je}(y) \phi_{je}^*(x) \quad (4.6)$$

Using the addition theorem for the rotation functions one finds

$$\begin{aligned} \Delta^{(1)}(x,y) &= \frac{1}{4\pi} \sum_{j=1}^{\infty} \frac{2j+2}{j(j+2)} \mathcal{D}_{\frac{1}{2}, \frac{1}{2}}^{j+\frac{1}{2}}(\varphi, \beta, \gamma) \\ &= \frac{1}{4\pi} e^{\frac{i}{2}(\gamma+\gamma')} \sum_{j=1}^{\infty} \left(\frac{1}{j} + \frac{1}{j+2}\right) d_{\frac{1}{2}, \frac{1}{2}}^{j+\frac{1}{2}}(\beta) \end{aligned} \quad (4.7)$$

Here (α, β, γ) are Euler angles found by applying the rotation $(\varphi_x, -\theta_x, -\varphi_x)$ followed by $(\varphi_y, \theta_y, -\varphi_y)$, where (θ_x, φ_x) and (θ_y, φ_y) are the spherical coordinates of x and y respectively.

The angles α, β, γ can be found from the relations

$$\cos \beta = \cos \theta_x \cos \theta_y + \sin \theta_x \sin \theta_y \cos \varphi \quad (4.8)$$

$$\sin(\gamma + \varphi_y) \sin \beta = -\sin \varphi \sin \theta_x \quad (4.9)$$

$$\sin(\gamma - \varphi_x) \sin \beta = \sin \varphi \sin \theta_y \quad (4.10)$$

where $\varphi = \varphi_y - \varphi_x$.

In terms of the plane variables $\underline{x}$ and $\underline{y}$ one finds

$$\cos \beta = 1 - \frac{2\lambda^2 r^2}{(\lambda^2 + x^2)(\lambda^2 + y^2)} \quad (4.11)$$

where

$$r = |\underline{x} - \underline{y}| \quad (4.12)$$

We have exhibited the instanton size λ explicitly which means that the relation (2.19) is modified to

$$\cos \theta = \frac{\lambda^2 - x^2}{\lambda^2 + x^2} \quad (4.13)$$

The phase factor in (4.7) can be found from

$$e^{i(\gamma + \varphi)} = \frac{1}{\cos^2 \frac{\beta}{2}} A \quad (4.14)$$

with

$$\begin{aligned}
 A &= \frac{1}{2} (\sin \theta_x \sin \theta_y + \cos \varphi (1 - \cos \theta_x)(1 - \cos \theta_y)) e^{-i\varphi} \\
 &+ \frac{1}{2} (\cos \theta_x + \cos \theta_y) = \\
 &= \frac{1}{2(\lambda^2 + x^2)(\lambda^2 + y^2)} \left[(x^2 + y^2 - r^2)(x^2 + y^2 + 2\lambda^2 - r^2) \right. \\
 &\quad \left. + 2(\lambda^4 - x^2 y^2) + i(x^2 + y^2 + 2\lambda^2 - r^2)(2(x^2 + y^2)r^2 - (x^2 - y^2)^2 - r^4)^{1/2} \right] \quad (4.15)
 \end{aligned}$$

To evaluate the sum appearing in (4.7) one observes that

$$d_{\frac{1}{2}, \frac{1}{2}}^{j+1/2}(\beta) = \cos \frac{\beta}{2} P_j^{(0,1)}(\beta) \quad (4.16)$$

where $P_j^{(0,1)}(\cos \beta)$ is a Jacobi polynomial. Defining

$$f_1(\cos \beta, z) = \sum_{j=1}^{\infty} \frac{1}{j} P_j^{(0,1)}(\cos \beta) z^j \quad (4.17)$$

and

$$f_2(\cos \beta, z) = \sum_{j=1}^{\infty} \frac{1}{j+2} P_j^{(0,1)}(\cos \beta) z^j \quad (4.18)$$

one finds the equations

$$z \frac{\partial}{\partial z} f_1(\cos \beta, z) = P^{(0,1)}(\cos \beta, z) - 1 \quad (4.19)$$

and

$$(z \frac{\partial}{\partial z} + 2) f_2(\cos \beta, z) = P^{(0,1)}(\cos \beta, z) - 1 \quad (4.20)$$

where

$$\begin{aligned} P^{(0,1)}(\cos \beta, z) &= \sum_{j=0}^{\infty} P_j^{(0,1)}(\cos \beta) z^j = \\ &= 2 [1 - 2z \cos \beta + z^2]^{-1/2} [1 + z + (1 - 2z \cos \beta + z^2)^{1/2}]^{-1} \end{aligned} \quad (4.21)$$

is the generating function for the Jacobi polynomials $P_j^{(0,1)}$.

Solving the differential equations (4.19) and (4.20) and putting $z = 1$ one finds:

$$\begin{aligned} f(\cos \beta) &\equiv f_1(\cos \beta, 1) + f_2(\cos \beta, 1) = \\ &= \int_0^1 dz (P^{(0,1)}(\cos \beta, z) - 1) (z + z^{-1}) \end{aligned} \quad (4.22)$$

This integral can be performed explicitly with the result:

$$f(\cos \beta) = - \log \frac{1 - \cos \beta}{2} - 3/2 \quad (4.23)$$

For $\Delta^{(1)}(x, y)$ we thus find the result:

$$\Delta^{(1)}(x, y) = - \frac{\sqrt{A}}{4\pi} \left[\log \frac{\lambda^2 |x - y|^2}{(\lambda^2 + x^2)(\lambda^2 + y^2)} + 3/2 \right] \quad (4.24)$$

with A given by (4.15).

We may proceed similarly with $\Delta^{(2)}(x, y)$ to find:

$$\begin{aligned}\Delta^{(2)}(x, y) &= \frac{1}{4\pi} \sum_{j=1}^{\infty} \frac{2j+3}{j(j+3)} \mathcal{D}_{1,1}^{j+1}(\varphi, \beta, \gamma) = \\ &= \frac{1}{4\pi} e^{i(\varphi+\gamma)} \sum_{j=1}^{\infty} \left(\frac{1}{j} + \frac{1}{j+3} \right) d_{1,1}^{j+1}(\beta)\end{aligned}\quad (4.25)$$

Now

$$d_{1,1}^{j+1}(\beta) = \cos^2 \frac{\beta}{2} P_j^{(0,2)}(\cos \beta) \quad (4.26)$$

and using the generating function for the Jacobi polynomials $P_j^{(0,2)}(\cos \beta)$

$$\begin{aligned}P^{(0,2)}(\cos \beta, z) &= \sum_{j=0}^{\infty} P_j^{(0,2)}(\cos \beta) z^j \\ &= 4 [1 - 2z \cos \beta + z^2]^{-1/2} [1 + z + (1 - 2z \cos \beta + z^2)]^{-2}\end{aligned}\quad (4.27)$$

one can again evaluate the sum over j

$$\begin{aligned}&\sum_{j=1}^{\infty} \left(\frac{1}{j} + \frac{1}{j+3} \right) P_j^{(0,2)}(\cos \beta) \\ &= \int_0^1 dz (P^{(0,2)}(\cos \beta, z) - 1) (z^{-1} + z^2) = -\log \frac{1 - \cos \beta}{2} - 11/6.\end{aligned}\quad (4.28)$$

The result for $\Delta^{(2)}$ is therefore

$$\Delta^{(2)}(x, y) = -\frac{A}{4\pi} \left[\log \frac{\lambda^2 |x-y|^2}{(\lambda^2 + x^2)(\lambda^2 + y^2)} + 11/6 \right] \quad (4.29)$$

where A is taken from (4.15).

The final result for the non-zero mode part of the z propagator is thus:

$$\begin{aligned} \Delta_{\alpha\beta}(x,y) = & -\frac{\sqrt{A}}{4\pi f} \left[\log \frac{\lambda^2 |x-y|^2}{(\lambda^2+x^2)(\lambda^2+y^2)} + 3/2 \right] \bar{w}_\alpha w_\beta \\ & - \frac{A}{4\pi f} \left[\log \frac{\lambda^2 |x-y|^2}{(\lambda^2+x^2)(\lambda^2+y^2)} + 11/6 \right] \bar{p}_\alpha(x) p_\beta(y) \end{aligned} \quad (4.30)$$

5. DILUTE GAS APPROXIMATION AND $1/n$ EXPANSION

In this section we compute the $n \rightarrow \infty$ limit in the dilute gas approximation and we compare it with the result that one obtains using the $1/n$ expansion. At first sight, as has been noticed in Ref. 3), the contribution of instantons cannot reproduce the behaviour of the $1/n$ expansion because of the exponential damping factor which comes from the classical action of an instanton. However, the expansion that one gets for the contribution of one instanton is actually divergent because of the integration over large scales. One can remove this divergence by integrating over λ only up to a cut-off λ_c . We can then pose the following question: Can one fix the cut-off λ_c in such a way that the dilute gas approximation matches the result of the $1/n$ expansion?

In order to analyze this question we will take the large n limit of (3.18) after having introduced a cut-off on the scale integration.

Thus we consider the following expression:

$$Z^{(1)} = \int d^2x \frac{2^4}{\pi e (n-2)!} \int_0^{\lambda_c} \frac{d\lambda}{\lambda^3} \left(\frac{\pi}{f(\lambda)\sqrt{e}} \right)^n e^{-\frac{2\pi}{f(\lambda)}} \quad (5.1)$$

where the factor $(1/f)^n$, coming from the zero modes, has been substituted by $(1/f(\lambda))^n$, a procedure which we expect to be justifiable by taking into account higher orders in perturbation theory. In this way we obtain an expansion for $Z^{(1)}$ that is renormalization group invariant because the subtraction point μ_0 has disappeared from (5.1)

By means of a few changes of variables and by using Eq. (3.21) one can rewrite $Z^{(1)}$ as follows:

$$Z^{(1)} = \int d^2x \frac{2^4 m^2}{\pi e (n-2)!} \left(\frac{n}{2\sqrt{e}}\right)^n \left(\frac{1}{n-2}\right)^{n+1} \Gamma(n+1, (n-2)B) \quad (5.2)$$

where Γ is the incomplete Γ -function:

$$\Gamma(a, b) = \int_b^\infty e^{-t} t^{a-1} dt \quad (5.3)$$

and where

$$B = -\log(m\lambda_c) \quad (5.4)$$

For the perturbative expansion to make sense we must require that $B > 0$. Since $\Gamma(n+1; (n-2)B) \leq n!$ we see that $Z^{(1)}$ goes to zero faster than $(2\sqrt{e})^{-n}$ when $n \rightarrow \infty$, independent of the value of the cut-off λ_c . For $B \leq 1$ we get the limiting expression

$$Z^{(1)} \sim \int \frac{16 e m^2}{\pi} n (2\sqrt{e})^{-n} d^2x \quad (5.5)$$

which is independent of the cutoff λ_c , whereas for $B > 1$ this just represents an upper bound. This expression cannot match the results of $1/n$ expansion which gives a power behaviour as $n \rightarrow \infty$. We notice that this is due to the factor $[1/f(\lambda)]^n$ which comes from the zero modes and not to the exponential coming from the classical action. Had this factor not been in $Z^{(1)}$ one would have obtained a limiting expression for $Z^{(1)}$ depending on the cut-off λ_c and one could have fixed it requiring agreement with the $1/n$ expansion. The same phenomenon occurs also in the Yang-Mills theory when the limit $n \rightarrow \infty$ of $Z^{(1)}$ is taken.

It would be very interesting to make a comparison between the $1/n$ expansion and the dense gas approximation that has recently been computed in two different papers⁸⁾ which we received while the present work was being completed.

ACKNOWLEDGEMENT

We thank P. Sorba for a useful discussion on the volume of the groups.

REFERENCES

- 1) H. Eichenherr, Nucl. Phys. B146 (1978) 215;
E. Cremmer and J. Scherk, Phys. Letters 74B (1978) 341;
V. Golo and A. Perelomov, ITEP preprint (1978).
- 2) A. D'Adda, P. Di Vecchia and M. Lüscher, Nucl. Phys. B146 (1978) 63,
and DESY 78/75 (1978), to be published in Nucl. Phys.
- 3) E. Witten, Harvard preprint (1978).
- 4) G. 't Hooft, Phys. Rev. D14 (1976) 3432.
- 5) S. Chadha, A. D'Adda, P. Di Vecchia and F. Nicodemi, Phys. Letters 72B
(1977) 103.
- 6) C. Bernard, UCLA preprint 79/TEP/3.
- 7) A. Jevicki, Nucl. Phys. B127 (1977) 125;
D. Förster, Nucl. Phys. B130 (1977) 38.
- 8) V.A. Pateev, I.V. Frolov and A.S. Schwarz, ITEP preprint (1979);
B. Berg and M. Lüscher, DESY preprint (1979).