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EXPLICIT SOLUTIONS OF THE KORTEWEG-de VRIES
EQUATION WITHOUT SCATTERING DATA

F. Lambert

CERN -- Geneva

ABSTRACT

We consider the problem of constructing solutions of the Korteweg-de Vries equation with the help of the inverse scattering theory for the Schrödinger equation on the line. Using a scheme proposed by Zakharov and Shabat we show how solutions of the full equation are related to solutions of the linearized equation. In this way we obtain solutions which are not recovered by the standard I.S.T. method. Explicit solutions, other than the multisolitons are derived without referring to a set of scattering data: they decrease exponentially at both ends of the line, at the cost of a singularity whose location varies with time.

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1. INTRODUCTION

The discovery, about ten years ago, of a remarkable connection between inverse problems of scattering and the solution of some important non-linear partial differential equations has produced an unexpected revival of interest in the inverse problem for the one-dimensional Schrödinger Hamiltonian^{1),2)}:

$$H = -\partial_x^2 + q(x), \quad -\infty < x < +\infty. \quad (1)$$

The simplest non-linear equation in two dimensions that can be associated with this scattering system is the Korteweg-de Vries (KdV) equation for the evolving potential $q(x,t)$:

$$q_t = \frac{1}{4} q_{xxx} - \frac{3}{2} q q_x \quad (2)$$

It is the first non-trivial equation which is obtained under the form of Lax's commutator equation $[B, H] = H_t$, by considering the isospectral deformations of H that are generated³⁾ by a linear differential operator B , and one has:

$$B = \partial_x^3 - \frac{3}{2} q \partial_x - \frac{3}{4} q_x. \quad (3)$$

Such isospectral deformations related to a varying regular potential are naturally characterized by their effect on the corresponding "scattering data". It turns out that potentials which evolve according to the KdV equation within the class of regular and rapidly decreasing functions have scattering data whose evolution is particularly simple. This result has led to the so-called I.S.T. (inverse scattering/spectral transform) method of solving the initial value problem for the KdV equation, provided that the initial value $q(x,0)$ is sufficiently well behaved⁴⁾ to produce suitable scattering data, by following these data in time and by recovering the potential $q(x,t)$ by inverse reconstruction.

This method, which was studied by many authors^{5),6)} who competed in deriving the most general I.S.T. solvable equations, can be understood as a non-linear generalization of the method of Fourier transforms for linear equations. The existence of I.S.T. solutions to the KdV equation can be proved under some restrictive set of initial conditions^{4),7)}. Explicit solutions are only obtained under very particular circumstances: they are the multi-soliton solutions which describe the interaction of a finite number of stable modes (solitons) and which correspond to the initial value of a reflectionless potential. In fact, it appeared⁸⁾ that these reflectionless potentials play a very special role for the

Schrödinger operator: they can be generated "out of zero" by repeated application of the Crum transformation⁹⁾ to the free Hamiltonian

$$H_0 = -\partial_x^2.$$

It is natural to see whether there exist further classes of explicit solutions of the KdV equation on the full line, which can be constructed with inverse methods, but which are not obtainable through application of the I.S.T. scheme. We show that such solutions, which decrease exponentially at both ends of the line do exist, that they can be obtained in closed forms analogous to the multi-solitons, but that, unlike the multi-solitons, they present singularities which prevent them from being interpreted in the framework of the standard scattering theory.

Our starting point is the algorithm of Zakharov and Shabat¹⁰⁾ for the construction of non-linear equations that can be brought under the Lax-form. Their approach constitutes an interesting alternative to the I.S.T. ideas. It first recognizes the fundamental role which is played by the Faddeev-Marchenko integral equation in the inverse reconstruction process: as it produces the causal transformation kernel which solves the inverse problem it provides us with a solution of a non-linear wave equation: Chudov's equation¹¹⁾. This solution is actually determined by a particular integral of the linearized wave equation (d'Alembert's equation). The remarkable Fredholm property of the Faddeev-Marchenko equation which allows in this way the construction of Volterra transformation operators which transform H_0 into a perturbed form H , holds also for a more general class of Wiener-Hopf integral equations¹²⁾. Furthermore, Zakharov and Shabat show that this property enables us to find Volterra transformation kernels in terms of which general higher order "free" differential operators L_0 with constant coefficients are transformed into a perturbed version L in which almost all coefficients are variable. By considering two such free and obviously commuting differential operators which transform into perturbed versions by the action of the same Volterra transformation they get a non-linear equation in commutator form which expresses the simple fact that the transformed differential operators should still commute. Solutions of this non-linear equation, which are found to correspond to solutions of a coupled system of non-linear partial differential equations, can then be obtained from integrals of the linearized system, after solution of an integral equation of the Wiener-Hopf type. This algebraic approach, which allows the construction of explicit solutions of non-linear equations of the Lax-form without referring to a scattering problem, has also found

generalizations to further classes of non-linear evolution equations¹³⁾. To be concrete we shall illustrate the Zakharov-Shabat procedure on a case which will lead us to the KdV equation.

Let us, however, first report the basic property of the Wiener-Hopf integral equations in connection with particular non-linear partial differential equations.

2. VOLTERRA TRANSFORMATION OPERATORS

Consider the kernel $F(x,y)$ of an integral operator F acting on functions ψ of the variable x , $-\infty < x < \infty$:

$$F\psi = \int_{-\infty}^{+\infty} dy F(x,y) \psi(y) \quad (1)$$

which is $n > 2$ times differentiable in x and y , and such that:

$$\sup_{x > a} \int_a^{\infty} dy |F(x,y)| < \infty, \quad \forall a > -\infty. \quad (2)$$

We consider the problem of representing the associated operator $I + F$ in the following factorized form:

$$I + F = (I + K_+)^{-1} (I + K_-) \quad (3)$$

where $K_{\pm}$ represent Volterra operators with $\pm$ triangular kernels:

$$\begin{aligned} K_+(x,y) &= 0 & x > y & \quad (\text{causal}) \\ K_-(x,y) &= 0 & x < y & \quad (\text{anticausal}) \end{aligned} \quad (4)$$

For such a factorization to hold we see that $K_+(x,y)$ must satisfy the Wiener-Hopf integral equation:

$$K_+(x,y) + F(x,y) + \int_x^{\infty} dz K_+(x,z) F(z,y) = 0, \quad x < y. \quad (5)$$

The other kernel is then given by

$$K_-(x,y) = F(x,y) + \int_x^{\infty} dz K_+(x,z) F(z,y), \quad x > y. \quad (6)$$

Let us assume that the operator $I + F$ can be factorized in this manner: let the integral equation (5) be uniquely solvable for all fixed values of x . Let furthermore $F(x,y)$ be a solution of the free wave equation:

$$\frac{\partial^2 F(x,y)}{\partial x^2} - \frac{\partial^2 F(x,y)}{\partial y^2} = 0 \quad (7)$$

which is Hilbert-Schmidt on $[x, \infty)$. It then follows from the Fredholm alternative that the corresponding causal kernel $K_+(x,y)$, solution of Eq. (5), satisfies the non-linear wave equation (Chudov's equation) (appendix 1):

$$\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} \right) K_+(x,y) = -2 \frac{dK_+(x,x)}{dx} \cdot K_+(x,y) \quad (8)$$

Such a situation is actually met in the theory of inverse scattering, where the integral equation (5) takes the form of the Faddeev-Marchenko equation, in which $F(x,y) = F(x+y)$ is a given integral of d'Alembert's equation with Fourier representation in terms of the scattering data. The corresponding solution $K_+(x,y)$ of Eq. (8) is the transformation kernel which expresses the Jost solution $f_+(x,k)$ in terms of the free solution $\exp(ikx)$ ¹⁴⁾. It is important to remark that the associated operator $U = I + K_+$ is a transformation operator¹⁵⁾ which transforms the free Hamiltonian H_0 into the perturbed one:

$$H = H_0 - 2 \frac{dK_+(x,x)}{dx} \quad (9)$$

This means that the operator U satisfies the operator equation:

$$H U = U H_0 \quad (10)$$

Going the other way round one may take this operator equation as a starting point and ask for the circumstances under which H_0 is transformed into $H = H_0 + q(x)$ by the action of a causal Volterra operator $U = I + K_+$. The answer is that $K_+(x,y)$ should satisfy Eq. (8) and that $q(x)$ should satisfy the compatibility condition:

$$q(x) = -2 \frac{dK_+(x,x)}{dx} \quad (11)$$

When $q(x)$ is an a priori given regular potential, this brings us back to the well-known Goursat problem for solving the forward scattering problem¹⁴⁾.

Yet, the problem of finding Volterra transformation operators can be generalized to the case of higher order differential operators than a Schrödinger Hamiltonian. Such a generalization is introduced with the method of Zakharov and Shabat.

3. THE METHOD OF ZAKHAROV AND SHABAT

Let us now look at the circumstances under which the free third order operator

$$B_0 = \partial_x^3 \quad (12)$$

transforms into a perturbed operator of the form:

$$B = \partial_x^3 + u_0(x) \partial_x^2 + u_1(x) \partial_x + u_2(x) \quad (13)$$

by the action of a causal Volterra transformation operator: $V = I + W$. Considering the operator equation:

$$B(I + W) = (I + W)B_0 \quad (14)$$

and equating the differential parts (including that of order zero) in both sides one gets a set of consistency conditions which fix the coefficients $u_i(x)$, $i = 0, 1, 2$ in Eq. (13) in terms of the kernel $W(x, y)$:

$$\begin{aligned} u_0(x) &= 0 \\ u_1(x) &= 3 \frac{dW(x, x)}{dx} \\ u_2(x) &= \frac{3}{2} \frac{d^2W(x, x)}{dx^2} + W(x), \\ W(x) &= \frac{3}{2} \frac{d}{dx} \left[\frac{\partial W(x, y)}{\partial x} \Big|_{x=y} + W^2(x, x) - \frac{\partial W(x, y)}{\partial y} \Big|_{x=y} \right] \end{aligned} \quad (15)$$

Subject to these conditions, the operator equation (14) reduces to the following partial differential equation for $W(x, y)$:

$$\begin{aligned} \frac{\partial^3 W(x, y)}{\partial x^3} + \frac{\partial^3 W(x, y)}{\partial y^3} + 3 \frac{dW(x, x)}{dx} \frac{\partial W(x, y)}{\partial x} + \frac{3}{2} \frac{d^2W(x, x)}{dx^2} W(x, y) \\ + W(x) W(x, y) = 0 \end{aligned} \quad (16)$$

Solutions of this third order analogue of the Chudov equation can again be obtained from well-behaved integrals of the linearized equation:

$$\frac{\partial^3 W(x, y)}{\partial x^3} + \frac{\partial^3 W(x, y)}{\partial y^3} = 0, \quad (17)$$

by solving the Wiener-Hopf integral equation (6). This result, which generalizes the above result for the Chudov equation, follows from the factorization property Eq. (3) of an operator $I + F$ into Volterra operators. It is an immediate consequence of Zakharov-Shabat's theorem¹⁰⁾.

As a next step we ask for the conditions under which the same Volterra transformation operator $V = I + W$ will transform the pair of free (and also commuting) differential operators $H_0 = -\partial_x^2$ and $B_0 = \partial_x^3$ respectively into the prescribed forms $H = H_0 + q(x)$ and B as given by expression (13). We thus consider the system of operator equations:

$$\begin{aligned} H(I+W) &= (I+W)H_0 \\ B(I+W) &= (I+W)B_0 \end{aligned} \quad (18)$$

Subject to the consistency conditions (15) and the condition:

$$q(x) = -2 \frac{dW(x, x)}{dx}, \quad (19)$$

this system reduces to the following system of partial differential equations:

$$\left\{ \begin{aligned} W_{xx} - W_{yy} + 2 \frac{dW(x, x)}{dx} W &= 0 \\ W_{xxx} + W_{yyy} + 3 \frac{dW(x, x)}{dx} W_x + \frac{3}{2} \frac{d^2 W(x, x)}{dx^2} W + w(x) W &= 0 \end{aligned} \right. \quad (20)$$

One can easily check that $w(x)$ vanishes identically as W satisfies the first equation. We are thus authorized to drop the $w(x)$ term in the second equation. It also follows that the resulting Volterra transformations will transform H_0 and B_0 into perturbed forms, which, according to conditions (15) and (19), are expressible in terms of $q(x)$ and its first derivative:

$$\begin{aligned} \mathcal{H}_0 & \longrightarrow \boxed{I+W} \longrightarrow \mathcal{H} = \mathcal{H}_0 + q(x) \\ \mathcal{B}_0 & \longrightarrow \boxed{I+W} \longrightarrow \mathcal{B} = \mathcal{B}_0 - \frac{3}{2} q \partial_x - \frac{3}{4} q_x \end{aligned} \quad (21)$$

We remark that the operator $\mathcal{B}$ is nothing else than the Lax operator B for which the Lax equation reduces to the KdV equation.

However, the main result is now that for any integral $F(x,y)$ of the linearized system:

$$\begin{aligned} F_{xx} - F_{yy} &= 0 \\ F_{xxx} + F_{yyy} &= 0 \end{aligned} \quad (22)$$

which is such^{*)} that the integral equation:

$$W(x,y) + F(x,y) + \int_x^\infty dz W(x,z) F(z,y) = 0, \quad (23)$$

has a unique solution $W(x,y)$, this solution is also a solution of the full system (20), leading thus to a required Volterra transformation $V = I + W$. Since for every such solution one has:

$$[\mathcal{B}, \mathcal{H}] = V [\mathcal{B}_0, \mathcal{H}_0] V^{-1} = 0 \quad (24)$$

it is clear that $q(x) = -2[dW(x,x)/dx]$ will solve the "time-independent" KdV equation:

$$[\mathcal{B}, \mathcal{H}] = \frac{1}{4} q_{xxx} - \frac{3}{2} q q_x = 0 \quad (25)$$

*) In order to satisfy the condition (2) one should restrict oneself to $F(x,y) = F(x+y)$. The function F should be sufficiently many times differentiable so as to ensure the existence of all the partial derivatives of $W(x,y)$ that are written. For solvability of the integral equation it suffices to require the condition:

$$\int_a^\infty dx (1+|x|) |F'(x)| < \infty, \quad \forall a > -\infty.$$

The final step consists of introducing the time variable. This is done in a rather trivial way. It suffices to repeat the above reasoning, starting with some integral operator F with kernel $F(x,y;t)$, acting on functions $\psi(x,t)$. Considering the pair of operators H_0 and $B_0 - \partial_t$ which, by the same Volterra transformation, are transformed into the operators $H = H_0 + q(x,t)$ and $B = \partial_x^3 - 3/2 q(x,t)\partial_x - 3/4 q_x(x,t) - \partial_t$, one is led to the following system for $W(x,y;t)$ ¹⁰⁾:

$$\begin{cases} W_{xx} - W_{yy} + 2 \frac{dW(x,x;t)}{dx} W = 0 \\ W_t - W_{xxx} - W_{yyy} - 3 \frac{dW(x,x;t)}{dx} W_x + \frac{3}{2} \frac{d^2 W(x,x;t)}{dx^2} W = 0 \end{cases} \quad (26)$$

Solutions of this system are again obtained by solving the Wiener-Hopf equation with an appropriate input kernel $F(x,y;t)$ which solves the linear system:

$$\begin{cases} F_{xx} - F_{yy} = 0 \\ F_t - F_{xxx} - F_{yyy} = 0 \end{cases} \quad (27)$$

To the solutions $W(x,y;t)$ there correspond solutions $q(x,t) = -2[dW(x,x;t)/dx]$ of the KdV equation:

$$[B - \partial_t, H] = q_t - \frac{1}{4} q_{xxx} + \frac{3}{2} q q_x = 0. \quad (28)$$

4. EXPLICIT SOLUTIONS OF THE KdV EQUATION

The linear system (27) is a homogeneous system with constant coefficients and we expect solutions under the form of a Fourier integral over some bounded measure. By introducing the characteristic coordinates $\{u = x + y, v = x - y\}$ we are led to solve the system:

$$\begin{cases} \frac{\partial^2 F}{\partial u \partial v} = 0 \\ \frac{\partial F}{\partial t} - 2 \frac{\partial^3 F}{\partial u^3} = 0 \end{cases} \quad (29)$$

subject to the conditions:

- i) condition (2) which should hold for fixed values of $t > 0$: this condition excludes from the general solution of (29a): $F(u,v;t) = \psi(u,t) + \phi(v,t)$, the presence of a ϕ -term.

ii) $F(u, t) = \psi(u, t)$ should satisfy the condition : 7)

$$\int_a^\infty du (1+|u|) / F_u(u, t) < \infty, \quad \forall a > -\infty, \quad t > 0. \quad (30)$$

which guarantees the solvability of the Faddeev-Marchenko equation.

We thus consider solutions of the form $F(u, t) = \psi(u, t)$, where $\psi(u, t)$ is a well-behaved solution of the basic equation:

$$\frac{\partial \psi(u, t)}{\partial t} - 2 \frac{\partial^3 \psi(u, t)}{\partial u^3} = 0. \quad (31)$$

This equation is actually very close to the linearized Korteweg-de Vries equation. Indeed according to the relation: $q(x, t) = -2[dW(x, x; t)/dx]$, we may write:

$$W_t - \frac{1}{4} W_{xxx} - \frac{3}{2} W_x^2 = 0 \quad (32)$$

as an equivalent to the KdV equation. The linear counterpart to this equation which results from neglecting the non-linear term and replacing the kernel $W(x, x; t)$ by its approximant $-F(u = 2x, t)$, is Equation (31). Separating the variables in this equation, we obtain solutions of the form:

$$\psi(u, t) = \sum_k e^{-2k^3 t} \left\{ A_k e^{-ku} + e^{k/2 u} \left[C_k \cos\left(\frac{\sqrt{3}}{2} k u\right) + S_k \sin\left(\frac{\sqrt{3}}{2} k u\right) \right] \right\}. \quad (33)$$

with summation over an appropriate set of k -values. This allows, as a particular subset, for solutions of the I.S.T. type:

$$\psi(u, t) = \frac{1}{\pi} \int_{-\infty}^{+\infty} dk R(k) e^{-2ik^3 t + iku} + \sum_{n=1}^N c_n e^{-\kappa_n u - 2\kappa_n^3 t}, \quad (34)$$

$\kappa_n, c_n > 0.$

However, there are also non-trivial solutions which correspond to the contribution of a finite set of negative k -values: $\{k_j = |k_j|\}$

$$\psi(u, t) = \sum_j e^{2|k_j|^3 t - \frac{1}{2}|k_j|u} \left[C_j \cos\left(\frac{\sqrt{3}}{2} k_j u\right) + S_j \sin\left(\frac{\sqrt{3}}{2} k_j u\right) \right] \quad (35)$$

In order to obtain a simple explicit solution of the KdV equation we consider the contribution of a single cosine term:

$$F_{\cos}(x+y, t) = C e^{2k^3 t - \frac{k}{2}(x+y)} \cos\left[\frac{\sqrt{3}}{2}(x+y)\right], \quad (36)$$

$k > 0.$

Insertion of this separable kernel into the integral equation:

$$W(x, y; t) + F(x+y, t) + \int_x^\infty dz W(x, z; t) F(z+y, t) = 0$$

leads, for $x = y$, to the solution:

$$W_{\cos}(x, x; t) = \frac{d}{dx} \log \Delta_{\cos}(x, t),$$

$$\Delta_{\cos}(x, t) = 1 + \frac{C}{2k} e^{2k^3 t - kx} \cos(\sqrt{3}kx + \frac{\pi}{3}) - 3 \frac{C^2}{16k^2} e^{4k^3 t - 2kx} \quad (37)$$

This yields the following solution of the KdV equation:

$$q_{\cos}(x, t) = -2 \frac{d^2}{dx^2} \log \Delta_{\cos}(x, t), \quad (38)$$

or, setting $c = C/4k$:

$$q_{\cos} = \frac{-16ck^2 \left[e^{2k^3 t - kx} \cos(\sqrt{3}kx - \frac{\pi}{3}) - 3ce^{4k^3 t - 2kx} + 3c^2 e^{6k^3 t - 3kx} \cos \sqrt{3}kx \right]}{\left[1 + 2ce^{2k^3 t - kx} \cos(\sqrt{3}kx + \frac{\pi}{3}) - 3c^2 e^{4k^3 t - 2kx} \right]^2}, \quad (39)$$

which corresponds to an oscillating initial value $q_{\cos}(x, 0)$ whose amplitude decreases exponentially at the ends of the line:

$$q_{\cos}(x, 0) \sim_{x \rightarrow +\infty} -16ck^2 e^{-kx} \cos(\sqrt{3}kx - \frac{\pi}{3}) + 48k^2 c^2 e^{-2kx} \quad (40)$$

$$q_{\cos}(x, 0) \sim_{x \rightarrow -\infty} -16 \frac{k^2}{3c} e^{-k|x|} \cos \sqrt{3}kx + \frac{16}{3c^2} k^2 e^{-2k|x|}$$

This solution is singular, if at a given time t , the denominator $\Delta_{\cos}(x, t)$ vanishes for some real value of x . Such zeros $x_0(t)$ are

solution of the equation:

$$3ce^{ik^3t - kx} = \cos(\sqrt{3}kx + \frac{\pi}{3}) \pm \sqrt{3 + \cos^2(\sqrt{3}kx + \frac{\pi}{3})}. \quad (41)$$

The right-hand side of this equation does not depend on t : with the $+$ sign it takes values which oscillate between 1 and 3, with the $-$ sign it takes values between -3 and -1 . Let us take $c > 0$, for definiteness. Then, whatever t , the left-hand side will take values which, for some positive or negative x , will pass through the strip between 1 and 3, and it follows from continuity that Eq. (41) will have at least one root $x_0(t)$ (Fig. 1). Actually, the number of the roots is odd, and it follows from a comparison of the period of the right-hand side with the decay rate of the left-hand side that there can be no more than one such root. Hence, the solution $q_{\cos}(x, t)$ will have one pole of order 2, whose location varies with time: it travels to the right for increasing values of t .

This kind of singularity was also found by Moses¹⁶⁾ in his particular solution of the KdV equation on the half-line $0 \leq x < \infty$ with the boundary condition $q(0, t) = 0$, which he obtained from the inverse scattering theory for the radial Schrödinger equation by using a particular Bargmann potential. In fact, the occurrence of second order poles in potentials which are obtained from solving the Marchenko equation, was already observed by Cornille¹⁷⁾ in connection with the N/D equations. They indicate that the "scattering data" have grown too big to correspond to a real scattering process. Though one can formally associate them to a singular potential with some double pole on the real axis, the singularity is such that it divides the (half) line in two regions $x < x_0$ and $x > x_0$ which are totally disconnected.

An equally simple explicit solution of the KdV equation can be obtained by considering the contribution of a single sine-term in (35), i.e., by taking:

$$F_{\sin}(x+y, t) = Se^{ik^3t - \frac{k}{2}(x+y)} \sin\left[\frac{\sqrt{3}}{2}k(x+y)\right], \quad (42)$$

$$k > 0.$$

The solution reads then ($s = S/4k$):

$$q_{\sin} = -16\Delta k^2 \frac{[e^{2k^3t-kx} \sin(\sqrt{3}kx - \frac{\pi}{3}) - 3\Delta e^{4k^3t-2kx} + 3\Delta^2 e^{6k^3t-3kx} \sin \sqrt{3}kx]}{[1 + 2\Delta e^{2k^3t-kx} \sin(\sqrt{3}kx + \frac{\pi}{3}) - 3\Delta^2 e^{4k^3t-2kx}]^2} \quad (43)$$

which is the expression (39) in which the cosines have been replaced by sines. One finally checks that the obtained explicit solutions are formally recovered from the customary two-soliton formula:

$$q_{\text{sol}}(x, t) = -2 \frac{d^2}{dx^2} \log \Delta_{\text{sol}}(x, t),$$

$$\Delta_{\text{sol}} = 1 + \frac{c_1}{2k_1} e^{-2k_1x-2k_1^3t} + \frac{c_2}{2k_2} e^{-2k_2x-2k_2^3t} + \frac{c_1 c_2}{4k_1 k_2} \left(\frac{k_1 - k_2}{k_1 + k_2} \right)^2 e^{-2(k_1+k_2)x-2(k_1^3+k_2^3)t} \quad (44)$$

by assigning to the positive bound state parameters k_i, c_i the "unphysical" values:

$$k_1 = \frac{k}{2} + i \frac{\sqrt{3}}{2} k$$

$$k_2 = \frac{k}{2} - i \frac{\sqrt{3}}{2} k$$

$$c_1 = c_2 = c \quad \text{or} \quad c_1 = -c_2 = i\Delta.$$

5. CONCLUSION

By using the method of Zakharov and Shabat in the simplest, scalar case, which is the one which proves adapted to the KdV equation, we have found solutions of this equation which fall beyond the scope of the I.S.T. method. Nevertheless, these solutions, as well as the customary I.S.T. solutions, are deduced from the key relation of the inverse scattering theory: the Faddeev-Marchenko equation. Both are seen to correspond in this approach to solutions of the linear equation $F_t - 2F_{uuu} = 0$, but the new ones come from solutions $F(u, t)$ that are not recovered by the method of Fourier transforms. Since they have still a fast decreasing asymptotic behaviour, they are obtained at the cost of singularities at finite values of x , which are second order poles.

Whether the above procedure can be modified and adapted to the Gel'fand-Levitan equation, so as to produce solutions¹⁶⁾ of the KdV equation on the half-line $0 < x < \infty$ with the boundary condition $q(x = 0) = 0$, for which the I.S.T. scheme fails to work¹⁸⁾ is still under investigation.

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APPENDIX 1

Consider the causal Volterra kernel $K_+(x,y)$, $K_+(x,y) = 0$ for $y < x$, which for $x < y$ is given by the solution of the Wiener-Hopf equation:

$$K_+(x,y) + F(x,y) + \int_x^\infty dz K_+(x,z) F(z,y) = 0,$$

in which $F(x,y)$ is a solution of the free wave equation:

$$\frac{\partial^2 F}{\partial x^2} - \frac{\partial^2 F}{\partial y^2} = 0,$$

which satisfies the condition:

$$\int_x^\infty dz \int_x^\infty dy |F(z,y)|^2 < \infty, \quad \forall x > -\infty.$$

It follows from straightforward differentiation of the integral equation that:

$$\begin{aligned} & \left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + 2 \frac{dK_+(x,x)}{dx} \right) K_+(x,y) \\ & + \int_x^\infty dz F(z,y) \left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial z^2} + 2 \frac{dK_+(x,x)}{dx} \right) K_+(x,y) \\ & = \left[F(z,y) \frac{\partial K_+(x,z)}{\partial z} - K_+(x,z) \frac{\partial F(x,z)}{\partial z} \right]_{z=\infty}. \end{aligned}$$

The right-hand side vanishes provided that one has: $F(x,\infty) = F(\infty,y) = 0$, and we are led to consider only solutions $F(x,y)$ of the form: $F(x,y) = F(x+y)$, with:

$$\int_a^\infty |F(t)| dt < \infty, \quad \int_a^\infty t |F'(t)| dt < \infty, \quad \forall a > -\infty,$$

in order to fulfil the above condition. It then follows from the Fredholm alternative that the homogeneous counterpart of the integral equation has no other $L_2(a,\infty)$ solution than the null-solution, so that one has:

$$\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + 2 \frac{dK_+(x,x)}{dx} \right) K_+(x,y) = 0, \quad x < y.$$

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FIG.1