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A NEW PERTURBATIVE APPROACH FOR THE ANHARMONIC OSCILLATOR

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A B S T R A C T

A new perturbative approach is presented and applied to the anharmonic oscillator. Estimating the large-order behaviour by means of the Lipatov method, the new perturbation expansion turns out to be convergent for the partition function at non-zero temperature. Although the new perturbation expansion is not convergent for the ground state energy, it nevertheless provides accurate approximations, even in situations which cannot be treated by standard perturbation techniques, like the strong coupling regime.

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1. INTRODUCTION

In 1976, Lipatov¹⁾ proposed a method for estimating the large-order behaviour of the coefficients in the perturbation expansions of Green's functions and other quantities of physical interest. By means of this method one can see immediately that the perturbation expansions in many quantum-field theories and quantum mechanical examples are badly divergent²⁾. For example, the coefficients E_n in the perturbation expansion of the ground-state energy $E(\alpha, \beta)$ of the anharmonic oscillator³⁾ ($\alpha = 1/2$) and the double-well potential⁴⁾ ($\alpha = -1/2$)

$$V(x) = \alpha x^2 + \beta x^4 \quad (1.1)$$

behave like:

$$E_n \sim n! a^n n^b C \quad \text{for } n \rightarrow \infty \quad (1.2)$$

with

$$a = \begin{cases} -3 & \text{for } \alpha = \frac{1}{2} \\ 3 & \text{for } \alpha = -\frac{1}{2} \end{cases} \quad (1.3)$$

$$b = \begin{cases} -\frac{1}{2} & \text{for } \alpha = \frac{1}{2} \\ 0 & \text{for } \alpha = -\frac{1}{2} \end{cases} \quad (1.4)$$

$$C = \begin{cases} -\pi^{-\frac{3}{2}} 6^{\frac{1}{2}} & \text{for } \alpha = \frac{1}{2} \\ -\frac{3}{\pi} & \text{for } \alpha = -\frac{1}{2} \end{cases} \quad (1.5)$$

There exist different methods to reconstruct from such divergent series the ground-state energy itself. For the case of the anharmonic oscillator, Loeffel, Martin, Simon and Wightman⁵⁾ have shown the convergence of the Padé approximation. The Borel summation procedure has been applied by Graffi, Grecchi and Simon⁶⁾. The reason for the bad convergence properties of the perturbation expansions is well known. In the case of the double-well potential [see Eq. (1.1) for $\alpha = -1/2$] it is the tunnelling between the two degenerate minima which is responsible for the divergence of the perturbation expansion. The same tunnelling effect can be seen in the case of the anharmonic oscillator [cf. Eq. (1.1) with $\alpha = 1/2$] if one considers the potential (1.1) for imaginary paths $x = iy_0$ ³⁾. Knowing exactly the origin of the divergencies of the perturbation expansion, one might ask if it

is possible to construct expansions which are convergent or at least less divergent. It has been pointed out already, in a paper by Simon⁷⁾, that the energy eigenvalues $E_n(\alpha, \beta)$ of the anharmonic oscillator (1.1) are analytic in the "harmonic" coupling constant α in a circle around $\alpha = 0$. Therefore an expansion around $\alpha = 0$ is indeed convergent and has the nice property of treating the double well ($\alpha < 0$) and the anharmonic oscillator ($\alpha > 0$) on the same footing -- in contrast to the standard perturbation theory! Moreover, owing to a simple scaling law, first observed by Symanzik (see Ref. 7)

$$E_n(\alpha, \beta) = \beta^{1/3} E_n(\alpha \beta^{-2/3}, 1) \quad (1.6)$$

it is completely sufficient to know the α dependence. According to Eq. (1.6) the expansion around $\alpha = 0$ corresponds to a strong coupling expansion ($\beta \rightarrow \infty$). The only drawback of the expansion around $\alpha = 0$ is that there is no general recipe -- as in standard perturbation theory -- for calculating the coefficients.

It is the purpose of this paper to develop a strategy for finding new perturbation expansions, which have better convergence properties than the standard one. We will derive a convergent perturbative approach for the partition function

$$Z(T, \alpha, \beta) = \sum_n e^{-E_n(\alpha, \beta) \cdot T} \quad (1.7)$$

in the case of the anharmonic oscillator and the double-well potential. The basic idea is the following.

We start with the path cum trace integral for the partition function, where we select out one integration mode^{*)} by fixing the harmonic term $\int_{-T/2}^{T/2} d\tau x(\tau)^2$ by means of a constraint:

$$Z(T, \alpha, 1) = \int_0^\infty dS e^{-\alpha S} G(T, S) \quad (1.8)$$

$$G(T, S) = \int [dx(\tau)] \delta\left(S - \int_{-\frac{T}{2}}^{\frac{T}{2}} d\tau x(\tau)^2\right) \cdot \exp\left[-\int_{-\frac{T}{2}}^{\frac{T}{2}} d\tau \left\{\frac{1}{2} \dot{x}(\tau)^2 + x(\tau)^4\right\}\right] \quad (1.9)$$

*) The method of introducing constraints into the path integral in order to select out collective modes with non-trivial dynamics was discussed by Gervais et al.⁸⁾.

Instead of expanding the exponential of the interaction term:

$$\exp \left[- \int_{-\frac{T}{2}}^{\frac{T}{2}} d\tau x(\tau)^4 \right] = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\int_{-\frac{T}{2}}^{\frac{T}{2}} d\tau x(\tau)^4 \right)^n$$

which leads to the standard perturbation expansion for the anharmonic oscillator, let us consider the following series:

$$\exp \left[- \int_{-\frac{T}{2}}^{\frac{T}{2}} d\tau x(\tau)^4 \right] = \exp [-B(\zeta, T)] \cdot \sum_n \frac{(-1)^n}{n!} \left(\int_{-\frac{T}{2}}^{\frac{T}{2}} d\tau x(\tau)^4 - B(\zeta, T) \right)^n$$

Here, $B(\zeta, T) \geq 0$ is some positive functional of the harmonic term $\zeta = \int_{-T/2}^{T/2} d\tau x(\tau)^2$, which will be specified below, when we study the large-order behaviour of the new perturbation expansion by means of the Lipatov method. To do that, we first consider the quantity

$$\tilde{Z}(T, \alpha, \lambda) = \int [dx(\tau)] \quad (1.10)$$

$$\cdot \exp \left[B(\zeta, T)(1-\lambda) + \int_{-\frac{T}{2}}^{\frac{T}{2}} d\tau \left\{ \frac{1}{2} \dot{x}(\tau)^2 + \alpha x(\tau)^2 + \lambda x(\tau)^4 \right\} \right]$$

which coincides with the partition function (1.8) at $\lambda = 1$. Moreover, $\tilde{Z}(T, \alpha, \lambda)$ is analytic in the new coupling constant λ for

$$0 < \operatorname{Re} \lambda < 1 \quad (1.11)$$

The new perturbation expansion is a power series in λ :

$$\tilde{Z}(T, \alpha, \lambda) = \sum_{n=0}^{\infty} \tilde{Z}_n(T, \alpha) \frac{(-1)^n}{n!} \lambda^n \quad (1.12)$$

The coefficients are defined by

$$\tilde{Z}_n(T, \alpha) = \int [dx(\tau)] \exp[-S_n[x(\tau)]] \quad (1.13)$$

$$S_n[x(\tau)] = B(\zeta, T) + \int_{-\frac{T}{2}}^{\frac{T}{2}} d\tau \left\{ \frac{1}{2} \dot{x}(\tau)^2 + \alpha x(\tau)^2 \right\} - n \log I[x(\tau)] \quad (1.14)$$

$$I[x(\tau)] = \int_{-\frac{T}{2}}^{\frac{T}{2}} d\tau x(\tau)^4 - B(\zeta, T) \quad (1.15)$$

The large-order behaviour ($n \rightarrow \infty$) of the coefficients $\tilde{Z}_n(T, \alpha)$ in the new perturbation expansion (1.12) is given by the solutions $x_c(\tau)$ of the Euler-Lagrange equations with the effective action integral $S_n[x(\tau)]$ (1.14):

$$\ddot{x}_c(\tau) = -V'(x_c(\tau)) \quad (1.16)$$

$$V(x_c(\tau)) = -\tilde{\alpha} x_c(\tau)^2 + \tilde{\beta} x_c(\tau)^4 \quad (1.17)$$

$$\tilde{\alpha} = \alpha + \left(1 + \frac{n}{I}\right) \frac{\partial B}{\partial \zeta} \quad (1.18)$$

$$\tilde{\beta} = \frac{n}{I} \quad (1.19)$$

Notice that the potential (1.17) is exactly the same type as we started with ($\alpha x^2 + x^4$); the only change is in the parameters, $\tilde{\alpha}$, $\tilde{\beta}$.

Next we come to the most essential point of the whole procedure. It is based on the observation that the large n -behaviour of the quantity

$$I[x_c(\tau)] = \int_{-\frac{T}{2}}^{\frac{T}{2}} d\tau x_c(\tau)^4 - B\left(\int_{-\frac{T}{2}}^{\frac{T}{2}} d\tau x_c(\tau)^2, T\right) \quad (1.20)$$

decides whether the new perturbation expansion (1.12) is convergent or not. If

$$|I[x_c(\tau)]| \leq n \cdot \text{const.} \quad \text{for } n \rightarrow \infty \quad (1.21)$$

the series (1.12) has a non-vanishing radius of convergence. If

$$|I[x_c(\tau)]| > O(n) \quad \text{for } n \rightarrow \infty \quad (1.22)$$

the radius of convergence is zero.

Therefore, the crucial question is: Is it possible to arrange the functional

$$B(\zeta, T) \quad , \quad \zeta = \int_{-\frac{T}{2}}^{\frac{T}{2}} dx \, x(\tau)^2$$

in such a way that (1.21) is fulfilled for every solution of the classical equation of motion? The answer is yes. For every well-behaved functional $B(\zeta, T)$ with

$$B(\zeta, T) \sim C(T) \zeta^3 \quad , \quad \text{for } \zeta \rightarrow \infty \quad (1.23)$$

the nearest singularities in the new coupling constant λ [see Eq. (1.12)] are found at

$$\lambda = -R(T)^{-1} \quad \text{and} \quad \lambda = 1 \quad (1.24)$$

where

$$R(T) = \frac{1}{2} \left\{ \left(1 + \frac{2}{3} \frac{1}{C(T)} \right)^{\frac{1}{2}} - 1 \right\} \quad (1.25)$$

The detailed calculations of the singularities (1.24) are given in Sections 2 and 3. Notice that the radius of convergence $R(T)^{-1}$ does not depend on α , the strength of the harmonic term in the potential $\alpha x^2 + x^4$. Therefore, the convergence of the new perturbation series holds also for the double-well potential ($\alpha < 0$). In Section 4 we investigate the question of what one can learn about the ground-state energy from this new perturbation theory. The ground-state energy $E(\alpha, 1)$ is projected out from the partition function in the limit

$$E(\alpha, 1) = \lim_{T \rightarrow \infty} -\frac{1}{T} \log Z(T, \alpha, 1) \quad (1.26)$$

The new perturbation expansion (1.12) implies a new perturbation expansion for the energy levels, if the quantity $\tilde{Z}(T, \alpha, \lambda)$ [cf. Eq. (1.12)] has also a well-defined limit of the type (1.26). However, this enforces us to specify the function $C(T)$ [see Eq. (1.23)] in the following way:

$$C(T) \sim C T^{-2} \quad \text{for } T \rightarrow \infty \quad (1.27)$$

Unfortunately, this choice lets the radius of convergence shrink to zero in the limit $T \rightarrow \infty$

$$R(T)^{-1} \sim \sqrt{6 \cdot C} T^{-1} \quad \text{for } T \rightarrow \infty \quad (1.28)$$

Therefore, we do not expect that the new perturbative approach converges for the energy eigenvalues. Nevertheless, numerical low-order calculations yield very accurate approximations: e.g. a 6% error for the ground-state energy of the anharmonic oscillator ($\alpha x^2 + x^4$) in the strong coupling limit $\alpha = 0$.

2. THE RADIUS OF CONVERGENCE OF THE NEW PERTURBATION EXPANSION

As was pointed out in the Introduction the large-order behaviour of the coefficients $\tilde{Z}_n(T, \alpha, \beta=1)$ [see Eq. (1.13)] in the new perturbation expansion (1.12) is determined by the solutions of the classical equations of motion

$$\ddot{x}_c(\tau) + V'(x_c(\tau)) = 0 \quad (2.1)$$

where

$$V(x_c(\tau)) = -\tilde{\alpha} x_c(\tau)^2 + \tilde{\beta} x_c(\tau)^4 \quad (2.2)$$

Since the parameters $\tilde{\alpha}$ and $\tilde{\beta}$ [see Eqs. (1.18) and (1.19)] can be positive as well as negative, we consider separately in the following the four possibilities, $\tilde{\alpha} > 0 \tilde{\beta} > 0$, $\tilde{\alpha} < 0 \tilde{\beta} < 0$, $\tilde{\alpha} < 0 \tilde{\beta} > 0$, $\tilde{\alpha} > 0 \tilde{\beta} < 0$, which are shown in Fig. 1.

Case A: $\tilde{\alpha} > 0 \tilde{\beta} > 0$

For the discussion of the solutions it is convenient to introduce the scale transformations

$$x_c(\tau) = \left(\frac{\tilde{\alpha}}{\tilde{\beta}} \right)^{\frac{1}{2}} y_c(t), \quad \tau = \tilde{\alpha}^{-\frac{1}{2}} \cdot t \quad (2.3)$$

Then the effective potential is the double well

$$-y^2 + y^4 \quad (2.4)$$

with extrema at $y = 0, \pm 1/\sqrt{2}$. One can easily verify that the three time-independent solutions

$$x_c(\tau) \equiv 0, \quad x_c(\tau) \equiv \pm \left(\frac{\zeta}{T} \right)^{\frac{1}{2}} \quad (2.5)$$

are not relevant for the large-order behaviour ($n \rightarrow \infty$) of the coefficients $\tilde{Z}_n(T, \alpha, \beta=1)$, since the critical quantity I [see Eq. (1.19)] cannot increase like n . Indeed it turns out that the periodic solutions in one of the two potential wells (see Fig. 1, Case A) determine the large-order behaviour:

$$t = \int_{y_c(0)}^{y_c(t)} dy \{ 2(E + y^2 - y^4) \}^{-\frac{1}{2}}, \quad -\frac{1}{4} < E < 0, \quad 0 < y_c < 1 \quad (2.6)$$

The classical energy E is related to the period T via Eq. (2.3)

$$\sqrt{2} \cdot T = I_1^0(E) \quad (2.7)$$

Here we introduce the following abbreviation for the integrals:

$$I_\nu^k(E) = 2^{-\frac{\nu}{2}} \oint dy y^k (E - V(y))^{-\frac{\nu}{2}}, \quad \nu = -1, 1, 3, \dots, \quad k = 0, 2, \dots \quad (2.8)$$

$$V(y) = -y^2 + y^4$$

The contour in the integral on the right-hand side of Eq. (2.8) encircles the two turning points $y = y_\pm(E)$ $\{E - V[y_\pm(E)] = 0\}$ of the classical motion. The large-order behaviour ($n \rightarrow \infty$) of the critical quantity I [see Eq. (1.20)] is given implicitly by the following set of equations, which are obtained from Eqs. (1.18), (1.19), and (2.3):

$$\tilde{\alpha} = T^{-2} \left(I_1^0(E) \right)^2 = \alpha + \left(1 + \frac{n}{I} \right) \frac{\partial B}{\partial \zeta} \quad (2.9)$$

$$\tilde{\beta} = \frac{n}{I} = (T \cdot \zeta)^{-1} I_1^0(E) I_1^2(E) \quad (2.10)$$

$$I = \zeta^2 T^{-1} \left(I_1^2(E) \right)^{-2} I_1^4(E) I_1^0(E) - B(\zeta, T) \quad (2.11)$$

The right-hand side of Eq. (2.11) will diverge only if

$$\zeta \rightarrow \infty \quad \text{for} \quad n \rightarrow \infty \quad (2.12)$$

Notice that $(I_1^2(E))^{-2} I_1^4(E)$ is always finite for $0 \leq E \leq 1/4$, whereas $I_1^0(E)$ only diverges for $E \rightarrow 0$ [see Eq. (2.8)]. On the other hand, owing to Eqs. (2.9) and (2.10):

$$T^{-1} I_1^0(E) = \frac{1}{2\zeta} \frac{\partial B}{\partial \zeta} I_1^2(E) + \left\{ \left(\frac{1}{2\zeta} \frac{\partial B}{\partial \zeta} I_1^2(E) \right)^2 + \alpha + \frac{\partial B}{\partial \zeta} \right\}^{\frac{1}{2}} \quad (2.13)$$

$I_1^0(E)$ can diverge only for $\zeta \rightarrow \infty$, which means

$$E \rightarrow 0 \quad \text{for} \quad \zeta \rightarrow \infty \quad (2.14)$$

The integrals $I_1^2(E)$ and $I_1^4(E)$ remain finite in that limit:

$$I_1^2(E=0) = \sqrt{2} \quad I_1^4(E=0) = \frac{2}{3} \sqrt{2} \quad (2.15)$$

Finally, we get for the large-order behaviour -- i.e. in the limit $n \rightarrow \infty$, $\zeta \rightarrow \infty$, $E \rightarrow 0$ -- for the critical quantity (2.11) and the effective action integral (1.14):

$$\frac{T}{n} = R - \alpha n^{-\frac{2}{3}} \left(\frac{4}{3} \right)^{\frac{2}{3}} R^{\frac{5}{3}} \left(\frac{1+R}{1+2R} \right)^{\frac{4}{3}} \cdot R \quad (2.16)$$

$$S_n = -n \log n + n - n \log R(T) + \alpha n^{\frac{1}{3}} R^{\frac{2}{3}} \left(6 \frac{1+R}{1+2R} \right)^{\frac{1}{3}} \quad (2.17)$$

where

$$R(T) = \frac{1}{2} \left\{ \left(1 + \frac{2}{3} \frac{1}{C(T)} \right)^{\frac{1}{2}} - 1 \right\} \quad (2.18)$$

and $C(T)$ is defined through the asymptotic formula (1.23) for $B(\zeta, T)$.

One readily observes that the saddle-point considered here produces a singularity at $\lambda = -R(T)^{-1}$ in the new coupling constant λ [see Eq. (1.12)]. It turns out that this is indeed the nearest singularity which defines the radius of convergence of the new perturbation expansion as long as $R(T) > 1$. The quantum fluctuations around the classical solution have no influence on the radius of convergence. Their calculation is postponed to Appendix A. The final result for the

contribution of this saddle point to the large order of the coefficients (1.13) in the new perturbation expansion is

$$\tilde{Z}_n(T, \alpha, \beta=1) \sim e^{-S_n} \frac{T}{\sqrt{\pi}} \left(\frac{I_1^0}{T} \right)^{\frac{5}{2}} \left(\frac{2}{3} \frac{1+R}{1+2R} \right)^{\frac{1}{2}} \bar{C} \exp \left[-\frac{1}{2} I_1^0 \right] \quad (2.19)$$

where

$$I_1^0 \sim T n^{\frac{1}{3}} \left(\frac{3}{12} \frac{1}{R} \frac{1+R}{1+2R} \right)^{\frac{1}{3}} \quad (2.20)$$

$$\bar{C} = 4 \cdot 2^{\frac{1}{4}} \quad (2.21)$$

3. SINGULARITIES IN THE NEW COUPLING CONSTANT λ AT $\lambda = 1$

We are going to discuss now the saddle-point contributions for the

Case B: $\tilde{\alpha} < 0$, $\tilde{\beta} < 0$

It turns out, that they lead to singularities in the new coupling constant λ [see Eq. (1.12)] at $\lambda = 1$. For the discussion of the solutions of the classical equations of motion [see Eqs. (2.1) and (2.2)], it is convenient to perform the scale transformation

$$x_c(\tau) = \left(\frac{\tilde{\alpha}}{\beta} \right)^{\frac{1}{2}} y_c(t), \quad \tau = (-\tilde{\alpha})^{-\frac{1}{2}} t \quad (3.1)$$

and the effective potential

$$y^2 - y^4 \quad (3.2)$$

is just the opposite of the double well, discussed as Case A in the previous section (see Fig. 1, Case B). Again we have three time-independent solutions $y \equiv 0, \pm 1/\sqrt{2}$. Of course, the trivial solutions $y \equiv 0$ lets the critical quantity I [see Eq. (1.20)] vanish.

The two other time-independent solutions

$$x_c(\tau) = \pm \left(\frac{\tilde{\alpha}}{\beta} \right)^{\frac{1}{2}} \quad (3.3)$$

lead to the following behaviour of the quantity (1.20)

$$I = \frac{\zeta^2}{T} - B(\zeta, T) \quad (3.4)$$

ζ has to be determined from Eqs. (1.18) and (1.19) for the parameters $\tilde{\alpha}$ and $\tilde{\beta}$:

$$\left(\alpha + \frac{\partial B}{\partial \zeta} \right) \left(\frac{\zeta}{T} - \frac{\partial B}{\partial \zeta} \right)^{-1} \left(\frac{\zeta^2}{T} - B(\zeta, T) \right) = n \quad (3.5)$$

Since $B(\zeta, T)$ is assumed to be well behaved for $\zeta > 0$ -- which means that it is finite for ζ finite -- the critical quantity (3.4) can diverge only if ζ diverges for $n \rightarrow \infty$. According to our assumption (1.23) on the asymptotic behaviour of $B(\zeta, T)$, this means

$$B(\zeta, T) \sim n \quad \text{for } n \rightarrow \infty \quad (3.6)$$

i.e.

$$I \sim -n \quad \text{for } n \rightarrow \infty \quad (3.7)$$

The effective action integral turns out to be in the leading order of n

$$S_n \sim n - n \log n \quad (3.8)$$

and produces a singularity at $\lambda = 1$ in the new coupling constant λ [see Eq. (1.12)].

There are also time-dependent solutions of the equations of motion with the effective potential (3.2) (see Fig. 1, Case B):

$$t = \int_{y_c(t)}^{y_c(t)} \frac{dy}{\sqrt{2}} (E - V(y))^{-\frac{1}{2}}, \quad V(y) = y^2 - y^4 \quad (3.9)$$

$$0 < E < \frac{1}{4} \quad -\frac{1}{\sqrt{2}} < y_c(t) < \frac{1}{\sqrt{2}} \quad (3.10)$$

The discussion of the time-dependent solution is almost the same as in Case A (see Section 1). In particular, here the equations corresponding to Eqs. (2.8)-(2.11) are

$$I_\nu^k(E) = 2^{\frac{\nu}{2}} \oint dy y^k (E - y^2 + y^4)^{-\frac{\nu}{2}}, \quad \nu = 1, 3, \dots; k = 0, 2, \dots \quad (3.11)$$

$$\tilde{\alpha} = -T^{-2} (I_1^0(E))^2 = \alpha + \left(1 + \frac{n}{I}\right) \frac{\partial B}{\partial \zeta} \quad (3.12)$$

$$\tilde{\beta} = \frac{n}{I} = - (T \zeta)^{-1} I_1^0(E) \cdot I_1^2(E) \quad (3.13)$$

$$I = \zeta^2 T^{-1} (I_1^2(E))^{-2} I_1^4(E) I_1^0(E) - B(\zeta, T) \quad (3.14)$$

Again, the critical quantity I [Eq. (3.14)] diverges only if $\zeta \rightarrow \infty$ for $n \rightarrow \infty$. To see this one has to realize that the combination of integrals

$$(I_1^2(E))^{-2} I_1^4(E) I_1^0(E)$$

is finite for all values of E : $0 \leq E \leq 1/4$. On the other hand, we can see from Eqs. (3.12) and (3.13) that

$$\alpha + T^{-2} (I_1^0(E))^2 + \frac{\partial B}{\partial \zeta} = (T \zeta)^{-1} \frac{\partial B}{\partial \zeta} I_1^0(E) I_1^2(E) \quad (3.15)$$

i.e. ζ can diverge only if $I_1^0(E)$ diverges, which is just the case for $E \rightarrow 0$. In this limit, the integrals $I_1^2(E)$ and $I_1^4(E)$ diverge too:

$$I_1^2(E) \sim \frac{1}{2} I_1^0(E) \quad (3.16)$$

$$I_1^4(E) \sim \frac{1}{4} I_1^0(E) \quad (3.17)$$

Finally, we get for the large n -behaviour of the critical quantity I [see Eq. (3.14)] and the effective action integral:

$$I \sim -n \quad (3.18)$$

$$S_n \sim -n \log(-n) + n \quad (3.19)$$

Thus, the saddle point considered here, also produces a singularity at $\lambda = 1$ in the new coupling constant λ [see Eq. (1.12)]. Again it turns out that quantum fluctuations around these saddle points have no influence on the position of this singularity. We are left with

Case C: $\tilde{\alpha} < 0$, $\tilde{\beta} > 0$ and Case D: $\tilde{\alpha} > 0$, $\tilde{\beta} < 0$

(See Fig. 1.) One can easily verify that the critical quantity $|I[x_c(\tau)]|$ [See Eq. (1.20)] cannot increase like n in these cases. Therefore, the classical solutions in cases C and D do not produce singularities in the new coupling constant λ .

4. LOW-ORDER CALCULATIONS OF THE GROUND-STATE ENERGY IN THE NEW PERTURBATION EXPANSION

The coefficients $\tilde{Z}_n(T, \alpha)$ in the new perturbation expansion (1.12) can be calculated from the first n coefficients in the standard perturbation expansion. This can be seen from Eq. (1.10), which is written as:

$$\tilde{Z}(T, \alpha, \lambda) = \frac{1}{2\pi i} \int_0^\infty d\zeta \exp[-\alpha\zeta + B(\zeta, T)(1-\lambda)] \cdot \int_{-i\infty}^{i\infty} d\alpha' e^{\alpha'\zeta} Z(T, \alpha', \lambda) \quad (4.1)$$

Here $Z(T, \alpha', \lambda)$ is the partition function for the anharmonic oscillator

$$\alpha' x^2 + \lambda x^4 \quad (4.2)$$

Let us first consider the zeroth-order approximation, where one has to insert for $Z(T, \alpha', \lambda=0)$ the partition function of the harmonic oscillator

$$Z(T, \alpha', \lambda=0) = \left\{ 2 \cdot \sinh \sqrt{\frac{\alpha'}{2}} T \right\}^{-1} \quad (4.3)$$

and for $B(\zeta, T)$ we take

$$B(\zeta, T) = C(T) \zeta^3 \quad (4.4)$$

The limit $T \rightarrow \infty$, which projects out the ground-state energy from the partition function is of special interest. One easily verifies for the zeroth order of the new perturbation expansion:

$$\lim_{T \rightarrow \infty} -\frac{1}{T} \log \tilde{Z}_0(T, \alpha) = \begin{cases} \infty & \text{for } C(T) > O(T^{-2}) \\ \tilde{E}_0(\alpha) & \text{for } C(T) = C \cdot T^{-2} \\ E_0(\alpha) = \sqrt{\frac{\alpha}{2}} & \text{for } C(T) < O(T^{-2}) \end{cases} \quad (4.5)$$

i.e. for $C(T) < 0$ ($1/T^2$) we are led back to the result of the standard perturbation expansion. On the other hand, we obtain a reasonable new perturbation expansion for the energy eigenvalues only if we choose

$$C(T) \sim C T^{-2} \quad \text{for } T \rightarrow \infty \quad (4.6)$$

In this case, we find for the zeroth order $\tilde{E}_0(\alpha)$ [see Eq. (4.5)] of the ground-state energy in the new perturbation expansion:

$$\tilde{E}_0(\alpha) = 2^{-\frac{1}{6}} 3^{-\frac{1}{2}} \left\{ 1 + \frac{\alpha^2}{3C} \left(-1 + \sqrt{1 + \frac{3C}{\alpha^2}} \right) \right\} \left(\frac{\alpha}{C} \right)^{-\frac{1}{2}} \left(-1 + \sqrt{1 + \frac{3C}{\alpha^2}} \right)^{-\frac{1}{2}} \quad (4.7)$$

Higher orders can be calculated from Eq. (4.1). We insert in the large T limit for the partition function

$$\tilde{Z}(T, \alpha', \lambda) \sim e^{-E(\alpha', \lambda)T} \quad \text{for } T \rightarrow \infty \quad (4.8)$$

Here $E(\alpha', \lambda)$ is the ground-state energy of the anharmonic oscillator (4.2). Then the integrals on the right-hand side of Eq. (4.1) are evaluated in the limit $T \rightarrow \infty$ by means of the saddle-point method. The saddle points in $\zeta' = \zeta \cdot T$ and α' are found at the minima of

$$(\alpha - \alpha')\zeta' + C\zeta'^3(1-\lambda) + E(\alpha', \lambda) \quad (4.9)$$

i.e. for

$$\zeta' = \frac{\partial}{\partial \alpha'} E(\alpha', \lambda) \quad (4.10)$$

$$\alpha' - \alpha = 3C(1-\lambda) \left(\frac{\partial}{\partial \alpha'} E(\alpha', \lambda) \right)^2 \quad (4.11)$$

Equation (4.11) is an implicit equation for

$$\alpha' = \alpha'(\alpha, \lambda, C) \quad (4.12)$$

which has to be solved in order to calculate the saddle-point contribution:

$$\begin{aligned} \tilde{E}(\alpha, \lambda) &= \lim_{T \rightarrow \infty} -\frac{1}{T} \log \tilde{Z}(T, \alpha, \lambda) \\ &= E(\alpha', \lambda) - 2C(1-\lambda) \left(\frac{\partial}{\partial \alpha'} E(\alpha', \lambda) \right)^3 \end{aligned} \quad (4.13)$$

For the practical calculation of the N-th order

$$\tilde{E}_N(\alpha) = \frac{1}{N!} \frac{\partial^N}{\partial \lambda^N} \tilde{E}(\alpha, \lambda) \Big|_{\lambda=0} \quad (4.14)$$

one first inserts the standard perturbation expansion

$$E(\alpha, \lambda) = \sum_{n=0}^N E_n(\alpha) \lambda^n \quad (4.15)$$

into Eq. (4.11) and solves for

$$\alpha' = \alpha'(\alpha, \lambda, C) = \sum_{n=0}^N \alpha'_n(\alpha, C) \lambda^n \quad (4.16)$$

The N-th order (4.14) then follows from Eqs. (4.13), (4.15) and (4.16).

There are two open questions concerning this new perturbation expansion for the energy eigenvalues:

- 1) What about the convergence of the series? We have seen in Section 2 that the radius of convergence for finite values of T was given by $R^{-1}(T)$ [see Eq. (1.25)]. If we choose $C(T)$ according to Eq. (4.6) -- in order to get a well-defined perturbation series for the energy eigenvalues -- the radius of convergence shrinks to zero for $T \rightarrow \infty$:

$$R^{-1}(T) \sim \sqrt{6C} \quad T^{-1} \quad \text{for } T \rightarrow \infty \quad (4.17)$$

Therefore, the new perturbation expansion for the energy eigenvalues is not expected to be convergent.

- 2) What is the meaning of the yet unspecified parameter C in our ansatz for the functional

$$B(\zeta, T) = C \cdot T^{-2} \zeta^3, \quad \zeta = \int_{-\frac{T}{2}}^{\frac{T}{2}} d\tau \times \langle \mathbf{r} \rangle^2 \quad (4.18)$$

which is subtracted from the interaction term [see Eq. (1.9)]? For finite values of T an increase of C leads to an increase of the radius of convergence [see, for example, Eq. (4.17)]. Our computer calculations indicate that an increase of C also improves the convergence of the perturbation expansion for the ground-state energy in the following sense: the usefulness of an asymptotic expansion

$$\tilde{E}(\alpha, \lambda) = \sum_n \tilde{E}_n(\alpha) \lambda^n \quad (4.19)$$

depends on the order $n = N$, for which the coefficients $\tilde{E}_n(\alpha)$ start to increase.

In order to get a reasonable approximation one has to cut off the asymptotic expansion for $n = N$. If N is small one only gets a poor approximation. Our computer calculations show that the maximal order N which can be taken into account, increases with C . An example is given in Table 1, where we have calculated the first 10 orders for the ground-state energy of the pure anharmonic oscillator^{*)} $[V(x) = x^4, \alpha = 0]$.

For the lowest value of C ($C = 6.75$) one has to cut off the asymptotic series already for $N = 3$. But for increasing values of C , the higher orders become smaller and smaller, providing thus a more and more accurate approximation for the ground-state energy. For example, for $C = 18.12$ one can sum up the first 10 orders, which approximate the exact value (see the last line in Table 1) up to an error of 6%. The moral of this example is: the proper choice of C depends on the highest order N , which one intends to take into account. If one is interested only in the first two orders ($N = 1$), there is a unique way to determine the optimal value of C , since the first two orders always yield an upper bound of the ground-state energy (see Appendix B)

$$E(\alpha, \beta) \leq \tilde{E}_0(\alpha) + \tilde{E}_1(\alpha) \quad (4.20)$$

Obviously, the best value of C minimizes the right-hand side of the inequality (4.20). The resulting bound is shown in Fig. 2 and compared with the known exact values for the ground-state energy. For $\alpha = 0$, the upper bound only deviates 2.2% from the exact value⁷⁾. For $\alpha > 0$ this deviation decreases monotonically to zero.

*) We would like to emphasize that this case cannot be treated with the methods of standard perturbation theory, since $\alpha \rightarrow 0$ corresponds to the strong coupling limit. This limit has been considered also by Parisi⁶⁾ and Bonnier⁹⁾.

APPENDIX A

CALCULATION OF QUANTUM FLUCTUATIONS

The quantum fluctuations around the saddle point discussed in Section 2 have no influence on the radius of convergence of the new perturbation expansion. Their calculation has therefore been postponed to this Appendix. For any classical solution the operator of the second derivatives of the effective action integral [see Eqs. (1.14), (1.15)] can be split into a "diagonal" and a "projecting" part:

$$\frac{\delta^2 S}{\delta x(t_1) \delta x(t_2)} = (M_D + M_P) \approx \frac{3}{2} \quad (A.1)$$

where

$$M_D = \left\{ -\frac{d^2}{dt^2} + 2 - 12 y_c(t)^2 \right\} \delta(t_1 - t_2) \quad (A.2)$$

and

$$M_P = u_1(t_1) u_1(t_2) + u_2(t_1) u_2(t_2) \quad (A.3)$$

$$u_1(t) = a y_c(t) \quad (A.4)$$

$$u_2(t) = b (y_c(t) + c y_c(t)^3) \quad (A.5)$$

$$a = 2^{\frac{5}{4}} \quad (A.6)$$

$$b^2 = \frac{4}{\sqrt{2}} \cdot 3 \cdot (1+2R)^{-1} (1+R)^{-1} \quad (A.7)$$

$$c = -2(1+R) \quad (A.8)$$

In calculating Eqs. (A.1) to (A.8), we have already performed the scale transformation (2.3), where the parameters $\tilde{\alpha}$ and $\tilde{\beta}$ were determined from Eqs. (2.9), (2.10), (2.11), and (2.16). $R(T)^{-1}$ is the radius of convergence [see Eq. (2.18)]. The calculation of the determinant of the diagonal operator (A.2) -- where the zero frequency mode has been extracted correctly -- is quite straightforward¹⁰⁾

$$\det M_D = \pi I_3^0(E) = \pi \cdot 2^{-\frac{3}{2}} \oint dy (E + y^2 - y^4)^{-\frac{3}{2}} \quad (A.9)$$

Of course, we need here the determinant of the total operator $M_D + M_P$. Since the projecting part is a sum of two projection operators on the states $|u_1\rangle$ and $|u_2\rangle$ [see Eqs. (A.3)-(A.5)], which are orthogonal to the zero-frequency mode $\dot{y}_c(t)$ of the diagonal operator, we can make use of the following identity:

$$\begin{aligned} \det (M_D + |u_1\rangle\langle u_1| + |u_2\rangle\langle u_2|) &= \\ &= \det M_D \cdot (1 + \langle u_1 | M_D^{-1} | u_1 \rangle) (1 + \langle u_2 | (M_D + |u_1\rangle\langle u_1|)^{-1} | u_2 \rangle) \end{aligned} \quad (A.10)$$

We are left with the calculation of the matrix elements, which appear on the right-hand side of Eq. (A.10). The first one is given by

$$\langle u_1 | M_D^{-1} | u_1 \rangle = \langle u_1 | f \rangle = \int_{-\frac{\tilde{T}}{2}}^{\frac{\tilde{T}}{2}} dt \, u_1(t) f(t) \quad (A.11)$$

where $\tilde{T} = \sqrt{a} T$ and $f(t)$ is a periodic solution of the differential equation:

$$\left(-\frac{d^2}{dt^2} + 2 - 12 y_c^2(t) \right) f(t) = u_1 = a y_c(t) \quad (A.12)$$

The solution f can be found by the ansatz:

$$f(t) = \dot{y}_c(t) F(y_c) \quad (A.13)$$

$$F'(y_c) = 2^{-\frac{3}{2}} (E + y_c^2 - y_c^4)^{\frac{3}{2}} \left(c - \int_0^{y_c} dy' \, u_1(y') \right) \quad (A.14)$$

The constant c has to be determined from the periodicity condition

$$f\left(-\frac{\tilde{T}}{2}\right) = f\left(\frac{\tilde{T}}{2}\right), \quad \text{i.e.} \quad \oint dy \, F'(y) = 0 \quad (A.15)$$

which yields

$$c = \frac{1}{2} \frac{I_3^2}{I_3^0} \quad (A.16)$$

Finally, we get for the matrix element (A.11), with $u_1 = a y_c$,

$$\langle u_1 | M_D^{-1} | u_1 \rangle = \frac{a^2}{4} \Delta_3^4 \quad (A.17)$$

with

$$\Delta_3^4 = I_3^4 - \frac{(I_3^2)^2}{I_3^0} \quad (\text{A.18})$$

The integrals I_v^k are defined in Eq. (2.8). In order to calculate the second matrix element in Eq. (A.10) we have to find a periodic solution of the differential equation:

$$\left(-\frac{d^2}{dt^2} + 2 - 12 y_c(t)^2\right) f(t) = u_2 - \langle u_1 | f \rangle u_1 \quad (\text{A.19})$$

which can be found again by an ansatz of the type (A.13) with

$$F'(y) = 2^{-\frac{3}{2}} (E + y^2 - y^4)^{-\frac{3}{2}} \left\{ C - \int_0^y dy' (u_2 - \langle u_1 | f \rangle u_1) \right\} \quad (\text{A.20})$$

C has to be determined again from the periodicity condition $\oint dy F'(y) = 0$. After some lengthy, but straightforward, calculation one obtains for the second matrix element in Eq. (A.10):

$$\begin{aligned} \langle u_2 | (u_D + |u_1\rangle \langle u_1|)^{-1} | u_2 \rangle = \\ = \left(\frac{b}{2}\right)^2 \left\{ \Delta_3^4 + C \Delta_3^6 + \frac{C^2}{4} \Delta_3^8 \right\} - \left(\frac{ab}{4}\right)^2 \left(\Delta_3^4 + \frac{C}{2} \Delta_3^6\right)^2 \left(1 + \frac{a^2}{4} \Delta_3^4\right)^{-1} \end{aligned} \quad (\text{A.21})$$

where Δ_3^4 is defined in Eq. (A.18) and

$$\Delta_3^6 = I_3^6 - \frac{I_3^4 \cdot I_3^2}{I_3^0}, \quad \Delta_3^8 = I_3^8 - \frac{(I_3^4)^2}{I_3^0} \quad (\text{A.22})$$

In the limit $n \rightarrow \infty$ the integrals I_3^k yield:

$$I_3^0 \sim -\bar{C}^{-2} e^{\sqrt{2} I_1^0} \quad (\text{A.23})$$

$$I_1^0 \sim T n^{\frac{1}{3}} \left(\frac{3}{12} \frac{1}{R} \frac{1+R}{1+2R} \right)^{\frac{1}{3}} \quad (\text{A.24})$$

$$\bar{C} = 4 \cdot 2^{\frac{1}{4}} \quad (\text{A.25})$$

$$\Delta_3^4 \sim I_3^4 \sim -\frac{1}{\sqrt{2}} \quad (\text{A.26})$$

$$\Delta_3^6 \sim I_3^6 \sim -\sqrt{2} \quad (\text{A.27})$$

$$\Delta_3^8 \sim I_3^8 \sim -\frac{4}{3}\sqrt{2} \quad (\text{A.28})$$

The resulting formula for the quantum corrections as shown in Eq. (2.19) contains three factors:

- 1) The contribution of the zero-frequency mode $T(I_1^0/T)^{5/2}$;
- 2) The determinant of the diagonal operator M_D (to the power $-1/2$)

$$\det M_D = \pi I_3^0 \sim -\pi \bar{c}^{-2} e^{\sqrt{2} I_1^0} \quad (\text{A.29})$$

- 3) The corrections of the determinant, due to the projecting part (to the power $-1/2$)

$$\begin{aligned} & \left(1 + \langle u_1 | M_D^{-1} | u_1 \rangle\right) \left(1 + \langle u_2 | (M_D + |u_1\rangle\langle u_1|)^{-1} | u_2 \rangle\right) = \\ & = -\frac{3}{2} \frac{1+2R}{1+R} \end{aligned} \quad (\text{A.30})$$

Finally we want to mention that higher order derivatives of the action integral, e.g. $\delta^3 S / \delta x(\tau_1) \delta x(\tau_2) \delta x(\tau_3)$, which are neglected in the saddle-point approximation, yield corrections of the order $O(n^{-1/2})$. This is true also in the limit $T \rightarrow \infty$.

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APPENDIX B

AN UPPER BOUND FOR THE GROUND-STATE ENERGY

The proof of the bound (4.20) on the ground-state energy is based on the convexity of the exponential function in the path integral for the partition function¹¹⁾:

$$\frac{Z(T, \alpha, 1)}{\tilde{Z}_0(T, \alpha)} = \left\langle \exp - \left(\int_{-\frac{T}{2}}^{\frac{T}{2}} d\tau x(\tau)^4 - B(\zeta, T) \right) \right\rangle \geq \exp - \left\langle \int_{-\frac{T}{2}}^{\frac{T}{2}} d\tau x(\tau)^4 - B(\zeta, T) \right\rangle \quad (B.1)$$

Here $\tilde{Z}_0(T, \alpha) = \tilde{Z}(T, \alpha, \lambda=0)$ [cf. Eq. (1.13)] is the zeroth order of the new perturbation expansion (1.12)

$$\tilde{Z}_0(T, \alpha) = \int [dx(\tau)] \exp [-S_0[x(\tau)]] \quad (B.2)$$

$$\sim \exp[-\tilde{E}_0(\alpha)T] \quad \text{for } T \rightarrow \infty$$

$$S_0[x(\tau)] = B(\zeta, T) + \int_{-\frac{T}{2}}^{\frac{T}{2}} d\tau \left\{ \frac{1}{2} \dot{x}(\tau)^2 + \alpha x(\tau)^2 \right\} \quad (B.3)$$

The mean value $\langle f[x(\tau)] \rangle$ is defined by

$$\langle f[x(\tau)] \rangle = \tilde{Z}_0(T, \alpha)^{-1} \int [dx(\tau)] f[x(\tau)] \exp[-S_0[x(\tau)]] \quad (B.4)$$

The bound (4.20) follows from (B.1) in the limit $T \rightarrow \infty$

$$\begin{aligned} E(\alpha, 1) &= \lim_{T \rightarrow \infty} -\frac{1}{T} \log Z(T, \alpha) \\ &\leq \lim_{T \rightarrow \infty} \left\{ -\frac{1}{T} \log \tilde{Z}_0(T, \alpha) + \left\langle \int_{-\frac{T}{2}}^{\frac{T}{2}} d\tau x(\tau)^4 - B(\zeta, T) \right\rangle \right\} \end{aligned} \quad (B.5)$$

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Table 1

Calculation of the ground-state energy of the anharmonic oscillator ($\alpha x^2 + x^4$) in the strong coupling régime ($\alpha = 0$) by means of the new perturbation expansion. The exact value is taken from Ref. 7.

n	C = 6.75	C = 11.5	C = 15.5	C = 18.1
0	0.6300	0.724	0.7754	0.8063
1	5.25×10^{-2}	-1.922×10^{-2}	5.521×10^{-2}	-7.339×10^{-2}
2	-1.63×10^{-2}	-1.903×10^{-2}	-2.541×10^{-2}	-2.962×10^{-2}
3	8.00×10^{-3}	-4.721×10^{-3}	-1.001×10^{-2}	-1.291×10^{-2}
4	-1.38×10^{-2}	-4.982×10^{-3}	-6.271×10^{-3}	-7.604×10^{-3}
5	2.52×10^{-2}	$+3.071 \times 10^{-4}$	-2.944×10^{-3}	-4.274×10^{-3}
6	-6.26×10^{-2}	-4.507×10^{-3}	-2.771×10^{-3}	-3.143×10^{-3}
7	1.83×10^{-1}	5.958×10^{-3}	-5.001×10^{-4}	-1.657×10^{-3}
8	-6.24×10^{-1}	-1.521×10^{-2}	-2.611×10^{-3}	-1.870×10^{-3}
9	2.41	3.707×10^{-2}	2.607×10^{-3}	-1.232×10^{-3}
N	3	5	7	9
$\sum_{n=0}^N E_n$	0.674	0.6723	0.6722	0.6717
E_{exact}	0.6678			

Figure captions

- Fig. 1 : The effective potentials [see Eq. (2.2)] for the four cases:
A: $\tilde{\alpha} > 0, \tilde{\beta} > 0$; B: $\tilde{\alpha} < 0, \tilde{\beta} < 0$; C: $\tilde{\alpha} < 0, \tilde{\beta} > 0$; D: $\tilde{\alpha} < 0, \tilde{\beta} < 0$.
- Fig. 2 : The bound for the ground-state energy [see Eq. (4.20)] compared with the exact values (solid line) of Ref. 7.

case A : $\tilde{\alpha} > 0, \tilde{\beta} > 0$

case B : $\tilde{\alpha} < 0, \tilde{\beta} < 0$

case C : $\tilde{\alpha} < 0, \tilde{\beta} > 0$

case D : $\tilde{\alpha} > 0, \tilde{\beta} < 0$

Fig. 1

Fig. 2