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CHARGED SECTORS AND SCATTERING STATES IN QUANTUM ELECTRODYNAMICS

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ABSTRACT

Charge operator and charged sectors in four-dimensional QED are investigated, both within a local, covariant (indefinite metric) and within a positive metric formulation of QED. Assuming a few basic, physical principles -- such as Gauss' law and a correspondence principle etc. -- we conclude that charge states determine non-Fock (coherent state) representation of the algebra generated by the asymptotic, electromagnetic field, that Lorentz boosts do not leave the charged sectors invariant (spontaneous breaking of boost symmetry) and that an unambiguous definition of the "mass" of the charged infra-particles is possible. This and other results represent first steps at extending the Haag-Ruelle scattering theory to charged infra-particles.

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1. INTRODUCTION

1.1 Purposes of the paper

Four types of super-selection rules must be distinguished in relativistic quantum field theory (rqft).

Vacuum super-selection rules¹⁾

They arise in rqft's with degenerate physical vacuum. They label all possible distinct vacuum sectors (with extremal vacuum) of the theory. Degeneracy of the physical vacuum is often, but not always accompanied by the spontaneous breaking of an internal symmetry group. The theory of vacuum super-selection rules (i.e., phase transitions in standard rqft's, respectively, the occurrence of θ vacua in gauge qft's with instantons) is well developed, and many examples have been analyzed rigorously²⁾.

Charge super-selection rules in standard rqft³⁾

Super-selection sectors labelled by "standard" charges (not satisfying Gauss' law) arise in rqft's described by -- amongst other local fields -- unobservable, local, charged fields (e.g., charged Fermi fields), or field bundles, which are not in the algebra of local observables. Sectors of non-zero total charge are constructed by applying local, charged fields, or field bundles, to the vacuum sector. An example is provided by the charged sectors of the Yukawa quantum field model in two or three space-time dimensions. The general theory of these super-selection rules has been developed in detail and great depth by Doplicher, Haag and Roberts (DHR)³⁾.

Topological-charge super-selection rules⁴⁾

They arise in theories with quantum solitons (quantum vortices or monopoles, respectively). The corresponding super-selection sectors are labelled by the eigenvalues of a topological charge operator. Generally, it appears to be possible to construct field bundles -- almost local relative to the basic, observable local fields -- which, when applied to the vacuum sector, yield states of non-zero topological charge. Examples are provided by quantum field models with soliton sectors (in two space-time dimensions always accompanied by phase transitions!), such as the $\lambda\phi_2^4$ model. In two space-time dimensions the present understanding of topological charges is quite perfect⁴⁾. In higher dimensions topological-charge super-selection rules are only present in gauge theories for which the basic existence problems are still open. A reasonably good heuristic understanding has, however, been achieved⁵⁾, and many results of DHR³⁾ seem to be applicable.

Charge superselection rules in gauge qft's with a local Gauss law

An example is the charge super-selection rule in QED. More generally, these super-selection rules concern gauge theories (gqft's) with unconfined charges related to the gauge field strength by a local Gauss law. These super-selection rules are still not well understood at all, even in the case of QED. There are reasons to believe that the problems met in QED are typical, since non-Abelian charges satisfying a local Gauss law are expected to be screened or confined. See also Sections 2 and 3.

The basic purpose of this article is to find as many constraints as possible on the form of a general theory, in particular the charge super-selection rule in QED⁶⁾, to propose a tentative form of such a theory and to discuss attempts at constructing a scattering theory for charged (infra-) particles⁷⁾⁸⁾.

The asymptotic condition in rqft's with massless particles has been the subject of numerous investigations since the early days of QED. After the classic papers of Bloch and Nordsieck⁹⁾ and Pauli and Fierz¹⁰⁾, a more complete understanding of that problem was aimed at in Refs. 11), 12), 13) and 14), on the level of perturbation theory. From the non-perturbative point of view, relevant contributions are Dollard's treatment of Coulomb scattering¹⁵⁾ and the rigorous analysis of non-relativistic field theory models¹⁶⁾⁸⁾. See also Ref. 17).

In rqft's in even space-time dimensions the existence of asymptotic limits (free local in/out fields) of local operators with non-vanishing matrix elements between the vacuum and massless fermion¹⁸⁾ and massless boson¹⁹⁾ one-particle states, respectively, has been recently established, on the vacuum sector, by Buchholz [following some earlier, heuristic proposals¹⁷⁾]. Buchholz has, in fact, constructed in/out scattering states with a "finite number of particles" (Fock representation) in terms of the asymptotic fields and determined a general property of the vacuum representation of the asymptotic field algebra¹⁹⁾. Existence and properties of representations of the in/out fields on sectors which cannot be reached by applying local field bundles to the vacuum sector remain open problems in Buchholz' treatment.

Existence (and some of the properties) of such representations in QED (charged collision states with infinitely many soft photons present, yielding non-Fock representations) is strongly suggested by perturbation expansions in QED and by infra-red models. Our analysis of the general structure of the charge super-selection rule in gauge qft's with a local Gauss law and unconfined (Abelian) charges and of the properties of collision states with infra-particles is based on the following inputs:

- a) The Gauss law: In QED the Gauss law takes the form

$$\text{div } \vec{E} = \rho ,$$

where $E_i = F_{0i}$ are the components of the electric field $\vec{E}$, assumed to be local, operator-valued, tempered distributions, and $\rho = j_0$ is the charge density, the 0 component of the quantized, local, locally conserved electric current. It has already been remarked in Ref. 3) that the Gauss law ought to yield important constraints.

- b) A careful definition and analysis of the electric charge operator; (see Section 2).
- c) The results of Buchholz¹⁹⁾ concerning the Haag-Ruelle theory for massless bosons (in the vacuum sector).
- d) A correspondence principle, saying that in the limit of very long wavelengths the quantum energy of an electromagnetic field configuration converges to the "classical energy" of that configuration.
- e) Conventional wisdom about collision states in QED.
- f) Experience with lattice gauge theories²⁰⁾.

Our main conclusions are:

- A. Scattering states of non-zero charge (if they exist) always determine representations of the algebra of the asymptotic, electromagnetic field disjoint from the Fock representation. Equivalent to this (as we shall show) is the assertion that charged particles are infra-particles. (See Section 2).
- A'. The possible representations of the asymptotic, electromagnetic field algebra determined by collision states (satisfying space-time translation covariance and the relativistic spectrum condition) are characterized.
- A''. Under various reasonable, dynamical hypotheses it is shown that collision states are generalized coherent states (up to hard photon contributions). Even if those hypotheses are questionable, collision states must be - in some suitable sense - close to being generalized coherent states.
- B. Under more stringent, dynamical hypotheses, super-selection sectors of non-zero total charge are shown not to be invariant under Lorentz boosts ("spontaneous breaking of boost symmetry").
- C. Physical, charged asymptotic fields (if they exist) cannot be local relative to the asymptotic, electromagnetic field. They are not boost-covariant.

D. It is outlined how one might construct a generalized Haag-Ruelle scattering theory for charged (infra-) particles.

We try to motivate our main assumptions as clearly as possible. This part of our work is necessarily heuristic, as the basic existence problem of gauge theories (with unconfined charges) in four dimensions is, of course, still widely open. The conclusions which we derive from our basic assumptions are, however, rigorous.

This paper neither presents a "solution of the infra-red problem in QED" nor a "general theory of the charge super selection rule". All we propose to do is to review and clarify the present situation and to rule out some concepts and ideas because they appear to contradict each other or basic principles.

If we succeed in renewing some general interest in the basic problems discussed in the subsequent sections and challenging the reader to do better, our goal will be achieved.

1.2 Summary of contents

The structure of this paper is inductive: We try to find a correct general theory and its main features by first trying to find as many well-accepted general principles and plausible assumptions as possible. (This makes our paper rather lengthy.)

It is clear that the lack of locality, e.g., of the charged fields, in a gauge theory with an unconfined charge satisfying a local Gauss law makes a general analysis, in particular of the collision theory, difficult. These difficulties already appear at a general level when one tries to formulate gauge quantum field theories (gqft's) in a positive metric Hilbert space without local charged fields, or to work in a local (and covariant) formulation without positivity (indefinite metric formulation). For proofs that a combination of strict locality and/or covariance (of charged fields) with positivity, in QED and other gqft's, is impossible see Refs. 21) and 22). Here the crucial fact is Gauss' law which, moreover, forbids the existence of local (physical) charged states²³⁾.

In Section 2 we show that, in the framework of a local and covariant formulation of QED:

i) The charged states define non-Fock representations of the asymptotic fields $A_\mu^{\text{in/out}}$. This is related to the fact that charged states always "contain an infinite number of in/out photons" (Section 2.3).

ii) The set $D^{\text{in/out}}$ of (asymptotic) states which are "local" with respect to $A_\mu^{\text{in/out}}$ (i.e., states of the form $B\psi_0$, B a local operator relative to $A_\mu^{\text{in/out}}$, respectively) have zero charge. Therefore, if $\mathcal{H}^{\text{in/out}}$ denotes the closure of $D^{\text{in/out}}$, $\mathcal{H}^{\text{in/out}} \neq \mathcal{H}$ (the total Hilbert space; see Section 2.2).

Remark: In our definition, $\mathcal{H}^{\text{in/out}}$ contains all states local relative to $A_\mu^{\text{in/out}}$, respectively, and therefore contains the standard spaces generated by the in/out fields $A_\mu^{\text{in/out}}$, presumably properly. In our setting the fields generating $\mathcal{H}^{\text{in}}$ from the vacuum need neither be free fields, nor some sort of asymptotic limits of the local fields of the theory; (only relative locality to A_μ^{in} is required).

Hence, our results imply that if some kind of asymptotic matter fields ψ^{in} exist in $\mathcal{H}$ such that the space generated by ψ^{in} , A_μ^{in} from the vacuum equals $\mathcal{H}$ the fields ψ^{in} cannot be local relative to A_μ^{in} : locality is necessarily lost at the level of asymptotic fields (Section 2.2). It is important to note that this is not a general feature common to all theories with massless bosons, but is strongly linked to Gauss' law. Another result is: Suppose that, in some gqft, the gauge field acquires a mass and that massive free field asymptotic limits of the gauge field strength exist. Then the representation of the asymptotic gauge fields in any space-time translation covariant state is a Fock representation. Thus, any such state is an eigenstate corresponding to eigenvalue 0 of any charge satisfying a local Gauss law.

In Section 3 we investigate the general properties of the representations of the asymptotic algebras $\mathcal{A}_{\text{e.m.}}^{\text{as}}$ (as = in/out), generated by $F_{\mu\nu}^{\text{as}}$ on the physical Hilbert space $\mathcal{H}_{\text{phys}}$. Our input, here, are the deep results of Buchholz¹⁹⁾ who has constructed asymptotic algebras assuming the basic axioms of rqft and the existence of massless one-particle states (in even space-time dimensions). In QED, his construction only works on the charge 0 (vacuum) sector; a straightforward consequence of his hypotheses and Gauss' law! Nevertheless it is reasonable [see also Refs.8) and 24) for plausibility arguments] to suppose that the asymptotic, electromagnetic algebras, $\mathcal{A}_{\text{e.m.}}^{\text{as}}$, have well-defined representations on the charged sectors, too. All these representations obviously have an energy-momentum operator satisfying the relativistic spectrum conditions. Representations of the algebra generated by a free electromagnetic field which have that property are called scattering representations. In Section 3.1 we show that scattering representations of $\mathcal{A}_{\text{e.m.}}^{\text{as}}$ are "locally Fock", both in momentum space

and in configuration space and that they allow for a separation of the total energy-momentum operator $(H, \vec{P})$ of the theory into a part $(H_{\text{e.m.}}^{\text{ph}}, \vec{P}_{\text{e.m.}}^{\text{ph}})$ referring to the asymptotic electromagnetic field (i.e., affiliated with $\mathcal{Q}_{\text{e.m.}}^{\text{as}}$) and a part $(H_{\text{e.m.}}^{\text{c}}, \vec{P}_{\text{e.m.}}^{\text{c}})$ describing the asymptotic dynamics of the charges. The mass spectrum of charged (infra-) particles is determined by $(H_{\text{e.m.}}^{\text{c}}, \vec{P}_{\text{e.m.}}^{\text{c}})$. In this framework, the Haag-Ruelle construction appears to have a natural generalization to the case of charged particles (Section 3.2). A solution of the important problem of identifying the charged one-(infra-) particle states in a theory in which the infra-red catastrophe forbids the existence of a sharp mass eigenvalue in the spectrum of the mass operator on charged sectors is given. Generalized coherent representations (i.e., ones determined by generalized coherent states) are shown to arise as a consequence of the correspondence principle which says that the quantum e.m. field energy ought to approach the classical field energy in the limit of very long wave lengths (Section 3.3). It is well known that irreducible, non-Fock coherent states do not define Lorentz covariant representations. A natural and important question is whether, for example, the whole (reducible) one-particle sector is Lorentz covariant. In Section 3.4 we show that the generalized coherent states usually postulated in perturbative calculations of the S-matrix in QED⁹⁾²⁴⁾ have covariance properties under Lorentz boosts which are incompatible with boost invariance of the charged sectors. Conversely, it is shown that those covariance properties and a minimality condition imply that the scattering states are the usual generalized coherent states.

An argument, based on experience with lattice gauge theories, in favour of those covariance properties is sketched.

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Finally we would like to add a few comments about charges in two and three dimensions obeying a local Gauss law. In Abelian gauge theories in two and three space-time dimensions (without Higgs mechanism) one expects that the electric charge is confined, i.e., there are no physical states of non-zero total electric charge. The intuitive reason for why this is so is that the one- and two-dimensional Coulomb potentials are confining. For some Abelian lattice gauge theories such results have been established²⁶⁾. On the other hand, it has been shown²⁷⁾ that in such theories the confining potential is never stronger than the one-, respectively two-dimensional Coulomb potentials. In four dimensions, Abelian gauge theories do not seem to confine the electric charge: This has been shown for a class of lattice models²⁷⁾ and is expected to remain true in the continuum limit (unless renormalization has mysterious effects).

If there is a Higgs mechanism then, in two space-time dimensions, fractional charges are still confined²⁸⁾, in higher dimensions there does not seem to be confinement, but there is also no local Gauss law for the electric charge carried by the fermions, and the super-selection structure appears to fit into the DHR theory.

In non-Abelian gauge theories in two and three dimensions, one expects permanent confinement of quarks and screening (or confinement) of all charges²⁹⁾, so there do not seem to exist any physical charged super-selection vectors.

In conclusion, the theory of charge super-selection rules with charges obeying a local Gauss law appears to be empty in two and three space-time dimensions, but is relevant and needed in four dimensions, at least in Abelian gauge theories.

2. CHARGE OPERATOR AND ASYMPTOTIC FIELDS IN A LOCAL FORMULATION OF QUANTUM ELECTRODYNAMICS

In this section we adopt a local formulation of QED⁶⁾ on an indefinite metric space, and discuss how the charge operator, defined on the dense set of local states, can be extended to a larger domain which contains the "local asymptotic states". It is shown that the properties of that (extended) operator are strictly related to the infra-red problem in QED, a consequence of Gauss' law. In particular, it is shown that charged states determine non-Fock representations of $A_{\mu}^{\text{in/out}}$, and that a modification of the standard asymptotic condition is needed, since asymptotic charged fields $\psi^{\text{in/out}}$ cannot be local with respect to $F_{\mu\nu}^{\text{in/out}}$.

2.1 The charge operator

We work in a local and covariant formulation of QED, for which we refer to Ref. 6)^{*)}. In particular, our discussion will rely on the following structure:

- A) The Hilbert space $\mathcal{H}$, with scalar product $(\cdot, \cdot)$, has an indefinite metric structure $\langle \cdot, \cdot \rangle = (\cdot, \eta \cdot)$ defined by a self-adjoint operator, η , with $\eta^2 = 1$.
- B) There exists a unique, translationally invariant vector Ψ_0 , vacuum vector, with $\langle \Psi_0, \Psi_0 \rangle = 1$.
- C) The space-time translations $U(a)$ are unitary operators in $\mathcal{H}$, and the spectrum of their generators is contained in $\bar{V}_+$.
- D) The Maxwell equations hold in the form

$$\square A_{\mu}(x) = j_{\mu}(x) \quad (2.1)$$

where $j_{\mu}(x)$ is the conserved electric current operator.

In order to avoid pathologies which may, in principle, arise due to domain questions, we will in the following make the technical assumption that there exists a net of local algebras $\mathcal{A} = \cup \mathcal{A}(\mathcal{O})$ ($\mathcal{O}$ denoting a double cone of space-time) of bounded operators associated to the local fields, with the vacuum as a cyclic vector and such that

*) Such a structure provides a minimal modification of the standard (positive metric) Wightman axioms in order to describe QED³⁰⁾, and one may prove that most of the results of axiomatic field theory³¹⁾³²⁾ can be generalized to this framework.

$$D_0 \equiv \mathcal{Q} \Psi_0 \quad (2.2)$$

is contained in the domain of the local fields^{*)}.

The interpretation of $j_\mu(x)$ as the electric current operator requires that the local charge

$$Q_R = \int j_0(\vec{x}, x_0) f_R(|\vec{x}|) \alpha(x_0) d^4x \equiv j_0(f_R \alpha), \quad (2.3)$$

where

$$f_R(|\vec{x}|) = f\left(\frac{|\vec{x}|}{R}\right), \quad f \in \mathcal{D}(\mathbb{R}^1), \quad f(x) = 1 \quad |x| \leq 1, \quad (2.4)$$

$$f(x) = 0 \quad |x| > 2, \quad \alpha(x_0) \in \mathcal{D}(\mathbb{R}^1), \quad \int \alpha(x_0) dx_0 = 1,$$

generates a $U(1)$ group of local automorphisms of the algebra $\mathcal{A}$, through the equation

$$\delta A = \lim_{R \rightarrow \infty} i [Q_R, A], \quad A \in \mathcal{A}. \quad (2.5)$$

In the following we will often drop the specification $R \rightarrow \infty$ in the limit when the symbol $\lim$ is not ambiguous.

It is important to stress that there is a large freedom in the expression for the local operator Q_R in Eq. (2.5); in fact for any λ the operator

$$Q_R^\lambda \equiv Q_R + \lambda \partial^\mu F_{\mu 0} (f_R \alpha), \quad (2.6)$$

where $F_{\mu\nu}$ is the electromagnetic field, defines the same automorphism on $\mathcal{A}$; a consequence of locality and Gauss theorem³⁰⁾. By the spectral condition (C) and Eq. (2.1) this automorphism cannot be spontaneously broken³³⁾ and therefore it

*) Actually we will need this property only for the local charge operators Q_R .

uniquely defines a (closable) charge operator Q ³⁴⁾ on the local states $\Psi = A\Psi_0$, $A \in \mathcal{A}$, by the equation

$$Q\Psi = \lim_{R \rightarrow \infty} [Q_R, A] \Psi_0, \quad (2.7)$$

where Q_R is anyone of the Q_R^λ of Eq. (2.6). Since Q can be proved to be an η symmetric operator on D_0 , it follows that Q is closable; (η is the metric operator).

As a consequence of the structure A)-D) many of the physical statements involve vectors of $\mathcal{H}$ ⁶⁾ which are, in general, non-local, i.e., they do not belong to D_0 . It is therefore very important to analyze the extensions of the charge operator to a domain larger than D_0 . In addition, one would like to recover the definition of the charge as a limit of local operators. Both problems are solved if for some λ

$$w\text{-}\lim_{R \rightarrow \infty} Q_R^\lambda \Psi_0 = 0. \quad (2.8)$$

Proposition 2.1

If, for a given λ , $w\text{-}\lim_{R \rightarrow \infty} Q_R^\lambda \Psi_0$ exists, then

$$w\text{-}\lim_{R \rightarrow \infty} Q_R^\lambda \Psi_0 = 0. \quad (2.8)$$

Moreover, for all vectors Ψ of the form $\Psi = A\Psi_0$, $A \in \mathcal{A}$,

$$Q\Psi = w\text{-}\lim_{R \rightarrow \infty} Q_R^\lambda \Psi. \quad (2.9)$$

Proof. By the spectral condition and locality, for any $\Psi \in D_0$ of the form $\Psi = A\Psi_0$, one has ³⁵⁾ $((\partial F)_R \equiv \partial^\mu F_{\mu 0} (f_R^\alpha))$

$$\lim \langle (\partial F)_R \Psi_0, A\Psi_0 \rangle = \frac{1}{2} \lim \langle \Psi_0, [(\partial F)_R, A] \Psi_0 \rangle = 0,$$

so that

$$\lim \langle Q_R^\lambda \Psi_0, A\Psi_0 \rangle = \lim \langle Q_R \Psi_0, A\Psi_0 \rangle. \quad (2.10)$$

On the other hand,

$$\lim \langle \Psi_0, Q_R A \Psi_0 \rangle = \lim \langle \Psi_0, Q_R E_0 A \Psi_0 \rangle, \quad (2.11)$$

where $E_0 \equiv \int_{p^2=0} dE(p)$ and $E(p)$ are the spectral projections of the space-time translations, and therefore the right-hand side of Eq. (2.11) vanishes by Eq. (2.1).

Thus for a dense set $\Psi \in D_0$

$$\langle \Psi, w\text{-}\lim Q_R^\lambda \Psi_0 \rangle = 0,$$

i.e.,

$$\eta \quad w\text{-}\lim Q_R^\lambda \Psi_0 = 0$$

and Eq. (2.8) follows from the non-degeneracy of η .

Moreover, for any $\Psi = A\Psi_0$, $A \in \mathcal{Q}$,

$$Q_R^\lambda \Psi = [Q_R^\lambda, A] \Psi_0 + A Q_R^\lambda \Psi_0.$$

Since, by Eq. (2.7)

$$s\text{-}\lim [Q_R^\lambda, A] \Psi_0 = Q\Psi$$

and by Eq. (2.8)

$$w\text{-}\lim A Q_R^\lambda \Psi_0 = 0,$$

we get

$$w\text{-}\lim Q_R^\lambda \Psi = Q\Psi.$$

□

Corollary 2.1

If $w\text{-}\lim Q_R^\lambda \Psi_0$ exists, then the operator Q^λ defined by

$$Q^\lambda \Psi = w\text{-}\lim Q_R^\lambda \Psi, \quad (2.12)$$

on the linear domain D^λ of all vectors, Ψ , which belong to D_{Q_R} for R sufficiently large ($R > R_0(\Psi)$) and such that $w\text{-}\lim Q_R^\lambda \Psi$ exists, defines a closable extension of Q . Moreover if Q is essentially η adjoint on D_0 , then

$$\overline{Q^\lambda} = \overline{Q}, \quad \forall \lambda,$$

the bar denoting the closure of the operator.

Proof.

For any $\Psi, \Phi \in D^\lambda$

$$\langle \Psi, Q_R \Phi \rangle = \langle Q_R \Psi, \Phi \rangle \quad R > R_0(\Psi), R_0(\Phi)$$

and therefore

$$\langle \Psi, Q^\lambda \Phi \rangle = \langle Q^\lambda \Psi, \Phi \rangle$$

i.e., Q^λ is η symmetric on D^λ and therefore closable.

If Q is essentially η adjoint on D_0 , then all its closed extensions coincide. $\square$

From the proof of Proposition 2.1 it follows:

Lemma 2.1

The limit $w\text{-}\lim Q_R^\lambda \Psi_0$ exists if there exists a constant L (independent of R) such that

$$\|Q_R^\lambda \Psi_0\| < L. \quad (2.13)$$

In the following we will discuss conditions which will guarantee the bound (2.13) *).

In the framework in which the basic field theoretical structure is formulated in terms of a local net of von Neumann algebras of bounded operators, Buchholz has shown the existence of the asymptotic limit of local operators, A , which make transitions between the vacuum and massless one-particle states. His treatment makes use of the positivity of the metric and the boundedness of the local operator A and therefore it does not cover the case of the asymptotic limit of the field A_μ in QED. We do not want to enter here into the discussion of this problem and we base our analysis on the following assumption:

*) The following analysis is still possible under the weaker assumption that the bounds hold for a suitable Hilbert norm which is finite on the local states and on the states of D_{in} (Eq. (2.19)).

E₁) There exists an operator valued (tempered) distribution $A_\mu^{\text{in}}(x)$ with a common dense domain D containing Ψ_0 such that

$$\square A_\mu^{\text{in}}(x) = 0,$$

D_0 is contained in the domain of $\partial^\mu A_\mu^{\text{in}}$, and on D_0

$$Q_R \equiv Q_R^{\lambda=-1} = \partial_0 \partial^\mu A_\mu (f_R^\alpha) = \partial_0 \partial^\mu A_\mu^{\text{in}} (f_R^\alpha) \quad (2.14)$$

E₂) A net of local algebras $\mathcal{A}^{\text{in}}(\mathcal{O})$ of bounded operators is associated to the operators $A_\mu^{\text{in}}(f)$, $f \in \mathcal{D}(\mathbb{R}^4)$, such that $\mathcal{A}^{\text{in}}(\mathcal{O}) \Psi_0 \subset D$ and the elements of $\mathcal{A}^{\text{in}} \equiv \cup \mathcal{A}^{\text{in}}(\mathcal{O})$ are local with respect to $A_\mu^{\text{in}}(f)$, in the following sense

$$[A_\mu^{\text{in}}(f), A] \Psi_0 = 0$$

whenever $A \in \mathcal{A}^{\text{in}}(\mathcal{O})$, and $\text{supp } f$ is space-like with respect to $\mathcal{O}$.

The validity of Eq. (2.14) is strongly suggested by the fact that $\partial^\mu A_\mu$ satisfies the free field equation and therefore it should coincide with its asymptotic limit!

We now discuss the validity of the bound (2.13) for $\lambda = -1$.

Proposition 2.2 *)

If the two-point function

$$F_{\mu\nu}(x-y) \equiv (A_\mu^{\text{in}}(x) \Psi_0, A_\nu^{\text{in}}(y) \Psi_0),$$

satisfies the ordinary cluster property for massless fields, namely, for all sufficiently large space-like vectors a of the form $a = (a_0, \vec{a})$, $|a_0| < 2d$,

$$|(A_\mu^{\text{in}}(f) \Psi_0, A_\nu^{\text{in}}(g_a) \Psi_0)| \leq C_{\mu\nu} |D(f \star g_a)| \quad (2.15)$$

with $C_{\mu\nu}$ a suitable constant, D the ordinary massless two-point function and $g_a(x) = g(x-a)$, then one has

$$\|Q_R \Psi_0\| < L. \quad (2.16)$$

*) This result has been proved under somewhat stronger assumptions by H. Reeh and M. Requardt³⁶⁾.

Proof (see Appendix A)

Remark:

One may easily prove that if $A^{\text{in}}(f)$ is a local field, any of the following conditions imply the validity of Eq. (2.15):

- 1) $\eta A_{\mu}^{\text{in}}(x) \eta$ is a local field with respect to $A_{\mu}^{\text{in}}(x)$.
- 2) There exists an operator M :

$$\mathcal{J}(\mathbb{R}^4) \ni f_{\mu} \rightarrow (Mf)_{\mu} \in \mathcal{J}(\mathbb{R}^4)$$

such that $M^2 = 1$ and

$$A_{\mu}^{\text{in}\star}(f^{\mu}) = A_{\mu}^{\text{in}}((Mf)^{\mu})$$

$$3) \quad \mathcal{H}_A \equiv \overline{\{ A^{\text{in}}(f) \Psi_0, f \in \mathcal{J}(\mathbb{R}^4) \}} = \eta \mathcal{H}_A$$

In the following we will take for granted that the bound (2.13) holds for Q_R . According to Proposition 2.1 and Corollary 2.1 we may therefore extend the charge Q from D_0 to the larger domain Δ on which the weak limit of Q_R exists:

$$Q \Delta = w\text{-}\lim Q_R \Delta. \quad (2.17)$$

The extended operator is closable by Corollary 2.1. In the following by charge operator we mean the closure of such an extension and Q and D_Q will henceforth always denote the closed operator and its domain.

It is worth while stressing that the above analysis and in particular Eqs. (2.14) and (2.18) do not imply that

$$Q \mathcal{H}' = 0$$

In fact, even if $\langle \mathcal{H}', Q_R \mathcal{H}' \rangle = 0$, the weak limit (2.18) is not expected to exist on $\mathcal{H}'$: if $D_0 \ni \Psi_n \rightarrow \Psi \in \mathcal{H}'$ and $\Psi \in D_Q$ is such that

$$Q \Psi_n = w\text{-}\lim Q_R \Psi_n \qquad \lim_n Q \Psi_n = Q \Psi$$

it is not at all guaranteed that

$$\lim_n w\text{-}\lim Q_R \Psi_n = w\text{-}\lim Q_R \lim_n \Psi_n \quad (2.18)$$

The existence of $w\text{-}\lim Q_R \Psi$ would be guaranteed if Ψ were a local state, but this is impossible if Ψ has a non-zero charge³⁰⁾.

A not sufficiently careful treatment of this delicate exchange of limits is, in our opinion, at the basis of an argument concerning the non-existence of physical charged states presented in Ref. 37).

2.2 Asymptotic fields, locality and charged states

In this section we investigate the structure of a set of states which have some locality property with respect to A_μ^{in} . More precisely we consider the vector space of states

$$D_{\mathcal{F}^{\text{in}}} \equiv \{ \Psi, \quad \Psi = A \Psi_0, \quad A \in \mathcal{F}^{\text{in}} \}, \quad (2.19)$$

where $\mathcal{F}^{\text{in}}$ is any local extension of $\mathcal{A}^{\text{in}}$, i.e., a net of local algebras of bounded operators local relative to A_μ^{in} . This means that for any $A \in \mathcal{F}^{\text{in}}$, localized in a region $\mathcal{O}$, $A \Psi_0$ is in the common domain, D , of $A_\mu^{\text{in}}(f)$ and

$$[A_\mu^{\text{in}}(f), A] \Psi_0 = 0. \quad (2.20)$$

Our discussion does not require any further characterization of $\mathcal{F}^{\text{in}}$, apart from the local structure, and therefore our conclusions have a rather general validity.

For simplicity, in the following we drop the index $\mathcal{F}$ and define

$$\mathcal{H}^{\text{in}} = \text{the (Hilbert space) closure of } D^{\text{in}} \equiv D_{\mathcal{F}^{\text{in}}}.$$

The definition and structure of $\mathcal{H}^{\text{in}}$ abstracts a property common to most situations in which an asymptotic in-space exists, namely the existence of asymptotic in-fields which are relatively local. Here, the elements of $\mathcal{F}^{\text{in}}$ need neither be (bounded functions of) free fields, nor the asymptotic limit of local fields of the theory, as in the case of a theory with a mass gap. (In fact, $\mathcal{F}^{\text{in}}$ turns out not to contain asymptotic, charged fields.)

In the following we investigate the charge structure of D^{in} or of some closely related space. In other words, we focus our attention on the consequences of a local structure of asymptotic states.

Within the framework specified so far we can prove the following^{*)}

Theorem 2.1. The elements of D^{in} are in the domain of the charge operator Q and

$$Q D^{\text{in}} = 0 \quad (2.21)$$

Proof. By E_1) one has

$$Q_R D^{\text{in}} = (-\partial_0 \partial_i A_i^{\text{in}}(f_R \alpha) + \Delta A_0(f_R \alpha)) D^{\text{in}} \equiv Q'_R D^{\text{in}}. \quad (2.22)$$

and for large R , the operator Q'_R is localized in the causal complement of any given bounded region $\mathcal{O}$. Therefore if $A \in \mathcal{A}^{\text{in}}$ is localized in $\mathcal{O}$, we have $A \Psi_0 \in D_{Q'_R}$, and by $E_1), E_2)$

$$Q_R A \Psi_0 = A Q'_R \Psi_0 = A Q_R \Psi_0. \quad (2.23)$$

Thus, since $w\text{-}\lim Q_R \Psi_0 = 0$ and A is a bounded operator also $w\text{-}\lim Q_R A \Psi_0$ exists and it is equal to zero. The Theorem now follows from Eq. (2.18). $\square$

Corollary 2.2. $\mathcal{H}^{\text{in}}$ is in the domain of Q and

$$Q \mathcal{H}^{\text{in}} = 0. \quad (2.24)$$

*) A weaker form of Theorem 2.1 can be proved without knowing the existence of $w\text{-}\lim Q_R \Psi_0$. In this case, $\Psi, \Phi \in D^{\text{in}}$, one has

$$\lim \langle \Psi, Q_R \Phi \rangle = 0,$$

a consequence of the Araki, Hepp, Ruelle theorem on the cluster property³⁸⁾. The virtue of the analysis given in Section 2.1, and in particular of the existence of $w\text{-}\lim Q_R \Psi_0$, is to guarantee that the above limits can be identified with the matrix elements of the charge operator Q .

Proof. If $D^{\text{in}} \ni \{\psi_n\} \rightarrow \psi \in \mathcal{H}^{\text{in}}$, then $Q \psi_n = 0$ and since Q is closed, $\psi \in D_Q$ and

$$Q \psi = 0 . \quad \square$$

Theorem 2.1 and Corollary 2.2 imply that if the charge operator is non-trivial $\mathcal{H}^{\text{in}} \neq \mathcal{H}$, and there cannot exist asymptotic fields ψ^{in} associated to charged particles which are local with respect to A_μ^{in} .

The lack of relative locality for asymptotic fields, required by the above results, reminds of a very similar situation in the Coulomb gauge where, at zeroth order, one treats A_i and ψ as free fields and

$$A_0(k) \sim \frac{1}{k^2} : \bar{\psi} \gamma_0 \psi : (k) . \quad (2.25)$$

As is clear in this simple case and in non-relativistic field theory models⁸⁾, the loss of locality has a deep physical interpretation in terms of the "infrared Coulomb tail" or the low energy photon cloud associated to a charged particle. Here we have found a rigorous proof of the existence of this phenomenon in QED.

One can extend the discussion given so far in this Section by noticing that the locality of the states of $\mathcal{H}^{\text{in}}$ only relative to Q_R was involved in the argument. This allows one to generalize the above results by replacing $\mathcal{Q}^{\text{in}}$ by different algebras. Since

$$Q_R = \text{div } \vec{E}^{\text{in}}(f_R \alpha) , \quad (2.26)$$

an interesting case is provided by the local algebras $\mathcal{Q}_{\text{tras}}^{\text{in}}$ of bounded operators associated to the operators $A_\mu^{\text{in}}(f_\mu)$, $f_\mu \in \mathcal{D}(R^4)$, $\partial^\mu f_\mu(x) = 0$, (in the same sense as specified by E_2). We then consider a local extension of $\mathcal{Q}_{\text{tras}}^{\text{in}} \equiv \cup \mathcal{Q}_{\text{tras}}^{\text{in}}(\mathcal{O})$, i.e. an algebra $\mathcal{F}_{\text{tras}}^{\text{in}}$ of bounded operators, such that if $A \in \mathcal{F}_{\text{tras}}^{\text{in}}$ and A is localized in $\mathcal{O}$, then $A\psi_0 \in D$ and

$$[A, A_\mu^{\text{in}}(f)] \psi_0 = 0 ,$$

for any f_μ with $\partial^\mu f_\mu(x) = 0$ and $\text{supp } f_\mu$ spacelike with respect to $\mathcal{O}$. As before we define

$$D_{\text{tras}}^{\text{in}} \equiv \{ \psi \mid \psi = A \psi_0 , \quad A \in \mathcal{F}_{\text{tras}}^{\text{in}} \} , \quad (2.27)$$

with Hilbert space closure denoted by $\mathcal{H}_{\text{tras}}^{\text{in}}$. In general, there is no inclusion relation between $\mathcal{H}_{\text{tras}}^{\text{in}}$ and $\mathcal{H}^{\text{in}}$ defined before. The physical property which characterizes $\mathcal{H}_{\text{tras}}^{\text{in}}$ is the locality with respect to $F_{\mu\nu}^{\text{in}}(f^{\mu\nu}) = A_{\mu}^{\text{in}}(\partial_{\nu} f^{\mu\nu})$.

Theorem 2.2. No asymptotic charged field ψ^{in} can exist which is local with respect to $F_{\mu\nu}^{\text{in}}$.

Proof. It suffices to notice that

$$Q_R = \partial^{\mu} F_{0\mu}^{\text{in}}(f_R^{\alpha}) + \square A_0^{\text{in}}(f_R^{\alpha}) \quad (2.29)$$

and that the second term on the right-hand side vanishes. The proof is therefore the same as for Theorem 2.1, since Q_R commutes with $\mathcal{F}_{\text{tras}}^{\text{in}}$, for R sufficiently large. $\square$

It is important to stress that the non-locality of asymptotic charged states with respect to $F_{\mu\nu}^{\text{in}}$ is a property quite different from the non-existence of physical states local with respect to $F_{\mu\nu}$. The latter is essentially a consequence of the validity of the Maxwell equations on the physical states, local charged states being allowed to exist in $\mathcal{H}$. Here, it is shown that existence of asymptotic charged states local with respect to $F_{\mu\nu}^{\text{in}}$ is impossible, whether they are physical or not. Equation (2.26) is essentially Eq. (2.14) since $\square A_{\mu}^{\text{in}} = 0$. As already remarked at the end of Section 2.1, the definition of the charge as weak limit of Q_R holds only on a suitable dense domain and it does not necessarily extend to all the physical states, i.e. to a dense domain of $\mathcal{H}'$. Therefore, the above Eq. (2.26) does not imply that $Q = 0$ on $\mathcal{H}_{\text{phys}} \equiv \mathcal{H}'/\mathcal{H}''$ even if $\text{div } \vec{E}^{\text{in}} = 0$ on $\mathcal{H}_{\text{phys}}$. We want to emphasize that one of the advantages of a formulation of QED based on local charged fields (with indefinite metric) is that the charge can be defined in terms of free (asymptotic) fields on a dense domain and therefore some of its properties are easier to investigate. One has, however, to be careful in extending such definition to vectors of $\mathcal{H}'$ since in approximating such vectors by local states the exchange of limits (2.18) is in general not allowed.

2.3 Non-Fock representations

In this Section we investigate the question of whether, in QED, the non-locality of the asymptotic charged states is related to the appearance of non-Fock representations induced by the charged states. To investigate this question we must formulate what we mean by a Fock representation with indefinite metric.

From the structure of the free Gupta-Bleuler formulation we easily extract the following algebraic set up.

A Fock representation of $\mathcal{Q}^{\text{in}}$ is a mapping

$\pi_F: \mathcal{Q}^{\text{in}} \rightarrow$ bounded operators on a Hilbert space $\mathcal{H}_F$ with a metric operator $\eta_F, \eta_F^2 = 1$

such that

- a) there exists a vector $\Psi_F \in \mathcal{H}_F$, invariant under space-time translations and cyclic with respect to $\pi_F(\mathcal{Q}^{\text{in}})$, i.e.

$$\mathcal{H}_F = \overline{\{ \pi_F(\mathcal{Q}^{\text{in}}) \Psi_F \}} \equiv D_F^{\text{in}}, \quad (2.30)$$

- b) the fields $A_\mu^{\text{in}}(f)$ and $(A_\mu^{\text{in}})^*(f)$ are represented in $\mathcal{H}_F$ by local operators $\pi_F(A_\mu^{\text{in}}(f)), \pi_F((A_\mu^{\text{in}})^*(f)) = \eta_F \pi_F(A_\mu^{\text{in}}(f)) \eta_F$, with a common dense domain $D_F \supset D_F^{\text{in}}$. For real f , $\pi_F(A_\mu^{\text{in}}(f))$ is η_F -essentially self-adjoint on D_F .

A vector $\Psi \in \mathcal{H}$ is said to provide a representation of $\mathcal{Q}^{\text{in}}$ equivalent to a Fock representation if there exists an isometric isomorphism

$$I: \mathcal{H}_\Psi \equiv \overline{\{ a^{\text{in}} \Psi, a^{\text{in}} \in \mathcal{Q}^{\text{in}} \}} \rightarrow \mathcal{H}_F, \quad (2.31)$$

such that

$$I a^{\text{in}} = \pi_F(a^{\text{in}}) I, \quad \text{on } \mathcal{H}_\Psi, \quad (2.32)$$

$$I \eta = \eta_F I, \quad \text{on } \mathcal{H}_\Psi, \quad (2.33)$$

$$I A I^{-1} = \pi_F(A), \quad A = A_\mu^{\text{in}}(f), \quad A_\mu^{\text{in}*}(f). \quad (2.34)$$

Theorem 2.3. Let Ψ be a state which provides a representation of $\mathcal{Q}^{\text{in}}$ equivalent to a Fock representation, then $\Psi \in D_Q$ and

$$Q \Psi = 0. \quad (2.35)$$

Proof. By assumption, there exists an isometric isomorphism $I: \mathcal{H}_\Psi \rightarrow \mathcal{H}_F$ such that if

$$D_F^{\text{in}} \equiv \{ \pi_F(a^{\text{in}}) \Psi_F, a^{\text{in}} \in \mathcal{Q}^{\text{in}} \}, \quad (2.36)$$

$I^{-1}D_F^{\text{in}} \equiv \mathcal{D}_0$ is dense in $\mathcal{H}_\Psi$. Now, by Propositions 2.1, 2.2 and Lemma 2.1

$$w\text{-}\lim \pi_F(Q_R) \Psi_F = 0,$$

$$w\text{-}\lim \pi_F(Q_R) D_F^{\text{in}} = 0.$$

This implies

$$w\text{-}\lim Q_R I^{-1} \Psi_F = 0$$

$$w\text{-}\lim Q_R \mathcal{D}_0 = 0$$

i.e. $\mathcal{D}_0$ is in the domain of the charge operator and

$$Q \mathcal{D}_0 = 0$$

Since $\mathcal{D}_0$ is dense in $\mathcal{H}_\Psi$ and Q is a closed operator, it follows that $\mathcal{H}_\Psi \subset D_Q$ and

$$Q \mathcal{H}_\Psi = 0. \quad \square \quad (2.37)$$

The above Theorem shows that charged states induce non-Fock representations of Q^{in} . Since no property of locality of such states has been required in Theorem 2.3, the statement applies to vectors of $\mathcal{H}'$ equally well.

The question, left open in Buchholz' treatment, of the existence of other representations of the massless (boson) fields besides the Fock representation has therefore a positive answer in the Gupta-Bleuler formulation of QED. This provides rigorous support to indications coming from the perturbation expansion of QED and from infrared models. A natural question arises: What kind of (positive metric) representations of the asymptotic, electromagnetic field algebra, compatible with the basic principles of r.q.f.t., may occur in charged sectors? We give an answer in the next section.

In conclusion, in this section we have shown that in a formulation of QED based on local charged fields, asymptotic charged fields $\psi^{\text{in/out}}$ cannot be local with respect to $F_{\mu\nu}^{\text{in/out}}$. This may be interpreted in the sense that the asymptotic limits $\psi^{\text{in/out}}$ must keep track of the Coulomb field associated to charged particles.

Furthermore charged states yield non-Fock representations of $A_\mu^{\text{in/out}}$. The analysis performed so far does not make clear whether the non-Fock character of such representations is due to the longitudinal and/or to the transverse photons.

In order to answer this question one has to go to the physical space $\mathcal{H}_{\text{phys}} = \mathcal{H}'/\mathcal{H}''$. The properties of the representations of $F_{\mu\nu}^{\text{in/out}}$ in $\mathcal{H}_{\text{phys}}$ will be discussed in the next section.

3. GENERAL DISCUSSION OF INFRARED REPRESENTATION

In this Section, we discuss a characterization of the physically relevant representations of the algebra generated by the asymptotic electromagnetic field. It turns out that the basic properties of such representations follow from general physical requirements. In particular, we show that physically reasonable assumptions imply that such representations are given by generalized coherent states, and that the Lorentz group is "broken" in each charged sector, in the sense that Lorentz boosts are not unitary operators on the charged sectors.

The basic input for this section is Buchholz' collision theory¹⁹⁾ for massless bosons mentioned in the introduction, here applied to a net of local algebras of bounded operators generated by the electromagnetic field, $F_{\mu\nu}$, in a positive metric formulation. (The photon is assumed to be a stable, massless particle).

We assume that charged states yield well-defined representations of the Weyl algebra of the asymptotic (free) electromagnetic field which Buchholz constructs, in the above framework, only on the charge 0 (vacuum) sector. See Refs. 17 and 24 for heuristic motivation. Moreover, we assume that the unitary representation of the space-time translation group of the interacting theory implements the space-time translation automorphisms of the asymptotic field algebra.

3.1 Scattering representations

We consider asymptotic, electromagnetic fields $F_{\mu\nu}^{as}$ defined as operator-valued tempered distributions on the physical Hilbert space $\mathcal{H}_{phys}$, satisfying the free Maxwell equations

$$\partial^\mu F_{\mu\nu}^{as} = 0, \quad \epsilon_{\mu\nu\rho\sigma} \partial^\nu F^{\rho\sigma} = 0. \quad (3.1)$$

(This physically reasonable starting point is also supported by the discussion given in Section 2, within a local formulation of QED. In that case, locality may allow one to use Buchholz' argument, based on Huyghens principle, to construct the asymptotic limit of A_μ , A_μ^{as} , on a dense set of states. The property

$$\square A_\mu^{as} = 0, \quad (3.2)$$

together with $\partial^\mu A_\mu = \partial^\mu A_\mu^{as}$, then imply that the operator $\mathcal{F}_{\mu\nu} \equiv \partial_\mu A_\nu^{as} - \partial_\nu A_\mu^{as}$ satisfies the Maxwell's equations as a quadratic form on $\mathcal{H}'$, and therefore in the sense of an operator equation on the quotient space $\mathcal{H}_{phys} \equiv \mathcal{H}'/\mathcal{H}''$).

The results discussed in this Section stay true if the discussion is based on the assumption that the $F_{\mu\nu}^{as}$ are defined as operator-valued tempered distributions on $\mathcal{H}_{phys}$ and satisfy the weaker equations

$$\square F_{\mu\nu}^{as} = 0, \quad (3.3)$$

instead of Eq. (3.1). In this case, the analysis is done with test functions $f_{\mu\nu} \in \mathcal{S}(R^4)$ satisfying the transversality condition

$$\partial^\mu f_{\mu\nu}(x) = 0. \quad (3.4)$$

For the validity of Eq. (3.3) in the vacuum sector one can rely on Buchholz' treatment¹⁹⁾.

As indicated, we assume that, in $\mathcal{H}_{phys}$, the space-time translations $F_{\mu\nu}^{as}(x) \rightarrow F_{\mu\nu}^{as}(x+a)$ are implemented by unitary operators, $U(a)$, whose generators $(H, \vec{P})$ satisfy the relativistic spectrum condition.

The construction of bounded operators (observables) associated to $F_{\mu\nu}^{as}$ requires the self adjointness of $F_{\mu\nu}^{as}(f)$, f real, on $\mathcal{H}_{phys}$. The following proposition gives a necessary and sufficient condition for the construction of the Weyl algebra associated to $F_{\mu\nu}^{as}$.

Proposition 3.1¹⁷⁾. If the operators $F_{\mu\nu}^{as}(f^{\mu\nu})$ on $\mathcal{H}_{phys}$ obey the canonical commutation relations (CCR) in infinitesimal form and satisfy the following inequality in the sense of quadratic forms

$$\pm F_{\mu\nu}^{as}(f^{\mu\nu}) \leq H + \Delta(f^{\mu\nu}), \quad f_{\mu\nu} \in \mathcal{S}_{real}(R^4), \quad (3.5)$$

where H is the generator of time translations and Δ is a constant, depending on $f^{\mu\nu}$, then the $F_{\mu\nu}^{as}(f^{\mu\nu})$ are essentially self adjoint on any core for H , and

$$W(f) \equiv \exp i F_{\mu\nu}^{as}(f^{\mu\nu}), \quad f_{\mu\nu} \in \mathcal{S}_{real}(R^4), \quad (3.6)$$

transform covariantly under the space-time translations and fulfill the Weyl relations.

Conversely, given a space-time covariant representation π of the Weyl algebra $\mathcal{A}^{as} \equiv \{W(f), f = \{f^{\mu\nu}\}, f^{\mu\nu} \in \mathcal{S}_{real}(R^4)\}$, with $(H, \vec{P})$ satisfying the relativistic spectrum condition, then there exist operators $F_{\mu\nu}^{as}(f^{\mu\nu})$ which are essentially self adjoint on any core for H and satisfy inequality (3.5) in the sense of quadratic forms.

Remarks. This result is a relatively straightforward combination of results in³⁹⁻⁴¹⁾. The Weyl algebra generated by a free, electromagnetic field has also been discussed in⁴²⁾.

We note that one can construct the electromagnetic potential from the field, provided the bound (3.5) holds:

Proposition 3.2. Given an operator-valued tempered distribution $F_{\mu\nu}(f)$ satisfying

$$\partial^\mu F_{\mu\nu} = 0, \quad \epsilon_{\mu\nu\rho\sigma} \partial^\nu F^{\rho\sigma} = 0,$$

the CCR and the bound (3.5), then one can construct an operator-valued tempered distribution $A_i(f)$, $i = 1, 2, 3$, such that

$$F_{0i} = \partial_0 A_i = -F_{i0}, \quad F_{ij} = \partial_i A_j - \partial_j A_i,$$

and

$$\partial^i A_i = 0 \tag{3.7}$$

(vector potential in the Coulomb gauge.)

Proof. If $\tilde{f} = p_0 g(p)$, $g \in \mathcal{S}(R^4)$ we define

$$A_i(f) = F_{0i}(g) \tag{3.8}$$

From the bound (3.5) and the CCR's it follows that, $\forall \Psi \in \mathcal{H}_{\text{phys}}$,

$$\|F_{0i}(g)\Psi\| \leq \|p_0^{-1/2} \tilde{f}\|_{L^2} \|H^{1/2}\Psi\| + \|f\|_{L^2} \|\Psi\|.$$

Since the set of functions $\tilde{f} = p_0 g(p)$ is dense in $\mathcal{S}(R^4)$, with respect to the norm $\|p_0^{-1/2} \tilde{f}(p)\|_{L^2}$, the field $A_i(f)$ can be extended to $\mathcal{S}(R^4)$ by continuity and the extension so obtained still satisfies the bound (3.5). $\square$

Definition 3.1. A representation π of $\mathcal{Q}^{\text{as}}$ is called a scattering representation if the space-time translation automorphisms are unitarily implementable, and the energy-momentum operator $(H, \vec{P})$ satisfies the relativistic spectrum condition. Let $\mathcal{O}$ be a bounded, open region in Minkowski space. We define $\mathcal{Q}^{\text{as}}(\mathcal{O})$ to be the local von Neumann algebra generated, in a scattering representation π , by the Weyl operators

$$\{W(f): f = f_{\mu\nu} \in \mathcal{S}_{\text{real}}(R^4), \text{supp } f \subset \mathcal{O}, \partial^\mu f_{\mu\nu} = 0\}.$$

Theorem 3.1 ¹⁷⁾. Let π be a scattering representation of $\mathcal{Q}^{as}$ then

- 1) π is "locally Fock in momentum space", i.e. when restricted to

$$\mathcal{Q}_p^{as} \equiv \{ W(f), f = \{ f^{\mu\nu} \}, f^{\mu\nu} \in \mathcal{S}_{real}(R^4), \text{supp } \tilde{f}(p) \cap \{ p: p_0 < p \} = \emptyset \}$$

π is quasi-equivalent to the Fock representation for all $p > 0$.

- 2) π is "locally Fock in configuration space", i.e. when restricted to $\mathcal{Q}^{as}(\mathcal{O})$, with $\mathcal{O}$ a bounded double cone, π is equivalent to the Fock representation.

Remark. The proof of this theorem is based on the powerful methods of Refs. 43 and 44, and a theorem in Ref. 45. The theorem implies that the number of photons with energy greater than $p > 0$ is an observable, i.e. the operator

$$N_p = \sum_{\lambda=1,2} \int_{|\vec{k}| \geq p} d^3k a_{\lambda}^*(\vec{k}) a_{\lambda}(\vec{k}) \quad (3.9)$$

is well defined in π . The theorem also implies that all measurements performed in a bounded region $\mathcal{O}$ can be described in terms of states with a finite number of photons. This extends a theorem of Buchholz ¹⁹⁾ who proves such a result for regions $\mathcal{O}$ contained in a future (resp. past) light cone, on the vacuum sector.

In the following an important rôle is played by scattering representations of $\mathcal{Q}^{as}$ which can be obtained from the Fock representation by a *-morphism σ ,

$$\pi \simeq \pi_F \circ \sigma. \quad (3.10)$$

In fact, in Section 3.4 we need:

Proposition 3.3. A sufficient condition for (3.10) is that the scattering representation π be countably decomposable into irreducible type I_{∞} factor representations.

Proof. By part 1 of Theorem 3.1, given a sequence $\{\rho_n\} \rightarrow 0$, $(\bigcup_n (\pi(\mathcal{Q}_{\rho_n}^{as})))^{\text{norm closed}}$ is a proper sequential type I_{∞} funnel in the sense of Takesaki ⁴⁶⁾. Therefore if π is countably decomposable, by Takesaki's theorem ⁴⁶⁾, there exists a *-morphism σ such that (3.10) holds.

3.2 Energy-momentum spectrum in a scattering representation

In order to get more detailed information about the energy-momentum spectrum in each charged sector and in particular for the identification of the (charged) one-particle states, we first prove that the energy operator can be separated into a photon energy operator and a charged particle energy operator. The above separation of the energy then allows a description of the scattering states in terms of charges and configurations of the asymptotic electromagnetic field with well-defined energy spectrum but with an undefined photon number.

Theorem 3.2. In a scattering representation π the energy-momentum operator can be decomposed in the following way

$$\begin{aligned} H &= H_{as}^{ph} + H_{as}^c, \\ \vec{P} &= \vec{P}_{as}^{ph} + \vec{P}_{as}^c, \end{aligned} \quad (3.11)$$

where $(H_{as}^{ph}, \vec{P}_{as}^{ph})$ are self-adjoint operators affiliated with $\pi(\mathcal{A}^{as})''$ and satisfy

$$\text{spect} (H_{as}^{ph}, \vec{P}_{as}^{ph}) = \bar{V}_+, \quad H_{as}^{ph} \leq H, \quad (3.12)$$

and $(H_{as}^c, \vec{P}_{as}^c)$ are self-adjoint operators affiliated ^{*)} with $\pi(\mathcal{A}^{as})'$ and satisfy

$$\text{spect} (H^c, \vec{P}^c) \subseteq \bar{V}_+. \quad (3.13)$$

Proof. We denote by $\overline{\pi(\mathcal{A}_\rho^{as})}$ the weak closure of $\pi(\mathcal{A}_\rho^{as})$. Since, for $\rho > 0$, $\pi(\mathcal{A}_\rho^{as})$ is quasi-equivalent to a Fock representation the space-time automorphisms are implemented by the unitary operators $\exp i[tH_\rho - \vec{x} \cdot \vec{P}_\rho]$, where

$$H_\rho = \sum_{\lambda=1,2} \int_{|\vec{k}| \geq \rho} d^3k |\vec{k}| a_\lambda^*(\vec{k}) a_\lambda(\vec{k}), \quad (3.14)$$

$$\vec{P}_\rho = \sum_{\lambda=1,2} \int_{|\vec{k}| \geq \rho} d^3k \vec{k} a_\lambda^*(\vec{k}) a_\lambda(\vec{k}), \quad (3.15)$$

*) The existence of operators $(H_{as}^{ph}, \vec{P}_{as}^{ph})$ affiliated with $\overline{\pi(\mathcal{A}^{as})}$ and implementing the space-time translations on $\pi(\mathcal{A}^{as})$ is also guaranteed by a general theorem of Borchers⁴⁷⁾. The main point of Theorem 3.2 is to provide an explicit construction of such operators and to prove properties related to the Fock character of $\pi(\mathcal{A}_\rho^{as})$, $\rho > 0$. For a similar construction in a concrete case see Ref. 8.

as weak integrals over the dense domain of states corresponding to a finite number of photons of energy $\geq \rho$. Clearly this implies that $(H_\rho, \vec{P}_\rho)$ are essentially self-adjoint on that domain and are affiliated with $\pi(\mathcal{Q}_\rho^{\text{as}})$.

Since $e^{isH_f} \in \overline{\pi(\mathcal{Q}_f^{\text{as}})}$ we have

$$e^{itH} e^{isH_f} e^{-itH} = e^{itH_f} e^{isH_f} e^{-itH_f} = e^{isH_f}$$

that is

$$[e^{itH}, e^{isH_f}] = 0. \quad (3.16)$$

Therefore

$$V_f(t) \equiv e^{itH} e^{-itH_f} \quad (3.17)$$

is a unitary group (in $\mathcal{H}_{\text{phys}}$) affiliated with $\overline{\pi(\mathcal{Q}_\rho^{\text{as}})}$. We denote by H_ρ^c the generator of $V_\rho(t)$. Since $H \geq 0$ and $H_\rho \geq 0$ and their spectral resolutions commute, H_ρ^c is essentially self-adjoint on $D_H \cap D_{H_\rho}$ and

$$H_f^c = H - H_f, \quad \text{on that domain.} \quad (3.18)$$

Clearly H_ρ^c is affiliated with $\overline{\pi(\mathcal{Q}_\rho^{\text{as}})}$.

Now, since $\pi(\mathcal{Q}_\rho^{\text{as}})$ is quasi-equivalent to a Fock representation, it is a factor representation of type I_∞ , and therefore

$$\mathcal{H}_{\text{phys}} = \mathcal{H}_f \otimes \mathcal{H}_f^c. \quad (3.19)$$

From the positivity of H and of H_ρ it then follows that also H_ρ^c is positive, so that

$$H_f \leq H. \quad (3.20)$$

The construction of P_ρ and P_ρ^c is done in a similar way. Since, by Eqs. (3.14) and (3.15),

$$H_f + \vec{\lambda} \cdot \vec{P}_f \geq 0, \quad \forall |\vec{\lambda}| \leq 1,$$

equation (3.19) implies that

$$H_f^c + \vec{\lambda} \cdot \vec{P}_f^c \geq 0 \quad \forall |\vec{\lambda}| \leq 1,$$

i.e. the spectrum condition (3.13) holds.

To conclude the proof of the theorem we prove that the strong limit, $s\text{-}\lim_{\rho \rightarrow 0} \exp itH_\rho$, exists and defines a unitary group $U(t)$, whose generator will be identified with H_{as}^{ph} . The same then holds for $V_\rho(t)$. To this purpose we notice that, for $\Psi \in D_H$,

$$\begin{aligned} \|(e^{itH_f} - e^{itH_{f'}})\Psi\|^2 &= 2\|\Psi\|^2 - (\Psi, e^{-itH_f} e^{itH_{f'}} \Psi) \\ &\quad - (\Psi, e^{-itH_{f'}} e^{itH_f} \Psi) = \\ &= \int_0^t ds (\Psi, e^{-is(H_f - H_{f'})} (H_f - H_{f'}) \Psi) \\ &\quad - (f \leftrightarrow f'), \end{aligned} \tag{3.21}$$

where we have used Duhamel's formula.

Now, as a consequence of $H_\rho \leq H$, the function

$$h_f(\Psi) = (\Psi, H_f \Psi) \leq (\Psi, H \Psi)$$

is bounded in ρ ; it is also a decreasing function since $H_{\rho'} - H_\rho \geq 0$ for $\rho' < \rho$. Thus $\lim_{\rho \rightarrow 0^+} h_\rho(\Psi)$ exists, i.e.

$$h_f(\Psi) - h_{f'}(\Psi) \rightarrow 0 \quad \text{as } f, f' \rightarrow 0^+. \tag{3.22}$$

By using the spectral representation we have

$$|(\Psi, e^{-is(H_f - H_{f'})} (H_f - H_{f'}) \Psi)| \leq (\Psi, (H_{f'} - H_f) \Psi)$$

and therefore by Eq. (3.22) the right-hand side of Eq. (3.21) vanishes in the limit $\rho, \rho' \rightarrow 0^+$.

Clearly $U(t) \equiv s\text{-}\lim_{\rho \rightarrow 0} e^{itH_\rho}$ belongs to $\overline{\pi(\mathcal{Q}^{as})}$ and its generator, H_{as}^{ph} , is positive. Moreover, since $\lim_{\rho \rightarrow 0^+} e^{itH_\rho^c} = e^{itH} e^{-itH_\rho}$, also $s\text{-}\lim_{\rho \rightarrow 0^+} e^{itH_\rho^c}$ exists and defines a self-adjoint operator H_{as}^c . By the argument given above,

$$H_{as}^c = H - H_{as}^{ph}, \quad \text{on } \mathcal{D}_H.$$

The construction of $\vec{P}_{as}^{ph}$ and $\vec{P}_{as}^c$ is done in a similar way and the relativistic spectrum conditions are preserved under taking strong limits.

The spectrum of $(H_{as}^{ph}, \vec{P}_{as}^{ph})$ is the entire $\bar{V}_+$, a consequence of Eq. (3.14) and the existence of the limit $\rho \rightarrow 0^+$. $\square$

The physical meaning of $(H_{as}^{ph}, \vec{P}_{as}^{ph})$ is to describe the energy and momentum of the asymptotic electromagnetic field configuration; consequently we can identify H_{as}^c and $\vec{P}_{as}^c$ as the energy and momentum operators associated to the charges and to fields without electromagnetic interactions. Thus, the above theorem offers a solution to the basic problem of how one can recover the mass spectrum of charged particles in a theory in which an infrared mechanism inhibits ⁸⁾ the existence of sharp hyperboloids in the spectrum of $(H, \vec{P})$. Since, physically, such a phenomenon is understood in terms of the infinite photon cloud surrounding each charged particle, the above separation of H into H_{as}^p and H_{as}^c reduces the analysis of the energy-momentum spectrum of charged particles to the analysis of the spectrum of $(H_{as}^c, \vec{P}_{as}^c)$.

We tentatively say that a theory describes charged particles of mass m if the spectrum of $(H_{as}^c, \vec{P}_{as}^c)$ contains an isolated one-particle hyperboloid of mass m . We denote such one-particle states by $\Psi_{(m)}$. Next, we show how, by exploiting Theorem 3.2, one can set up a framework in which the Haag-Ruelle construction of scattering states has a natural generalization on one-charged-particle states. To this purpose we consider a translation-covariant "field" operator Ψ_ϵ which interpolates between the vacuum, Ω , and the one particle states $\Psi_{(m)}$:

$$(\Psi_{(m)}, \Psi_\epsilon \Omega) \neq 0 \quad (3.23)$$

and define

$$\Psi_\epsilon(x) = e^{i(t H_{as}^c - \vec{x} \cdot \vec{P}_{as}^c)} \Psi_\epsilon e^{-i(t H_{as}^c - \vec{x} \cdot \vec{P}_{as}^c)}.$$

$\epsilon = -1$ for $as = in$, $\epsilon = +1$ for $as = out$.

Proposition 3.4. The asymptotic limits

$$s\text{-}\lim_{\epsilon t \rightarrow \infty} e^{itH} e^{-itH_{as}^{ph}} \Psi_\epsilon(f, t) \Omega, \quad (3.24)$$

where

$$\Psi_\epsilon(f, t) = (2\pi)^{-3/2} \int d^4p \tilde{\Psi}_\epsilon(\vec{p}, p_0) e^{-it\sqrt{\vec{p}^2 + m^2}} \tilde{f}(\vec{p}, p_0),$$

where f is a test function with the properties that

$$\text{supp } \tilde{f} \cap \{(\vec{p}, p_0) ; p_0^2 - \vec{p}^2 = m^2\} \neq \emptyset$$

but $\text{supp } \tilde{f}$ is disjoint from the rest of the spectrum of $(H_{as}^C, \vec{P}_{as}^C)$, exist and define "asymptotic" states $\Psi_{in/out}(f)$ which belong to the one particle hyperboloid.

Remark. One might recognize in the factor $e^{-itH_{as}^{ph}}$ the analog of the distorted wave factor introduced by Dollard¹⁵⁾ to discuss the Coulomb scattering.

Proof. The existence of the limit follows easily from $e^{itH} e^{-itH_{as}^{ph}} = e^{itH_{as}^C}$ and the spectral properties of the operator $\Psi_\epsilon(x)$, by standard arguments.

The construction of many (charged) particle scattering states depends on an analysis of the cluster properties of the field $\Psi_\epsilon(x)$, which lies outside the approach of this section.

We conclude with a relevant result that says that the above construction of "charged one-particle states", in particular the hypothesis that $(H_{as}^C, \vec{P}_{as}^C)$ contain an isolated mass hyperboloid, is compatible with the Folk theorem that charged particles are infra-particles⁷⁾.

Theorem (= Proposition 4.10 of Ref. 8 and following remarks)

The spectrum of $(H, \vec{P})$ contains an isolated mass hyperboloid of mass $m > 0$ if and only if $(H_{as}^C, \vec{P}_{as}^C)$ contains an isolated mass hyperboloid of mass m and the representation of $\mathcal{A}^{as}$ determined by a charged one particle state, $\Psi(m)$, is quasi-equivalent to the Fock representation.

Hence charged particles are infra-particles if they are surrounded by clouds of infinitely many soft photons.

3.3 Correspondence principle and generalized coherent states

From the experience of infrared models and perturbation theory we know that the infrared problem is governed by the low energy photon emission, i.e. by a phenomenon which can reasonably be treated at the classical level. Information obtained from the rigorous analysis of non-relativistic field theory models or from perturbative methods indicate that a characteristic feature of these phenomena is that charged states define generalized coherent representations of the asymptotic algebra. The purpose of this Section is to show that this property is largely independent of

the detailed dynamics of the model and is, in fact, related to a basic physical principle. The crucial idea is to exploit the property that in the limit of large wavelengths the quantum theory should predict the same results as the classical theory (correspondence principle).

To formulate this property more precisely, we consider a scattering representation π and a vector $\theta \in D_H$. The classical, electromagnetic energy of the state θ is defined by

$$\begin{aligned} E_{\theta}^d &= \frac{1}{2} \sum_{\lambda=1,2} \int d^3k [(\theta, \vec{E}(\vec{k})\theta)^2 + (\theta, \vec{B}(\vec{k})\theta)^2] \\ &= \sum_{\lambda=1,2} \int d^3k |\vec{k}| |(\theta, a_{\lambda}(\vec{k})\theta)|^2 \equiv \int d^3k E_{\theta}^d(\vec{k}), \end{aligned} \quad (3.25)$$

where $a_{\lambda}(\vec{k})$, $a_{\lambda}^*(\vec{k})$ are the destruction and creation operators introduced in Theorem 3.1. On the other hand, the quantum electromagnetic energy of θ is

$$E_{\theta}^q = \sum_{\lambda=1,2} \int d^3k |\vec{k}| (\theta, a_{\lambda}^*(\vec{k}) a_{\lambda}(\vec{k})\theta) \equiv \int d^3k \xi_{\theta}^q(\vec{k}) \quad (3.26)$$

and one always has the inequality

$$E_{\theta}^d(\vec{k}) \leq \xi_{\theta}^q(\vec{k}), \quad (3.27)$$

(for almost all $\vec{k}$'s in R^3).

Definition 3.2. The state θ is said to satisfy the correspondence principle if $\theta \in D_H$, and

$$\int_{|\vec{k}| \leq p} d^3k [\xi_{\theta}^q(\vec{k}) - E_{\theta}^d(\vec{k})] = O(p^{1+\epsilon}), \quad (3.28)$$

for some positive ϵ , i.e. the difference between the quantum and the classical energy of θ goes to zero in the limit of large wavelengths as $k^{1+\epsilon}$ *).

*) Equation (3.28) is essentially equivalent to the requirement that the energy density for wavelength λ , $E_{\theta}(\lambda) = \int E_{\theta}(\vec{k}) \delta(|\vec{k}| - \lambda^{-1}) d^3k$ has the same low frequency behaviour in the classical and in the quantum case:

$$|E_{\theta}^d(\lambda) - \xi_{\theta}^q(\lambda)| \sim O(\lambda^{-\epsilon}).$$

Definition 3.3. A scattering representation, π , is said to satisfy the correspondence principle if (almost) all its factor subrepresentations (obtained by central decomposition) contain a dense set of vectors $\mathcal{D}$ satisfying the correspondence principle.

The existence of such a dense set $\mathcal{D}$ in a factor scattering representation is guaranteed by the existence of a cyclic vector θ satisfying the correspondence principle.

Theorem 3.3. Let π be a scattering representation of $\mathcal{Q}^{\text{as}}$ satisfying the correspondence principle. Then

- a) each factor representation π_i obtained from π by central decomposition is quasi equivalent to $\pi_F \circ \sigma_i$, where π_F is the Fock representation and σ_i is the $\ast$ automorphism defined by

$$\sigma_i(a_\lambda(\vec{k})) = a_\lambda(\vec{k}) + F_\theta^\lambda(\vec{k}), \quad F_\theta^\lambda(\vec{k}) = (\theta, a_\lambda(\vec{k})\theta) \quad (3.29)$$

(generalized coherent $\ast$ automorphism) with θ any vector of π_i satisfying the correspondence principle.

- b) the weak closure of $\pi_i(\mathcal{Q}^{\text{as}})$ is a factor of type I_∞ .

- c) for any two states, $\theta, \theta' \in \pi_i$, satisfying the correspondence principle

$$|F_\theta(\vec{k}) - F_{\theta'}(\vec{k})|^2 \quad \text{is integrable at } \vec{k} = 0.$$

For the proof we require

Lemma 3.1. Let π_i be a factor scattering representation containing a cyclic vector θ satisfying the correspondence principle. Then

$$\frac{1}{|\vec{k}|} [\mathcal{E}_\theta^q(\vec{k}) - \mathcal{E}_\theta^d(\vec{k})] \quad \text{is integrable at } \vec{k} = 0.$$

Proof. By a change of variables and integration by parts we get

$$\begin{aligned} \int_{\delta \leq |\vec{k}| \leq 1} \frac{1}{|\vec{k}|} [\mathcal{E}_\theta^q(\vec{k}) - \mathcal{E}_\theta^d(\vec{k})] d^3k &= \int_\delta^1 \frac{1}{k} [\bar{\mathcal{E}}_\theta^q(k) - \bar{\mathcal{E}}_\theta^d(k)] k^2 dk = \\ &= \int_1^{1/\delta} t F(t) dt = [t G(t)]_1^{1/\delta} - \int_1^{1/\delta} G(t) dt \end{aligned} \quad (3.30)$$

where

$$-G(x) = \int_x^\infty F(t) dt = \int_{|\vec{k}| \leq 1/x} (\xi_\theta^q(\vec{k}) - E_\theta^q(\vec{k})) d^3k \leq \frac{1}{x^{1+\epsilon}} \quad (3.31)$$

is well defined, because $\theta \in D_H$, and the inequality follows from the correspondence principle. Hence the right-hand side of Eq. (3.30) has a limit when $\delta \rightarrow 0^+$. $\square$

Proof of Theorem 3.3. Let θ be a vector $\in \pi_i$, $(\mathcal{H}_\theta, \pi_\theta)$ the cyclic representation determined by θ , and τ_θ the *automorphism

$$a_\lambda(\vec{k}) \rightarrow \tau_\theta(a_\lambda(\vec{k})) = a_\lambda(\vec{k}) - F_\theta^\lambda(\vec{k}). \quad (3.32)$$

We consider the operator e^{itN_ρ} , with

$$N_\rho = \sum_{\lambda=1,2} \int_{|\vec{k}| \geq \rho} d^3k \ a_\lambda^*(\vec{k}) a_\lambda(\vec{k}), \quad \rho > 0. \quad (3.33)$$

It is now shown that, on $\mathcal{H}_\theta$, the strong limit of $\tau_\theta(e^{itN_\rho})$, as $\rho \rightarrow 0$, exists and defines a continuous, unitary group with a generator whose spectrum is the non-negative integers. This is based on an argument similar to that given in Theorem 3.2, using Lemma 3.1 to estimate expectations of $\tau_\theta(N_\rho)$ in a dense set of vectors in $\mathcal{H}_\theta$, uniformly in $\rho > 0$. Hence the representation $\pi_\theta \circ \tau_\theta$ of $\mathcal{A}^{as}$ has a total number operator. Details of these arguments can be found in ⁸⁾ Chapter 4, Section 4.1. By a theorem of Dell'Antonio, Doplicher and Ruelle⁴³⁾, $\pi_\theta \circ \tau_\theta$ is therefore quasi-equivalent to a Fock representation or, equivalently, π_θ is quasi-equivalent to $\pi_F \circ \tau_\theta^{-1}$ (generalized coherent state representation).

Since for any θ , τ_θ is an automorphism, $\pi_F \circ \tau_\theta^{-1}$ is of type I_∞ and therefore the original representation π_i is also of type I_∞ ⁴⁸⁾. Since π_i is a factor representation, it follows that for any $\theta' \in \pi_i$, satisfying the correspondence principle, $\pi_{\theta'}$ is quasi equivalent to π_θ . Property c) then follows from a well known theorem^{42,49)}. $\square$

3.4 Breaking of the Lorentz group in QED

The results of perturbation theory, the experience from non-relativistic field theory models and the general results proved in Section 2 and in Section 3.3 justify the assumption that charged states define generalized coherent state representations of the asymptotic algebra. See also the end of Section 3.5. In this Section we will discuss the Lorentz transformation properties of charged sectors under the

assumption that they are described by standard^{9,24,37)} generalized, coherent states. The transformation properties of the generalized coherent states have been discussed by Roepstorff⁴²⁾ who proved that the Lorentz transformations are not implementable in any non-Fock irreducible, coherent representation of the asymptotic (electromagnetic field) algebra $\mathcal{Q}^{as}$. Now, a one particle charged sector cannot yield an irreducible representation of $\mathcal{Q}^{as}$ since other observables, associated to the charged particle can be added which are in the commutant of $\pi(\mathcal{Q}^{as})$. In particular, the "electron" momentum commutes with $\mathcal{Q}^{as}$ (Theorem 3.2) and therefore the one electron representation can be decomposed over the momentum spectral projections.

Proposition 3.5. Let π be a scattering representation of $\mathcal{Q}^{as}$. Then

$$\pi(\mathcal{Q}^{as}) = \int_{\text{spec}(H_{as}^c, \vec{P}_{as}^c)}^{\oplus} d\mu(p) \pi_p(\mathcal{Q}^{as}) \quad (3.34)$$

where the direct integral refers to the spectral decomposition of the energy-momentum operator $(H_{as}^c, \vec{P}_{as}^c)$ of the charges, constructed in Theorem 3.2.

Proof. By Theorem 3.2 $(H_{as}^c, \vec{P}_{as}^c)$ generates an abelian von Neumann algebra in the commutant, $\pi(\mathcal{Q}^{as})'$, of $\mathcal{Q}^{as}$. The proposition now follows from standard results. $\square$

Next, let $\mathcal{K}^{(1)}$ be the space of scattering states with one charged (infra-) particle present, constructed in Proposition 3.4, that is

$$\mathcal{K}^{(1)} = \{ \Psi \in \mathcal{K}_{\text{phys}} : [(H_{as}^c)^2 - (\vec{P}_{as}^c)^2] \Psi = m^2 \Psi \}.$$

The spectrum of $\vec{P}^c$ on $\mathcal{K}^{(1)}$ is absolutely continuous (a well known consequence of Lorentz covariance). Let $\pi^{(1)}$ denote the representation of $\mathcal{Q}^{as}$ on $\mathcal{K}^{(1)}$. Then Proposition 3.5 yields.

Proposition 3.6.

$$\mathcal{K}^{(1)} = \int^{\oplus} d^3p \mathcal{K}_p^{(1)}, \quad \text{and}$$

$$\pi^{(1)} = \int^{\oplus} d^3p \pi_p^{(1)}, \quad (p^0 = \sqrt{\vec{p}^2 + m^2}).$$

In our opinion [see also⁸⁾], such a theory has a reasonable (charged) particle interpretation only if the representations $\pi_p^{(1)}$ are type I_∞ with the property that the centre, $\mathcal{Z}_p$, of $\pi_p^{(1)}(\mathcal{Q}^{as})''$ is finite [or countably infinite; describing the spectrum of the charge operator on $\mathcal{H}^{(1)}$]. Let

$$\pi_p^{(1)} = \bigoplus_n \pi_{p,n}^{(1)}$$

be the central decomposition. We assume, moreover, that $\pi_{p,m}^{(1)}(\mathcal{Q}^{as})' = M_{p,n}$, where $M_{p,n}$ is the algebra of all matrices on a finite dimensional Hilbert space (describing spin - and internal degrees of freedom of a charged particle). Thus, in this situation, $\pi_p^{(1)}$ can be decomposed into a countable sum of irreducible representations $\pi_{p,n}^{(1)}$, $n = 1, 2, \dots$, of finite multiplicity. By Takesaki theorem (Proposition 3.3),

$$\pi_{p,n}^{(1)} = \pi_F \circ \sigma_{p,n}, \quad (3.35)$$

where $\sigma_{p,n}$ is a *-automorphism of $\mathcal{Q}^{as}$. Using Lorentz covariance, one concludes that, for all $p \in \mathbb{R}^3$, $\mathcal{Z}_p$ and $M_{p,n}$ are isomorphic (i.e. independent of p). Thus

$$\pi^{(1)} = \bigoplus_n \left\{ \bigoplus_{m=1}^{\ell_n} \int d^3p \pi_F \circ \sigma_{p,n} \right\}, \quad (3.36)$$

with $\ell_n < \infty$.

Henceforth it is generally assumed that $\pi^{(1)}$ is of the form (3.36). We shall drop, for fixed n , the subscript n , writing σ_p for $\sigma_{p,n}$.

The standard, generalized coherent representations^{9,24)} are obtained when σ_p is given by

$$\sigma_p(a_\lambda(\vec{k})) = a_\lambda(\vec{k}) + ei \frac{p \cdot \varepsilon_\lambda(\vec{k})}{p \cdot k \sqrt{k_0}} W(\vec{k}), \quad (3.37)$$

where e is the electric charge $\varepsilon_\lambda(\vec{k})$, $\lambda = 1, 2$, are the polarization vectors, and W is a C_0^∞ function with $W(\vec{k}) = 1$, for $|\vec{k}| \leq K$; Eq. (3.37) can be rewritten as

$$\begin{aligned} \sigma_p(a_i(\vec{k})) &= a_i(\vec{k}) + ei \frac{p_j}{p \cdot k} \left(\delta_{ij} - \frac{k_i k_j}{k^2} \right) W(\vec{k}), \\ a_i(\vec{k}) &\equiv a_\lambda(\vec{k}) \varepsilon_i^\lambda(\vec{k}) / \sqrt{k_0}, \end{aligned} \quad (3.38)$$

(standard coherent state automorphism); (in section 3.5 we discuss a limit $W \rightarrow 1$).

Theorem 3.4. If the space of one electron states in QED is a separable Hilbert space, $\mathcal{H}^{(1)}$, formed by standard, generalized coherent states, then the Lorentz transformations are not implementable in $\mathcal{H}^{(1)}$.

Lemma 3.2. Let τ_Λ be the Lorentz automorphism defined on the asymptotic algebra $\mathcal{A}^{as}$ by

$$\tau_\Lambda (F_{\mu\nu}(x)) = \Lambda^\alpha_\mu \Lambda^\beta_\nu F_{\alpha\beta}(\Lambda^{-1}x), \quad (3.39)$$

$\forall \Lambda \in \mathcal{L}^+_{+}$, then the operators $a_i(k)$, (uniquely) defined in Proposition 3.2, transform in the following way

$$\tau_\Lambda (a_i(g^i)) = \Lambda^j_i a_j(g^i_\Lambda) - \frac{k_i}{k_0} \Lambda^k_0 a_k(g^i_\Lambda) \quad (3.40)$$

where $g^i_\Lambda(k) \equiv g^i(\Lambda k)$.

Proof of Lemma 3.2. We start by choosing $g^i(k) = k_0 f^i(k)$. Then, since

$$A_i(k_0 f^i(k)) = F_{0i}(f^i(k))$$

we have

$$\begin{aligned} \tau_\Lambda (A_i(g^i)) &= \Lambda^\alpha_0 \Lambda^\beta_i F_{\alpha\beta}(f^i_\Lambda) \\ &= \Lambda^l_i A_l(g^i_\Lambda) - \Lambda^k_0 A_k\left(\frac{k_i}{k_0} g^i_\Lambda\right) \end{aligned} \quad (3.41)$$

and by Proposition 3.2, Eq. (3.41) extends to all g^i in $\mathcal{S}(\mathbb{R}^4)$. $\square$

Proof of Theorem 3.4. We will show that

$$\tau_\Lambda \circ \sigma_P \circ \tau_\Lambda^{-1} = \gamma_\Lambda \circ \sigma_{\Lambda P} \quad (3.42)$$

where γ_Λ is the *automorphism defined by

$$\gamma_\Lambda (a_i(\vec{k})) = a_i(\vec{k}) - \sum_j e_i \frac{\Lambda^{-1}_{j0}}{(\Lambda k)_0} \left(\delta_{ij} - \frac{k_i k_j}{k_0^2} \right) W(\vec{k}). \quad (3.43)$$

Then, since the second term on the right-hand side of Eq. (3.43) is not square integrable on the light cone, γ_Λ is not implementable in any generalized coherent representation and therefore $\sigma_{\Lambda p}$ is not equivalent to $\tau_\Lambda \circ \sigma_p \circ \tau_\Lambda^{-1}$. Now, if τ_Λ is implementable in $\mathcal{H}^{(1)}$ by a unitary operator U_Λ and the energy momentum vector transforms covariantly under U_Λ

$$U_\Lambda (H, \vec{P}) U_\Lambda^{-1} = \Lambda (H, \vec{P})$$

(a property we always assume) then also $(H_{a_1}^c, \vec{P}_{a_1}^c)$ transforms covariantly under U_Λ , since so does $(H_{a_1}^{ph}, \vec{P}_{a_1}^{ph})$ by construction (Theorem 3.2). Therefore, by the decompositions (3.34) and (3.36) over the spectrum of $\vec{P}_{a_1}^c$, one must have

$$U_\Lambda : \mathcal{H}_{\pi_p} \rightarrow \mathcal{H}_{\pi_{\Lambda p}} \quad (3.44)$$

in contradiction with the inequivalence of $\sigma_{\Lambda p}$ and $\tau_\Lambda \circ \sigma_p \circ \tau_\Lambda^{-1}$.

Equation (3.42) is proved by computation. Using Lemma 3.2 one has

$$\begin{aligned} \tau_\Lambda \circ \sigma_p \circ \tau_\Lambda^{-1} (a_i(\vec{k})) &= a_i(\vec{k}) + \\ \sum_j \frac{ei}{p \cdot \Lambda^{-1} k} \left(\Lambda_i^\ell - \frac{k_i}{k_0} \Lambda_0^\ell \right) \left(p_\ell - \frac{(\Lambda^{-1} k)_j p_j (\Lambda^{-1} k)_\ell}{(\Lambda^{-1} k)_0^2} \right), \end{aligned} \quad (3.45)$$

whereas

$$\sigma_{\Lambda p} (a_i(\vec{k})) = a_i(\vec{k}) + \frac{ei}{\Lambda p \cdot k} \sum_j \left((\Lambda p)_i - \frac{(\Lambda p)_j k_j k_i}{k_0^2} \right). \quad (3.46)$$

The difference between the two is easily computed and shown to be of the form given in Eq. (3.43), by considering only the components orthogonal to $\vec{k}$. $\square$

The content of Theorem 3.4 has a very simple and basic physical interpretation. The automorphism σ_p describes the large wavelength behaviour of the radiation field associated to an electron with momentum p . Our assumption (3.37) is the statement that such low frequency behaviour is essentially described by semi-classical QED. Our computation in the proof of Theorem 3.4 is the comparison between the Lorentz transform of the low frequency radiation field associated to an electron of momentum p and the low frequency radiation field associated to an electron with momentum Λp . The two quantities differ by a radiation field whose infrared behaviour is singular as k^{-1} , as it must be by purely semi-classical considerations.

Theorem 3.4 has a straightforward generalization to all sectors with total charge different from zero. In the general case, γ_Λ depends in fact on the sum of the charges

$$\gamma_\Lambda(a_i(\vec{k})) = a_i(\vec{k}) - \frac{i \sum_n e_n}{(\Lambda \kappa)_0} \Lambda_j^{-1} \circ \left(\delta_{ij} - \frac{k_i k_j}{k_0^2} \right)$$

and therefore γ_Λ is the identity if the total charge is zero. No breaking of the Lorentz boosts is therefore present in the zero charge sector. This is consistent with Buchholz' results and the Lorentz invariance of the vacuum.

Clearly the non-implementability of the Lorentz transformations is related to the way charged states arise. The construction of such states involves an abstraction with respect to the actual physical situation in which, from an operational point of view, the creation of a charge q is always accompanied by the creation of a charge $-q$. In fact, one would like to have a description of the physics of one charge q , which is largely independent from what happened to the associated charge $-q$. This abstraction is at the origin of the breaking of boost invariance discussed above. Only the radiation field associated to the two charges $+q, -q$, is Lorentz covariant, whereas any description of one of the charges which disregards the other leads to a radiation field which depends on the reference frame in which the other (far separated) charge is at rest. The separation of the two charges given by the automorphism σ_p of Eq. (3.37) corresponds to choosing the frame in which the charge $-q$ is at rest and the automorphism γ_Λ describes just the radiation field of the Lorentz boosted charge $-q$.

Theorem 3.4 implies that in a positive metric formulation of QED satisfying our assumptions the Lorentz boost symmetry is spontaneously broken if the total charge is different from 0. One should keep in mind that this breaking cannot be experimentally proved by making measurements in any bounded region of space time (or by any apparatus which has a finite energy resolution), a consequence of the locally Fock property of the scattering representations (Theorem 3.1). This is, in fact, a general feature of spontaneously broken symmetries, since measurements done in bounded regions of space-time cannot ultimately decide whether one deals with a symmetric or a non-symmetric "phase". The choice between the two descriptions is ultimately a question of economy of the physical picture which one wants independent of the size of the space-time region considered.

Theorem 3.4 does not necessarily imply that no Lorentz covariant (indefinite metric) formulation of QED exists, since, in general, the Lorentz transformations are not described by bounded operators (see e.g. the free Gupta-Bleuler formulation) and therefore there is no guarantee that they define unitary operators on the physical space. It is actually possible that the Lorentz transformations defined on the dense set of local states, do not extend to the physical (non local) charged states.

Theorem 3.4 implies that, in a positive metric formulation, no covariant field exists which interpolates between the vacuum and the charged one-particle states. Therefore in the construction of asymptotic charged fields not only locality (Section 2), but also covariance in the physical space is lost.

3.5 Special gauge transformations and standard coherent states

The topic of this section is to derive the forms (3.37) and (3.38) of the *automorphisms σ_p of $\mathcal{Q}^{as}$, i.e. the property that the σ_p are the standard, generalized coherent "automorphisms" (or that charged one-particle states are generalized coherent states) from a covariance property of charged one-particle states under a class of special gauge transformations and a minimality condition. We then try to motivate these conditions dynamically using arguments based on lattice gauge theories. They make it plausible that the charged sectors in any gauge theory with unconfined charges cannot be invariant under Lorentz boosts.

Our first point is to show how to remove the ultraviolet cutoff K in the definition of standard coherent states [Eqs. (3.37) and (3.38)].

We fix a positive K_0 , $0 < K_0 < K < \infty$ and define $\sigma_p^{K_0}$ and $\sigma_p^{(K_0, K)}$ by Eqs. (3.37) and (3.38) with $W = W_{K_0}$ the characteristic function of $\{k \in \mathbb{R}^3: |\vec{k}| \leq K_0\}$ and $W = W_{K_0, K}$ the characteristic function of the shell $\{k \in \mathbb{R}^3: K_0 \leq |\vec{k}| \leq K\}$, respectively. $\mathcal{H}_p^{(1)}$ and $\sigma_{p, n} = \sigma_p$ are now still defined as in Proposition 3.6, Eq. (3.36), respectively, but with $\sigma_p = \sigma_p^{K_0}$.

Clearly $\sigma_p^{(K_0, K)}$ is unitarily implemented on $\mathcal{H}_p^{(1)}$ by

$$C_p^{(K_0, K)} = \exp \left[i e \sum_{\lambda=1}^2 \int \frac{d^3 k}{\sqrt{2|\vec{k}|}} W_{K_0, K}(\vec{k}) \left\{ \frac{p \cdot \epsilon_\lambda(\vec{k})}{p \cdot k} a_\lambda^*(\vec{k}) + h.c. \right\} \right] \quad (3.47)$$

We denote by $:C_p^{(K_0, K)}:$ the Wick ordered version of $C_p^{(K_0, K)}$; $C_p^{(K_0, K)} = e^{\frac{1}{2} K_p^{(K_0, K)}}$, where

$$K_p^{(K_0, K)} = \sum_{\lambda=1}^2 e^2 \int \frac{d^3 k}{2|\vec{k}|} W_{K_0, K}(\vec{k}) \left(\frac{p \cdot \epsilon_\lambda(\vec{k})}{p \cdot k} \right)^2 = O(\log K) \quad (3.48)$$

and by $\Omega_p^{K_0} \in \mathcal{H}_p^{(1)}$ the cyclic vector corresponding to $(\omega_0 \circ \sigma_p, \mathcal{Q}^{as})$, where ω_0 is a Fock vacuum state for $\mathcal{Q}^{as}$, by the GNS construction. We now introduce the vectors

$$\Psi_p^K(x, t) = e^{i(t\sqrt{\vec{p}^2 + m^2} - \vec{x} \cdot \vec{p})} e^{i(tH^{ph} - \vec{x} \cdot \vec{p}^{ph})} : C_p^{(K_0, K)} : \Omega_p^{K_0} \quad (3.49)$$

which belong to $\mathcal{H}_p^{(1)}$, and consider

$$\Psi_f^K(x,t) = \int^\oplus d^3p \Psi_p^K(x,t) f(\vec{p}) \in \mathcal{H}^{(1)}, \quad (3.50)$$

with f a C_0^∞ function in momentum space. Let g be a test function on configuration space whose Fourier transform, $\tilde{g}$, is C_0^∞ , and define

$$\begin{aligned} \Psi_f^K(g,t) &= \int d^3x g(x) \Psi_f^K(x,t) \\ &= e^{itH_{as}^{ph}} \Psi_{f_t}^K(g,0) \in \mathcal{H}^{(1)}, \end{aligned} \quad (3.51)$$

where $f_t(\vec{p}) = e^{it\sqrt{\vec{p}^2+m^2}} f(\vec{p})$.

Theorem 3.5. For f and g as specified above and $\Psi_f^K(g,t)$ given by (3.51), with K_0 large enough (depending on $\text{supp } f$),

$$s\text{-}\lim_{K \rightarrow \infty} \Psi_f^K(g,t) \equiv \Psi_f(g,t) \in \mathcal{H}^{(1)}$$

exists.

Proof. (See Appendix B).

Remarks: In perturbative calculations of scattering amplitudes in QED, using generalized coherent states, the cutoff function W introduced in (3.37), (3.38) is conventionally chosen to be $\equiv 1$ ^{9,24}. This choice is unjustified, because it gives rise to ultraviolet singularities in the asymptotic states which are incompatible with the space-time translation covariance of scattering states!

Theorem 3.5 shows that the choice $W \equiv 1$ is possible, provided the asymptotic localizations in configuration space of the charge and the cloud of soft photons are correlated and proper Wick ordering is taken into account. Thus Theorem 3.5 substantiates a part of the conventional wisdom concerning charged scattering states in QED, with the above proviso taken into account.

In this section, the main interest in Theorem 3.5 derives, however, from the fact that the states $\Psi_f^K(g,t)$ have remarkable covariance properties under certain gauge transformations and Lorentz boosts. We now derive those covariance properties, using the calculations presented in the proof of Theorem 3.4 and definitions (3.49) and (3.50).

For $K < \infty$, $\Psi_p^K(x,t)$ — see (3.49) — is a vector in $\mathcal{H}_p^{(1)}$, so that

$$\tilde{\Psi}_p^K(x,t) \equiv \left\| \Psi_p^K(x,t) \right\|_{\mathcal{H}_p^{(1)}}^{-1} \Psi_p^K(x,t)$$

is a unit vector in $\mathcal{H}_p^{(1)}$; [this just undoes the Wick ordering; see (3.48)].

Next, we note that

$$\omega_p^{(x,t)}(A) = \lim_{K \rightarrow \infty} (\tilde{\Psi}_p^K(x,t), A \tilde{\Psi}_p^K(x,t))_{\mathcal{H}_p^{(1)}} \quad (3.52)$$

exists, for all $A \in \mathcal{Q}^{as}$, so that

$$\begin{aligned} \omega_f^{(x,t)}(A) &= \lim_{K \rightarrow \infty} (\tilde{\Psi}_f^K(x,t), A \tilde{\Psi}_f^K(x,t))_{\mathcal{H}_p^{(1)}} \\ &\equiv \lim_{K \rightarrow \infty} \int d^3p |f(\vec{p})|^2 (\tilde{\Psi}_p^K(x,t), A \tilde{\Psi}_p^K(x,t))_{\mathcal{H}_p^{(1)}} \\ &= \int d^3p |f(\vec{p})|^2 \omega_p^{(x,t)}(A) \end{aligned} \quad (3.53)$$

exists, for all $A \in \mathcal{Q}^{as}$.

We define $\gamma_\Lambda^{(x,t)}$ [see Eq. (3.43)] by

$$\begin{aligned} \gamma_\Lambda^{(x,t)}(a_i(\vec{k})) &= a_i(\vec{k}) - \\ &- ei \frac{\Lambda^{-1} \delta}{(\Lambda k)_0} \left(\delta_{ij} - \frac{k_i k_j}{k_0^2} \right) e^{i(t k_0 - \vec{x} \cdot \vec{k})} \end{aligned} \quad (3.54)$$

Using the calculations presented in the proof of Theorem 3.4 and taking the limit $K \rightarrow \infty$ we find

$$\omega_p^{\Lambda^{-1}(x,t)}(\tau_\Lambda^{-1}(A)) = \omega_{\Lambda p}^{(x,t)}(\gamma_\Lambda^{(x,t)}(A)), \quad (3.55)$$

for all $p = (\sqrt{\vec{p}^2 + m^2}, \vec{p}) \in V_+^m$, all $\Lambda \in L_+$ and all $A \in \mathcal{Q}^{as}$. Thus

$$\omega_f^{\Lambda^{-1}(x,t)}(\tau_\Lambda^{-1}(A)) = \omega_{f_\Lambda}^{(x,t)}(\gamma_\Lambda^{(x,t)}(A)), \quad (3.56)$$

for all test functions f on the mass shell V_+^m , all Λ and all $A \in \mathcal{Q}^{as}$.

In a physicist's language, Eq. (3.59) says the following, (which, in fact, has a precise mathematical meaning within the framework of von Neumann's infinite direct product representations of $\mathcal{Q}^{as}$, i.e. of the Weyl relations): Let $\tilde{\Psi}_f(x,t)$ be the cyclic vector corresponding to $(\omega_f^{(x,t)}, \mathcal{Q}^{as})$, let $U(\Lambda)$ be the "unitary" operator (on the infinite tensor product space) implementing τ_Λ , let $\exp i\alpha_\Lambda(x,t)$ be the "unitary" operator implementing $\gamma_\Lambda^{(x,t)}$ and let $S(\cdot)$ be the appropriate, finite dimensional representation of $L_+^\uparrow$ which acts on the spinor indices of $\tilde{\Psi}_f(x,t)$, i.e. a spin $\frac{1}{2}$ representation in the case of electrons and positrons, and commutes with all elements of $\mathcal{Q}^{as}$. Then

$$U(\Lambda) \tilde{\Psi}_f(\Lambda^{-1}(x,t)) = S(\Lambda) e^{-i\alpha_\Lambda(x,t)} \tilde{\Psi}_{f_\Lambda}(x,t) \quad (3.57)$$

so that, formally,

$$U(\Lambda) \Psi_f(q,t) = S(\Lambda) \int d^3x g(x) e^{-i\alpha_\Lambda(\Lambda(x,t))} \Psi_{f_\Lambda}(\Lambda(x,t)),$$

(an equation that can be made precise by means of the limiting argument $K \rightarrow \infty$).

Next, we notice that $\alpha_\Lambda(x,t)$ is precisely the generator of the gauge transformation which, after a formal Lorentz transformation of the electromagnetic vector potential $A^\mu(x,t)$ in the Coulomb gauge restores the Coulomb gauge, i.e.

$$A_\Lambda^\mu(\Lambda^{-1}(x,t)) = \Lambda^\mu_\nu A^\nu(x,t) - \frac{1}{e} \partial^\mu \alpha_\Lambda(x,t) \quad (3.58)$$

where both, A and A_Λ , are in the Coulomb gauge. Thus Eqs. (3.55) to (3.57) show that the covariance properties of $\Psi_f(x,t)$ under the special gauge transformations generated by $\alpha_\Lambda(x,t)$ are exactly the ones of a generalized vector carrying charge $-e$, localized at (x,t) .

It goes without saying that there ought to be a deep, dynamical reason for those covariance properties. Before speculating about such a dynamical origin we show that, when the covariance properties (3.55) to (3.57) are combined with a rather plausible minimality condition, they are equivalent to $\omega_p^{(x,t)}$ being the standard generalized coherent states.

For this purpose we consider the general situation where charged infra-particle states (such as constructed in Proposition 3.4) can be localized in configuration spaces in the sense that they are given in terms of a family of states, $\phi_\mu^{(x,t)}$, on $\mathcal{Q}^{as}$ with

$$\phi_{\mu}^{(x,t)}(A) = \int d\mu(p) \phi_p^{(x,t)}(A), \quad A \in \mathcal{Q}^{as} \quad (3.59)$$

where, by the Lorentz covariance of $(H_{a_3}^C, \vec{P}_{a_3}^C)$, $d\mu(p)$ is given by $|f(p)|^2 d\Omega_m(p)$, $f \in L^2(V_+^m, d\Omega_m)$, with $d\Omega_m$ the invariant measure on V_+^m , and the states $\phi_p^{(x,t)}$ are assumed to have the following two properties:

$$(M) \quad \phi_{p=0}^{(x,t)} = \phi_0^{(x,t)}$$

is a state given by a vector

$$\Phi_0(x,t) = e^{i(tH_{a_3}^{ph} - \vec{x} \cdot \vec{P}_{a_3}^{ph})} \Phi_0$$

in the Fock space $\mathcal{F}$ for $\mathcal{Q}^{as}$,

$$\phi_0^{(x,t)}(A) = (\Phi_0(x,t), A \Phi_0(x,t))_{\mathcal{F}}, \quad A \in \mathcal{Q}^{as}, \quad (3.60)$$

i.e. the state $\phi_{p=0}^{(x,t)}$ corresponding to one charged infra-particle at rest can be chosen such that, with probability 1, it contains only finitely many photons.

(C) For all $p \in V_+^m$, $\phi_p^{(x,t)}$ has the following covariance properties under Lorentz transformations:

$$\phi_p^{\Lambda^{-1}(x,t)}(\tau_{\Lambda}^{-1}(A)) = \phi_{\Lambda p}^{(x,t)}(\gamma_{\Lambda}^{(x,t)}(A)) \quad (3.61)$$

for all $A \in \mathcal{Q}^{as}$, where $\gamma_{\Lambda}^{(x,t)}$ is given by (3.57). Equivalently,

$$(C') \quad \phi_f^{\Lambda^{-1}(x,t)}(\tau_{\Lambda}^{-1}(A)) = \phi_{f_{\Lambda}}^{(x,t)}(\gamma_{\Lambda}^{(x,t)}(A)) \quad (3.62)$$

for arbitrary $f \in L^2(V_+^m, d\Omega_m)$, $A \in \mathcal{Q}^{as}$, with $\phi_f^{(x,t)} \equiv \phi_{\mu=|f|^2 d\Omega_m}^{(x,t)}$; (i.e. $\phi_f^{(x,t)}$ transforms like a state of charge $-e$).

We now consider the states $\phi_p \circ \sigma_p^{-1}$, $p \in V_+^m$, where $\phi_p = \phi_p^{(x,t)=(0,0)}$, and σ_p is the $*$ automorphism defined in (3.37) and (3.38) with $W \equiv 1$.

Clearly $\phi_p \circ \sigma_p^{-1}$ is still a well-defined state on $\mathcal{Q}^{as}$. We propose to compute its covariance properties under Lorentz transformations:

$$\begin{aligned}
 \phi_p \circ \sigma_p^{-1} (\tau_\Lambda^{-1}(A)) &= \phi_p \circ \tau_\Lambda^{-1} \circ (\tau_\Lambda \circ \sigma_p^{-1} \circ \tau_\Lambda^{-1})(A) \\
 &= \phi_{\Lambda p} \circ \gamma_\Lambda \circ (\tau_\Lambda \circ \sigma_p^{-1} \circ \tau_\Lambda^{-1})(A) \\
 &= \phi_{\Lambda p} \circ \gamma_\Lambda \circ (\gamma_\Lambda^{-1} \circ \sigma_{\Lambda p}^{-1})(A) \\
 &= \phi_{\Lambda p} \circ \sigma_{\Lambda p}^{-1}(A)
 \end{aligned} \tag{3.63}$$

Now, given $p \in V_+^m$, we choose $\Lambda = \Lambda_p$ such that $\Lambda_p \vec{p} = 0$. We also set $A = \tau_{\Lambda_p}(B)$, $B \in \mathcal{Q}^{as}$. Then

$$\begin{aligned}
 \phi_p \circ \sigma_p^{-1} (\tau_{\Lambda_p}^{-1}(A)) &= \phi_{p=0} \circ \sigma_{p=0}^{-1}(A) \\
 &= (\Phi_0, A \Phi_0), \quad \Phi_0 \in \mathcal{F},
 \end{aligned}$$

by Eq. (3.63), condition (M) and definition (3.37) of σ_p . Hence

$$\phi_p \circ \sigma_p^{-1}(B) = (U(\Lambda_p) \Phi_0, B U(\Lambda_p) \Phi_0)_{\mathcal{F}}$$

i.e.,

$$\phi_p(B) = (U(\Lambda_p) \Phi_0, \sigma_p(B) U(\Lambda_p) \Phi_0)_{\mathcal{F}}.$$

Clearly $U(\Lambda_p) \Phi_0 \in \mathcal{F}$, so ϕ_p determines the standard generalized coherent representation of $\mathcal{Q}^{as}$, and $\phi_p = \omega_0 \circ \sigma_p$ if $\Phi_0 = \Omega_0$ is the Fock vacuum. Thus we have proven.

Theorem 3.6. Suppose that:

$$(M) \quad \phi_{p=0}(A) = (\Phi_0, A \Phi_0)_{\mathcal{F}}, \quad \Phi_0 \in \mathcal{F},$$

$$(C) \quad \phi_p^{\Lambda^{-1}(x,t)} (\tau_\Lambda^{-1}(A)) = \phi_{\Lambda p}^{(x,t)} (\gamma_\Lambda^{(x,t)}(A))$$

for all $A \in \mathcal{Q}^{as}$. Then

$$\phi_p^{(x,t)}(A) = (\Phi_p, \sigma_p(\tau_{(x,t)}(A)) \Phi_p)_J$$

with $\Phi_p = U(\Lambda_p) \Phi_0$, $\Phi_0 \in \mathcal{F}$, and σ_p the standard, generalized coherent *automorphism of $\mathcal{A}^{as}$ introduced in Eqs. (3.37) and (3.38).

Remarks: We recall that Theorem 3.5 and Eqs. (3.55) to (3.57) prove the converse of Theorem 3.6.

Finally we wish to add some speculative comments on the dynamical origin of the breaking of the Lorentz boost symmetry on charged sectors and the covariance property (C) of charged one - (infra) particle states. In the subsequent discussion we are guided by experience with lattice gauge theories in which the formal objects considered below can be given a precise mathematical meaning^{20,27}). Furthermore we insist on a gauge-invariant formalism. We consider a space-like surface Σ and two points $\xi = (x,t)$ and $\eta = (y,s)$ on Σ . Let $B_{\xi,\eta}$ be the space of all paths in Σ starting at ξ and ending at η . Such paths are denoted $b_{\xi,\eta}$. Let $d\mu_{\xi,\eta}(b_{\xi,\eta})$ be some finite, complex measure on $B_{\xi,\eta}$. Let, moreover, $\bar{\Psi}(\xi)$ and $\Psi(\eta)$ be formal, interacting, charged Dirac fields.

We introduce the formal, bilocal, formally gauge-invariant "field" (known to exist in lattice gauge theories)

$$T_{\Sigma}(\mu_{\xi,\eta}) \equiv \int_{B_{\xi,\eta}} d\mu_{\xi,\eta}(b_{\xi,\eta}) \bar{\Psi}(\xi) \exp \left[ie \int_{b_{\xi,\eta}} A^{\mu}(x) dx_{\mu} \right] \Psi(\eta).$$

We now assume that, for some Lorentz invariant family of space-like surfaces Σ and, given Σ , suitable sequences of measures $d\mu_{\xi,\eta}$ on $B_{\xi,\eta}$, the limit

$$\omega^* \lim_{\substack{\eta \rightarrow \infty \\ \eta \in \Sigma}} T_{\Sigma}(\mu_{\xi,\eta}) \equiv T_{\Sigma}(\mu_{\xi,\infty}) \quad (3.64)$$

exists in the sense that, after smearing in ξ , the r.h.s. of Eq. (3.64) intertwines the vacuum representation of the algebra generated by the interacting electromagnetic field with a charge $+e$ representation [i.e. $T_{\Sigma}(\mu_{\xi,\infty})$ is a "charged field bundle"].

Then we may define the vector

$$\theta_{\Sigma}(\mu_{\infty}; f) = \int_{\Sigma} d\xi f(\xi) T_{\Sigma}(\mu_{\xi,\infty}) \Omega \quad (3.65)$$

where $\Omega \in \mathcal{H}_{\text{phys}}$ is the physical vacuum, and f belongs to a space of test functions on Σ . One checks easily that if a limit (3.64) and a vector $\theta_{\Sigma}(\mu_{\infty}; f)$ as in (3.65) exist and the charge operator Q is obtained as a weak limit of local charges, Q_R , when $R \rightarrow \infty$, as investigated in Section 2, then indeed

$$Q \theta_{\Sigma}(\mu_{\infty}; f) = e \theta_{\Sigma}(\mu_{\infty}; f), \quad (3.66)$$

i.e. $\theta(\mu_{\infty}; f)$ is a vector of charge $+e$. Of course one would expect that the vector $\theta_{\Sigma}(\mu_{\infty}; f)$ depends crucially on the choice of Σ and, given Σ , on the family of measures $\mu_{\xi, \infty}$. One can at least argue what the localization properties of the measures $\mu_{\xi, \infty}$ ought to be: For abelian lattice gauge theories one can show that when the measures $\mu_{\xi, \eta}$ are supported on paths in a strictly localized tube connecting ξ and η the diameter of which is bounded uniformly in η (with ξ fixed) then the mean energy of the vector $T_{\Sigma}(\mu_{\xi, \eta})\Omega$ tends to ∞ , linearly in the distance between ξ and η , and smearing in ξ (and η) does not improve the situation²⁵⁾. Thus, in the limit $\eta \rightarrow \infty$, we expect that the paths in the support of $\mu_{\xi, \infty}$ "cover" all of Σ . This suggests that the vector $\theta_{\Sigma}(\mu_{\infty}; f)$ cannot be Lorentz boost covariant. Let Λ be a Lorentz boost. For an observer in a frame boosted by Λ^{-1} the state $\theta_{\Sigma}(\mu_{\infty}; f)$ looks like the state $\theta_{\Lambda\Sigma}(\mu_{\infty, \Lambda}; f_{\Lambda})$ for the observer in the rest frame. Since the space-like surfaces Σ and $\Lambda\Sigma$ are separated from each other the more the larger the distances, the two states $\theta_{\Sigma}(\mu_{\infty}; f)$ and $\theta_{\Lambda\Sigma}(\mu_{\infty, \Lambda}; f_{\Lambda})$ have little chance to determine unitarily equivalent representations of the algebra of local, gauge-invariant observables, although they have both total charge $+e$. Thus the formal picture sketched here suggests that at least the Lorentz boosts will not leave the charged sectors invariant, as they change the boundary conditions (the state of the charge distribution) at ∞ completely. This conclusion appears to be quite unavoidable in a gauge-invariant formalism.

Going back to the formal objects introduced in (3.64) and (3.65) and applying an even more formal, naive version of a strong convergence asymptotic condition (in the Coulomb gauge), to the vectors $\theta_{\Sigma}(\mu_{\infty}; f)$ -- see also Proposition 3.4 -- we argue that the asymptotic limit of $\theta_{\Sigma}(\mu_{\infty}; f)$ ought to have precisely the boost covariance properties hypothesized in (C) of Theorem 3.6. Since we have no rigorous arguments to offer at this point we feel it is unnecessary to present the formal manipulations on which our conjectures are based, but we note that, at least, the hypotheses and conclusions of Section 3.4 are not in conflict with the general wisdom extracted from lattice gauge theories.

3.6 The centre of the von Neumann algebra generated by $\mathcal{Q}^{\text{as}}$ in the physical representation

In this section we report a result of¹⁷⁾ (quoted in Ref. 19) which says that, under some rather plausible hypotheses, the weak closure of the algebra $\mathcal{Q}^{\text{as}}$ generated by the bounded functions of the asymptotic, electromagnetic field on the total Hilbert space $\mathcal{H}_{\text{phys}}$ is not a factor, but a type I_{∞} von Neumann algebra with a large centre, and that this centre consists of all bounded functions of the momenta of the asymptotic, charged infra-particles. In particular, the probability distribution of the momenta of the asymptotic, charged infra-particles in a scattering state is, in principle, determined by the ensemble of all measurements of the asymptotic, electromagnetic field alone. Our hypotheses from which we derive this conclusion are in agreement with the general folklore of^{9,12,13,24)} and the results of the previous section.

Let $\pi^{(1)}$ denote the representation of $\mathcal{Q}^{\text{as}}$ on the "one-particle sector"

$$\mathcal{H}^{(1)} = \{ \Psi \in \mathcal{H}_{\text{phys}} : [(\mathcal{H}_{\text{as}}^c)^2 - (\vec{\mathcal{P}}_{\text{as}}^c)^2] \Psi = m^2 \Psi \} .$$

We assume that

$$\pi^{(1)} = \int^{\oplus} d^3p \, \pi_p^{(1)} , \quad (3.67)$$

with

$$\pi_p^{(1)} = \pi_F \circ \sigma_{p,1} \oplus \pi_F \circ \sigma_{p,2}$$

where $\sigma_{p,1} \equiv \sigma_p$ and $\sigma_{p,2}$ are *-automorphisms of $\mathcal{Q}^{\text{as}}$, as discussed in Section 3.4, Proposition 3.6. Charge conjugation invariance says that

$$\sigma_{p,2} = \sigma_p^{-1}$$

The situation considered here is expected in a theory of electrons and positrons interacting with the quantized radiation field, and no other charged particles present. But the results reported below extend to the more general situation described after Proposition 3.6.

We need a few definitions: Let

$$(p)_{n_+} = (p_1, \dots, p_n) , \quad p_j \in V_+^m , \quad j = 1, \dots, n_+ ,$$

$$(q)_{n_-} = (q_1, \dots, q_n) , \quad q_j \in V_+^m , \quad j = 1, \dots, n_- ,$$

denote the momenta of n_+ "particles" of charge $+e$ and n_- "particles" of charge $-e$,

$$d(p)_{n_{\pm}} = \prod_{j=1}^{n_{\pm}} d^3 p_j ,$$

and

$$\sigma_{(p)_{n_+}(q)_{n_-}} \equiv \sigma_{p_1} \circ \dots \circ \sigma_{p_{n_+}} \circ \sigma_{q_1}^{-1} \circ \dots \circ \sigma_{q_{n_-}}^{-1} . \quad (3.68)$$

We now make the ansatz (somewhat stronger than necessary) that σ_p is the standard, generalized coherent *automorphism defined in Eqs. (3.37) and (3.38) [with some $W \in \mathcal{S}(\mathbb{R}^3)$, for simplicity] and that the representation π of $\mathcal{Q}^{as}$ on the total Hilbert $\mathcal{H}_{phys}$ is quasi-equivalent to the representation

$$\pi_0 = \bigoplus_{n_+, n_- = 0}^{\infty} \int^{\oplus} d(p)_{n_+} d(q)_{n_-} \pi_F \circ \sigma_{(p)_{n_+}(q)_{n_-}} \quad (3.69)$$

The first part of this ansatz concerning σ_p is quite well motivated by the results of Section 3.5. Eq. (3.69) is in accordance with the general folklore concerning the structure of scattering states in QED [assuming asymptotic completeness and absence of stable, charged bound states^{17,24}]. The multiplicities in π relative to π_0 are supposed to arise from the spin and internal degrees of freedom of the charged particles and the degrees of freedom of neutral particles. Given n_+ and n_- , they are independent of $(p)_{n_+}$ and $(q)_{n_-}$.

Lemma 3.5.

Assume that σ_p is given by Eqs. (3.37) and (3.38). Then the automorphism $\sigma_{(p)_{n_+}(q)_{n_-}}$ is independent of the order of the factors on the r.h.s. of Eq. (3.68). Furthermore, the condition that $\sigma_{(p)_{n_+}(q)_{n_-}}$ be unitarily implementable on Fock space implies that, almost surely with respect to the measure $\bigoplus_{n_-=0}^{\infty} d(q)_{n_-}$, $(q)_{n_-}$ is a permutation of $(p)_{n_+}$, in particular $n_+ = n_-$.

Remark: Clearly, Lemma 3.5 implies that if

$$\pi_F \circ \sigma_{(p)_{n_+}(q)_{n_-}} \cong \pi_F \circ \sigma_{(\tilde{p})_{\tilde{n}_+}(\tilde{q})_{\tilde{n}_-}}$$

then, almost surely with respect to $\bigoplus_{\tilde{n}_+, \tilde{n}_-=0}^{\infty} d(\tilde{p})_{\tilde{n}_+} d(\tilde{q})_{\tilde{n}_-}$, $(\tilde{p})_{\tilde{n}_+}$ is a permutation of $(p)_{n_+}$, $(\tilde{q})_{\tilde{n}_-}$ one of $(q)_{n_-}$ and $\tilde{n}_{\pm} = n_{\pm}$.

Proof: The independence of $\sigma_{(p)_{n_+} (q)_{n_-}}$ of the order of the factors on the r.h.s. of Eq. (3.68) follows directly from the definition (3.37) and (3.38) of σ_p . By Eqs. (3.37), (3.38) and (3.68), $\sigma_{(p)_{n_+} (q)_{n_-}}$ is unitarily implementable on Fock space if and only if the function

$$e \left\{ \sum_{j=1}^{n_+} \frac{p_j \cdot \varepsilon_\lambda(\vec{k})}{p_j \cdot k} - \sum_{l=1}^{n_-} \frac{q_l \cdot \varepsilon_\lambda(\vec{k})}{q_l \cdot k} \right\} W(\vec{k})$$

is square integrable with respect to the invariant measure on the light cone $d\Omega(k) = (2|k|)^{-1} d^3k$ ⁴⁸⁾. Since $W \in \mathcal{J}(R^3)$ with $W(0) = 1$, this -- it is easy to check -- is equivalent to $(q)_{n_-}$ being a permutation of $(p)_{n_+}$, almost surely with respect to $\bigoplus_{n_-=0}^{\infty} d(q)_{n_-}$, and hence $n_+ = n_-$. $\square$

Let $\mathcal{J}$ be the abelian algebra defined by

$$\mathcal{J} = \bigoplus_{n_+, n_- = 0}^{\infty} L^{\infty}(R^{3(n_+ + n_-)}) \quad (3.70)$$

Corollary 3.1:

The ansatz (3.69) and the hypotheses of Lemma 3.5 imply that the centre of the von Neumann algebra $\pi(\mathcal{Q}^{as})''$ is isomorphic to $\mathcal{J}$.

Corollary 3.2:

Under the same hypotheses and with $\mathcal{J}$ interpreted as the algebra generated by the momenta of the asymptotic, charged infra-particles we have:

- 1) The energy-momentum operator of the charged (infra-) particles is affiliated with $\pi(\mathcal{Q}^{as})''$.
- 2) Given a vector $\Psi \in \mathcal{H}_{phys}$, the state $(\Psi, \cdot \Psi)$ on $\mathcal{Q}^{as}$ is given by

$$\bigoplus_{n_+, n_- = 0}^{\infty} \int^{\oplus} d(p)_{n_+} d(q)_{n_-} \rho((p)_{n_+}, (q)_{n_-}) \omega_{(p)_{n_+} (q)_{n_-}}(\cdot),$$

where $\omega_{(p)_{n_+} (q)_{n_-}}$ is a normalized density matrix for

$$\pi_F \circ \sigma_{(p)_{n_+} (q)_{n_-}}(\mathcal{Q}^{as})'', \quad \rho((p)_{n_+}, (q)_{n_-}) \geq 0,$$

for all n_+, n_- and

$$\sum_{n_+, n_-} \int d(p)_{n_+} d(q)_{n_-} \rho((p)_{n_+}, (q)_{n_-}) = 1.$$

Moreover $\{\rho((p)_{n_+}, (q)_{n_-})\}_{n_+, n_- = 0}^{\infty}$ is uniquely determined (in measure) by

$$\{(\Psi, A\Psi) : A \in \mathcal{Q}^{\infty}\}.$$

Remark: Corollaries 3.1 and 3.2 follow from Lemma 3.5 by standard results⁴⁷⁾.

The results reported here show that the conventional folklore concerning the scattering states in QED implies that the configuration of momenta of the charged (infra-) particles is determined, in each state, by the configuration of the asymptotic electromagnetic field, in the sense made precise in the last part of Corollary 3.2. Assuming this property for both, the in -- and the out -- electromagnetic fields, i.e. for the representations of $\mathcal{Q}^+$ and $\mathcal{Q}^-$ on $\mathcal{H}_{\text{phys}}$, the scattering amplitudes -- summed over polarizations of the charged particles (and all parameters of neutral particles) -- can be reconstructed from measurements of the electromagnetic in -- and out -- fields, i.e. an extension of Buchholz' collision theory¹⁹⁾ for the electromagnetic field to all charged sectors would solve a major part of the problem of constructing a complete collision theory for QED.

Needless to say that one ought to find general, dynamical reasons for the hypotheses on which the above conclusions are based (even when they do not appear practical for explicit calculations!)

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APPENDIX A

Proof of Proposition 2.2

We have to estimate

$$\|Q_R \Psi_0\|^2 = \int F_{\alpha\beta}(x-y) \partial^\alpha \partial^\beta \partial_0^2 f\left(\frac{|\vec{x}|}{R}\right) f\left(\frac{|\vec{y}|}{R}\right) \alpha(x_0) \alpha(y_0) d^4x d^4y \quad (A.1)$$

where we have used the unitarity of the space-time translations.

We first consider the case in which α and β are both equal to zero or both range from 1 to 3. By a change of variables the above integral can be written as

$$R^3 \left(\int_{D_1} + \int_{D_2} \right) F_{\alpha\beta}(\vec{z}, z_0) \Delta^{\alpha\beta} \Delta g\left(\frac{\vec{z}}{R}, z_0\right) d^4z$$

where $D_1 \equiv \{z \mid |\vec{z}| < R_0, |z_0| < 2d\}$, $D_2 \equiv \{z \mid |\vec{z}| > R_0, |z_0| < 2d\}$, R_0 is sufficiently large so that the bound (15) hold on D_2 ,

$$g\left(\frac{\vec{z}}{R}, z_0\right) \equiv (f \star f)\left(\frac{|\vec{z}|}{R}\right) (\alpha \star \alpha)(z_0)$$

and $\Delta_{\alpha\beta} = \Delta$ if $\alpha=\beta=0$ and $\Delta_{\alpha\beta} = \partial_\alpha \partial_\beta$ otherwise.

The integral on D_1 is bounded by a constant C times R^{-3} , since

$$(\Delta_{\alpha\beta} \Delta)_{\vec{z}, z_0} = R^{-3} (\Delta_{\alpha\beta} \Delta)_{\vec{z}/R, z_0}.$$

On the other hand, by using condition (15) on D_2 we can bound the integral on D_2 by

$$\begin{aligned} R^{-1} \left| \int_{D_2} F_{\alpha\beta}(\vec{z}, z_0) (\Delta_{\alpha\beta} \Delta)_{\vec{z}/R, z_0} g\left(\frac{\vec{z}}{R}, z_0\right) d^4z \right| &\leq \\ &\leq R^{-1} \left| \int_{D_2} |\vec{z}|^{-2} (\Delta_{\alpha\beta} \Delta g)\left(\frac{\vec{z}}{R}, z_0\right) d^4z \right| \end{aligned}$$

$$\leq R^{-1} \sup |\Delta_{\alpha\beta} \Delta g| \int_{D_2 \cap \text{supp } g} |\vec{z}|^{-2} d^4z \leq L.$$

It remains to estimate the terms (no sum over i)

$$\int [F_{0i}(x-y) \partial_0^x \partial_i^y + F_{i0}(x-y) \partial_0^y \partial_i^x] f\left(\frac{|\vec{x}|}{R}\right) f\left(\frac{|\vec{y}|}{R}\right) \alpha(x_0) \alpha(y_0) d^4x d^4y$$

$$= (A_0(\partial_0 h) \Psi_0, A_i(\partial_i h) \Psi_0) + (A_i(\partial_i h) \Psi_0, A_0(\partial_0 h) \Psi_0).$$

By exploiting the positivity of the two point function $\langle A(f)\Psi_0, A(g)\Psi_0 \rangle$:

$$\| [A_0(\partial_0 h) - A_i(\partial_i h)] \Psi_0 \|^2 \geq 0$$

we obtain

$$| (A_0(\partial_0 h) \Psi_0, A_i(\partial_i h) \Psi_0) + (A_i(\partial_i h) \Psi_0, A_0(\partial_0 h) \Psi_0) | \leq$$

$$\leq (A_0(\partial_0 h) \Psi_0, A_0(\partial_0 h) \Psi_0) + (A_i(\partial_i h) \Psi_0, A_i(\partial_i h) \Psi_0)$$

and therefore the integral (A.1) is bounded by a constant independent of R.

Remark . It is worthwhile to notice that condition (2.15) involves the properties of the metric operator η , since $A_\mu^*(f) = \eta A_\mu(f) \eta$ and therefore it may depend on the way a Hilbert space structure is associated to the Wightman functions^{*)}.

*) The relevance of the Hilbert space structure associated to a given set of Wightman functions becomes very evident in two-dimensional QED where one can show that different Hilbert space structures lead to physically different theories³⁷⁾.

APPENDIX B

Proof of Theorem 3.5

Proof: By Eq. (3.51) it is enough to show that

$$s\text{-}\lim_{K \rightarrow \infty} \Psi_f^K(q, 0) \equiv \Psi_f(q, 0) \in \mathcal{H}^{(1)}$$

exists. Clearly this follows from the existence of the limit of

$$(\Psi_f^K(q, 0), \Psi_f^{K'}(q, 0)) \quad (B.1)$$

as $K \rightarrow \infty$, with $K \leq K' < \infty$.

Let $K_0 < \infty$ be fixed; (a choice of K_0 , depending on $\text{supp } f$, will be made later). Let $\{E(\Delta): \Delta \text{ Borel sets in } \mathbb{R}^3\}$ be the spectral projections of $\vec{P}_{a_3}^{ph}$. By Eqs. (3.49) to (3.51),

$$\begin{aligned} (\Psi_f^K(q, 0), \Psi_f^{K'}(q, 0)) &= \int d^3p |f(\vec{p})|^2 \int d(\Omega_p^{K_0}, E(q) \Omega_p^{K_0})_{\mathcal{H}_p^{(1)}} \times \\ &\times \left(: C_p^{(K_0, K)} : \Omega_0, |\tilde{g}(\vec{p} + \vec{P}_{a_3}^{ph} + \vec{q})|^2 : C_p^{(K_0, K')} : \Omega_0 \right) \quad (B.2) \end{aligned}$$

where $\Omega_0 \in \mathcal{F}$ is the Fock vacuum.

Lemma 3.3:

The measure $d(\Omega_p^{K_0}, E(q) \Omega_p^{K_0})$ is of rapid fall off, i.e. for arbitrary $p \in \mathbb{R}^3$ and all $K_0 < \infty$, for arbitrary $N = 0, 1, 2, \dots$,

$$(|\vec{q}|^2 + 1)^N d(\Omega_p^{K_0}, E(q) \Omega_p^{K_0})_{\mathcal{H}_p^{(1)}}$$

is a finite measure.

Remark: It is not hard to show that, in fact,

$$d(\Omega_p^{K_0}, E(q) \Omega_p^{K_0})_{\mathcal{H}_p^{(1)}} \leq K_{N, p, K_0} [|\vec{q}|^2 + 1]^{-N} d^3q$$

for some $K_{N, p, K_0} < \infty$, but we do not need this inequality.

Proof of Lemma 3.3: By a simple explicit computation with coherent states one verifies that

$$(\Omega_p^{K_0}, |\vec{p}_{as}^{ph}|^{2N} \Omega_p^{K_0}) < \infty$$

for all $N < \infty$; (note that $\sigma_p(\vec{p}_{as}^{ph})$ can be computed explicitly. This reduces the proof to a standard computation on Fock space). From the above inequality and the spectral theorem the lemma follows. $\square$

Next we investigate the second factor on the r.h.s. of (B.2). The operators $:C_p^{(K_0, K)}:$, $:C_p^{(K_0, K')}:$ are Wick ordered Weyl operators on the Fock space $\mathcal{F}$.

Thus $:C_p^{(K_0, K)}: \Omega_0$ and $:C_p^{(K_0, K')}: \Omega_0$ have a strongly convergent expansion in vectors with a fixed number, n , of photons. From the definition (3.47) and (3.48) of $:C_p^{(K_0, K)}:$ and this expansion we infer the straightforward estimate

$$\begin{aligned} & (:C_p^{(K_0, K)}: \Omega_0, |\tilde{g}(\vec{p} + \vec{p}_{as}^{ph} + \vec{q})|^2 :C_p^{(K_0, K')}: \Omega_0) \\ & \leq \sum_{n=0}^{\infty} \frac{(e^2 K (\vec{p}^2 + m^2) m^2)^n}{n!} \int \prod_{i=1}^n \frac{d^3 k_i}{(|\vec{k}_i|^2 + 1)^{3/2}} |\tilde{g}(\vec{p} + \sum_{i=1}^n \vec{k}_i + \vec{q})|^2 \quad (B.3) \\ & \quad K_0 \leq |\vec{k}_i| \leq K \end{aligned}$$

On the r.h.s. of Eq. (B.3) we obviously obtain an upper bound by replacing $|\tilde{g}(\vec{p} + \sum_{i=1}^n \vec{k}_i + \vec{q})|^2$ by $(|\sum_{i=1}^n \vec{k}_i + \vec{\ell}|^2 + 1)^{-3/2}$ with $\vec{\ell} = \vec{p} + \vec{q}$, and the constant K by some other constant $K' < \infty$. Then the estimate for the n^{th} term on the r.h.s. of (B.3) is completed by

Lemma 3.4: For each $\vec{\ell} \in \mathbb{R}^3$,

$$I_{n, K_0} \equiv \int_{K_0 \leq |\vec{k}_i| < \infty} \prod_{i=1}^n \frac{d^3 k_i}{(|\vec{k}_i|^2 + 1)^{3/2}} \frac{1}{(|\sum_{i=1}^n \vec{k}_i + \vec{\ell}|^2 + 1)^{3/2}} \leq (K''_{K_0})^n n!$$

with $K''_{K_0} \searrow 0$, as $K_0 \nearrow \infty$.

Proof: By Lebesgue dominated convergence it suffices to prove that

$$I_n \equiv I_{n, K_0=0} \leq (K'')^n n!$$

Clearly

$$\begin{aligned} I_n &= \int \prod_{i=1}^{n+1} \frac{d^3 \vec{\ell}_i}{(|\vec{\ell}_i|^2 + 1)^{3/2}} \delta \left(\sum_{i=1}^{n+1} \vec{\ell}_i + \vec{\ell} \right) \\ &= \int d^3 x \, e^{i \vec{\ell} \cdot \vec{x}} F(\vec{x})^{n+1} \end{aligned}$$

where

$$F(\vec{x}) \equiv (2\pi)^{-3/2} \int \frac{d^3 k}{(|\vec{k}|^2 + 1)^{3/2}} e^{i \vec{k} \cdot \vec{x}}.$$

By the Paley-Wiener theorem,

$$|F(\vec{x})| \leq O(e^{-|\vec{x}|}), \quad \text{for } |\vec{x}| \gg 1.$$

Furthermore, $F(\vec{x}) = F(|\vec{x}|)$ is rotation-invariant. Hence

$$\begin{aligned} F(\vec{x}) &= (2\pi)^{-1/2} \int_0^\infty \frac{r^2 dr}{(r^2 + 1)^{3/2}} \int_{-1}^1 dz \, e^{i r |\vec{x}| z} \\ &= 2 (2\pi)^{-1/2} \int_0^\infty \frac{r^2 dr}{(r^2 + 1)^{3/2}} \frac{\sin r |\vec{x}|}{r |\vec{x}|} \\ &\leq K''' \int_0^1 (t + |\vec{x}|)^{-1} dt \leq K_1 \log |\vec{x}|^{-1} + K_2 \end{aligned}$$

for $|\vec{x}| \ll 1$, by a change of variables and a simple estimate. Thus

$$|F(\vec{x})| \leq K_3 \log [|\vec{x}|^{-1} + 2] e^{-|\vec{x}|}$$

whence

$$\begin{aligned} I_n &\leq \int d^3 x \, |F(\vec{x})|^{n+1} \leq 4\pi (K_3)^{n+1} \int_0^\infty R^2 (\log [R^{-1} + 2])^{n+1} e^{-(n+1)R} dR \\ &\leq (K'')^n n! \end{aligned}$$

By Lemma 3.4 we may choose K_0 so large that for $p \in \text{supp } f$, with $\text{supp } f$ compact, the r.h.s. of (B.3) converges. Using now Lemma 3.3 we conclude that, for $f \in C_0^\infty$, the integrand on the r.h.s. of (B.2) is bounded above by an absolutely integrable function. A further application of Lebesgue dominated convergence then yields (B.1). This completes the proof of Theorem 3.5.

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