Ref.TH.2626-CERN

ANGULAR ENERGY FLOW BEYOND LOWEST ORDER

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ABSTRACT

The "angular energy flow" - or the average energy fraction outside a cone of opening angle 2δ about the principal jet axis in the final state of deep inelastic reactions - is calculated in QCD to all orders in the leading log approximation.

Ref.TH.2626-CERN
9 February 1979

1. INTRODUCTION

In a recent paper¹⁾ we have suggested a method by which models for the strong interaction, in particular QCD, can be checked even with low statistics data: it is the "angular energy flow" $F(\delta)$, i.e., the average fraction of energy outside a cone of opening angle 2δ about the principal jet axis in the final state of lepton-induced reactions. $F(\delta)$, being an energy weighted cross-section, is free of infra-red singularities. Assuming that the Lee-Kinoshita-Nauenberg cancellation of mass singularities holds for QCD in general, $F(\delta)$ is also free from such singularities for $\delta > 0$.

We have presented¹⁾ an explicit calculation in lowest non-trivial order of QCD, which approximates $e^+e^- \rightarrow \text{hadrons}$ by $e^+e^- \rightarrow q\bar{q}g$. We had further argued that the "dressing" of the gluon propagator in higher orders leads to the replacement of α_s , renormalized at fixed momentum, by an effective coupling $\alpha_s(E)$, where E = total centre-of-mass energy. Since one expects most of the energy to be radiated off at small angles δ , it is quite desirable to extend the calculation to small δ .

On the other hand, our ansatz is based purely on perturbation theory and therefore it can at best be extended down to angles which are not significantly affected by non-perturbative effects: $\delta > \delta_{\text{non-pert.}}$ = opening angle of a limited p_T jet.

Using a simple two-jet model with $\langle p_T \rangle \sim 450$ MeV, one finds that $\delta_{\text{non-pert.}} \sim 10^\circ$ at $E \sim 30$ GeV, falling swiftly with rising energy like $\sim \langle p_T \rangle / E$: e.g., $\delta_{\text{non-pert.}} \sim 3^\circ$ at $E \sim 100$ GeV. Thus one expects that at these energies considerations based on perturbation theory can, in principle at least, be applied to very small angles indeed.

Yet the expansion parameter in this case is not just $\alpha_s(E)$, but $\alpha_s(E) \log^2 \delta$ (and also $\alpha_s(E) \cdot \log \delta$), which is by no means small for small δ . Therefore, one has to go beyond a lowest-order calculation and has to sum all the higher-order contributions which are not a priori small. To do it explicitly is a rather formidable task which we do not want to undertake here. Instead we will base our estimates of higher order contributions to the angular energy flow $F(\delta)$ on the by now well-known leading log approximation²⁾, i.e., that in each order of α_s we keep only those terms which contain the highest power of large logarithms, in our case of $\log \delta$.

The procedure we will follow is quite analogous to procedures which were developed by various authors³⁾⁻⁵⁾ for the study of deep inelastic leptonproduction, in particular to the one adopted by Amati, Petronzio and Veneziano⁴⁾.

But before we do so we will show how to derive the leading log result given in Ref. 1) in a quick way: to order α_s the two diagrams of Fig. 1 give contributions, where the axis of maximal thrust is given by the $\bar{q}$ and q momenta, respectively. At first one might think that the integration over the angle between the gluon and quark or antiquark down to an angle δ generates just a term $\sim \alpha_s \log \delta$. Yet a more careful analysis shows this to be incorrect: using the Altarelli-Parisi probability function⁶⁾ for finding, say, a quark inside a quark with fraction of energy x , one can immediately write down the probability to find the quark outside an angle δ :

$$P_q(\delta) \sim \alpha_s \int_{\delta}^{\frac{\pi}{2}} \theta d\theta \left(\frac{\theta}{\theta^2}\right)^2 \int_{x_{\min}}^{x_{\max}} dx \frac{1+x^2}{1-x} \quad (1)$$

Therefore we obtain for the fraction of energy carried by the quark:

$$F_{q(\bar{q})}(\delta) \sim \frac{\alpha_s}{3\pi} \int_{\delta}^{\frac{\pi}{2}} \theta d\theta \left(\frac{\theta}{\theta^2}\right)^2 \int_{x_{\min}}^{x_{\max}} dx \left(\frac{1+x^2}{1-x}\right) x \quad (2)$$

For $\theta \geq \delta > 0$, $x_{\max} < 1$ and no actual singularity occurs, i.e., θ provides the necessary cut-off for the integration over x as $x \rightarrow 1$:

$$F_{q(\bar{q})}(\delta) \underset{\substack{\uparrow \\ \text{leading} \\ \log}}{\sim} - \frac{2\alpha_s}{3\pi} \int_{\delta} d\theta \frac{\log \theta}{\theta} \sim \frac{\alpha_s}{3\pi} \log^2 \delta \quad (3)$$

Since the gluon carries hardly any energy in the leading log approximation

$$z_{\text{gluon}} \sim 1 - x_{q(\bar{q})} \rightarrow 0$$

formula (3) already gives the wanted energy weighted quantity¹⁾:

$$F_{LL}(\delta) \equiv F_{\text{Lead. Log}}(\delta) \underset{\delta \rightarrow 0}{\sim} F_q(\delta) + F_{\bar{q}}(\delta) \sim \frac{2\alpha_s}{3\pi} \log^2 \delta \quad (4)$$

After this repetition we will proceed as follows⁴⁾: first we will identify the skeleton diagrams giving leading logarithms and then discuss the effects emerging when these diagrams are dressed; finally, we will perform the summation of the leading log terms.

2. IDENTIFICATION OF LEADING SKELETON DIAGRAMS

As in the case of leptonproduction, discussed in great detail by many authors^{4),5),7)}, only ordinary (uncrossed) ladder diagrams provide the maximal number of logarithms, if one works, as we will, in a physical gauge like the axial gauge. (There appears a term $\sim 1/k \cdot n$ in the sum over the polarization of a gluon with momentum k_μ , when expressed in an axial gauge fixed via a vector n_μ . Yet this term does not introduce an infra-red singularity into $F_{LL}(\delta)$ - which would be gauge dependent - since $F_{LL}(\delta)$ is an energy weighted quantity.) Analogous to the (flavour) singlet contribution in leptonproduction two types of ladders occur, which are sketched in Figs 2a and 2b. As discussed above, the second power of $\log \delta$ comes from the upper end of the integration over the quark momenta (2), (3), i.e., only the hard quarks contribute to it: the gluons on the other hand hardly carry any fraction of energy in the leading log approximation, and thus they cannot feed a finite fraction of energy into the lower part of the ladder in Fig. 2b; therefore, it does not contribute to an energy weighted cross-section like the one we are considering. For that reason we have to keep only diagrams of the type shown in Fig. 2a (diagrams of the type shown in Fig. 2b of course participate in the "dressing" process of the diagrams of Fig. 2a and will be included there).

Since only the quark and antiquark from the primarily produced $q\bar{q}$ pair give leading contributions $\sim \alpha_s \log^2 \delta$ to $F(\delta)$, one can view the reaction as a kind of random walk process for the quark and antiquark as far as $F(\delta)$ is concerned.

If n gluons are radiated off by the q line and m gluons by the $\bar{q}$ line, one finds:

$$\begin{aligned} F_{LL}^{(n,m)}(\delta) &\sim \frac{1}{2} \left(\frac{\alpha_s C_F}{2\pi} \right)^{n+m} 2^{n+m} (\log^2 \delta)^{n+m} \frac{1}{1 \cdot 2 \cdot \dots \cdot (2n)} \frac{1}{1 \cdot 2 \cdot \dots \cdot (2m)} \\ &= \frac{1}{2} \left(\frac{\alpha_s C_F}{2\pi} \right)^{n+m} (\log \delta^2)^{2(n+m)} \frac{1}{2^{n+m}} \frac{1}{2^{n+m}} \frac{1}{n!} \frac{1}{m!} \quad (5) \end{aligned}$$

with $C_F = 4/3$ for QCD, and where we have - as it is commonly done in the

leading log approach - used:

$$\log(a\delta) \sim \log \delta \quad a = \text{fixed number}$$

To calculate $F_{LL}(\delta)$ in order N (in α_s) one has to collect all contributions with $n + m = N$:

$$\begin{aligned} F_{LL}^{(N)}(\delta) &= \sum_{n+m=N} F_{LL}^{(n,m)}(\delta) \sim \\ &\sim \frac{1}{2} \left(\frac{2\alpha_s}{3\pi} (\log \delta^2)^2 \right)^N \frac{1}{2^{2N}} \frac{1}{N!} \sum_{\nu=0}^N \binom{N}{\nu}^2 \end{aligned} \quad (6)$$

which can be rewritten for convenience in terms of a "reduced" $\tilde{F}_{LL}^{(N)}$:

$$\begin{aligned} F_{LL}^{(N)}(\delta) &\equiv \left(\frac{2\alpha_s}{3\pi} (\log \delta^2)^2 \right)^N \tilde{F}_{LL}^{(N)} \\ \tilde{F}_{LL}^{(N)} &= \frac{1}{2} \frac{1}{2^{2N}} \frac{1}{N!} \sum_{\nu=0}^N \binom{N}{\nu}^2 \end{aligned} \quad (7)$$

To evaluate $\tilde{F}_{LL}^{(N)}$ further we will apply an approximation which is one of the basic formulae in statistical mechanics:

$$\binom{N}{\nu} \approx \frac{2^N}{\sqrt{\pi N/2}} \exp \left[-\frac{2(\nu - \frac{N}{2})^2}{N} \right] \quad (8)$$

Eq. (8) actually yields a rather good approximation already for small values of N : $N \geq 1$!

Using (8) we obtain:

$$\tilde{F}_{LL}^{(N)} \approx \frac{1}{\pi} \frac{1}{N!} \frac{1}{N} \sum_{\nu=0}^N \exp \left[-\frac{4(\nu - \frac{N}{2})^2}{N} \right] \quad (9)$$

which for large N can still further be simplified:

$$\begin{aligned}\tilde{F}_{LL}^{(N)} &\sim \frac{1}{\pi} \frac{1}{N!} \frac{1}{N} \int_0^N dv \exp\left[-\frac{4(v-\frac{N}{2})^2}{N}\right] \\ &\sim \frac{1}{\pi} \frac{1}{N!} \frac{1}{N} \frac{\sqrt{N\pi}}{2} = \frac{1}{2} \frac{1}{N!} \frac{1}{\sqrt{\pi N}} \equiv \frac{1}{N!} C_N\end{aligned}\quad (10)$$

with $C_N = \frac{1}{2\sqrt{\pi N}}$

Explicit calculation shows that Eq. (10) is a rather good approximation already for $N \geq 1$: using expression (7) we find

$$\tilde{F}_{LL}^{(1)} = \frac{1}{4}, \quad \tilde{F}_{LL}^{(2)} = \frac{3}{32} \approx 0.094, \quad \tilde{F}_{LL}^{(3)} = \frac{5}{192} \approx 0.026 \quad (11)$$

whereas the interpolating formula (10) yields:

$$\tilde{F}_{LL}^{(1)} \sim 0.28, \quad \tilde{F}_{LL}^{(2)} \sim 0.10, \quad \tilde{F}_{LL}^{(3)} \sim 0.027 \quad (12)$$

3. DRESSING OF THE BARE LADDERS

Dressing a ladder diagram leads to a diagram as shown in Fig. 3a. Ward identities then show that in the axial gauge (where there are no ghosts) vertex renormalization cancels quark self-energy renormalization. Therefore, such dressed diagrams can be simplified, as sketched in Fig. 3b. The effect of the dressing of the gluon propagator can, as is well known, be taken into account by the substitution

$$\alpha_s \rightarrow \alpha_s(E)$$

Therefore:

$$F_{LL}^{(N)} \sim \frac{1}{2\sqrt{\pi}} \frac{1}{N!} \frac{1}{\sqrt{N}} \left(\frac{2\alpha_s(E)}{3\pi} (\log \delta^2)^2 \right)^N \quad (13)$$

4. SUMMATION

Summation of all leading log contributions $F_{LL}^{(N)}(\delta)$ can now be performed yielding a finite result; while doing so it has to be kept in mind that $N > 0$ since the lowest order ($\sim O(\alpha_s^0)$) $q\bar{q}$ final state cannot contribute to $F_{LL}(\delta)$ for $\delta > 0$ -- assuming, as we do here, that δ is still larger than the typical opening angle δ_{jet} of a limited p_T jet:

$$\begin{aligned} F_{LL}(\delta) &= \sum_{N=1}^{\infty} F_{LL}^{(N)}(\delta) \sim \frac{1}{2\sqrt{\pi}} \sum_{N=1}^{\infty} \frac{1}{N!} \frac{1}{\sqrt{N}} \left(\frac{2\alpha_s(E)}{3\pi} (\log \delta^2)^2 \right)^N < \\ &< \frac{1}{2\sqrt{\pi}} \left[\exp \left(\frac{2\alpha_s(E)}{3\pi} (\log \delta^2)^2 \right) - 1 \right] \end{aligned} \quad (14)$$

At this point we will replace δ^2 by $y \equiv 1 - \cos \delta/2$. The reason for this substitution is the following: firstly, $y \approx \delta^2/4$ as $\delta \rightarrow 0$ and, therefore, $\log y \approx \log \delta^2/4 \approx \log \delta^2$ in the leading log approximation where one neglects terms like $\sim \log 4$. Secondly, explicit calculation¹⁾ has shown, for the lowest order expression:

$$F_{exact}^{(1)}(\delta) \approx F_{LL}^{(1)}(\delta) \quad (15)$$

within a few per cent if one replaces $\log \delta^2$ by $\log y$. That means that by using $\log y \equiv \log[(1-\cos \delta)/2]$ instead of $\log \delta^2$ one takes the non-leading terms quite efficiently into account, at least to order α_s . One might therefore expect (or at least hope) that such a substitution extends the range of validity of the leading log expression (14) also to larger angles δ . Thus, we write down:

$$F_{LL}(\delta) \sim \frac{1}{2\sqrt{\pi}} \sum_{N=1}^{\infty} \frac{1}{N!} \frac{1}{\sqrt{N}} \left(\frac{2\alpha_s(E)}{3\pi} \log^2 y \right)^N < \\ < \frac{1}{2\sqrt{\pi}} \left[\exp\left(\frac{2\alpha_s(E)}{3\pi} \log^2 y \right) - 1 \right] \quad (16)$$

In Fig. 4 we have plotted $F_{LL}(\delta)$, calculated up to order $N = 3$ with $\alpha_s(E) \sim 0.15$ to be definite; for comparison the lowest order expression $F_{LL}^{(1)}(\delta)$ is also plotted.

Expression (16) makes (at best) sense only as long as $F_{LL}(\delta) \lesssim 1$ of course, since it is meant to describe a fraction of energy. Therefore, $\delta_{\min} \gtrsim 3^\circ$ for $\alpha_s(E) \sim 0.15$ (i.e., $E \sim 100$ GeV); on the other hand, at such energies, non-perturbative effects are expected to set in at such angles, as mentioned above, which have been completely ignored in our ansatz, and they will provide the necessary cut-off. Since such effects will presumably not set in very abruptly, they will also modify the perturbative expression at slightly larger angles, maybe up to angles of the order of $\sim 5^\circ$. This, of course, cannot be derived in our purely perturbative approach.

5. CONCLUSIONS

Formula (16), which goes beyond lowest order, not only yields significant corrections to the lowest order expression for medium values of δ , it also provides us with a more reliable guide with which the range of applicability of such a formula can be estimated.

ACKNOWLEDGEMENTS

It is a pleasure to thank T. Walsh and F. Steiner for many valuable discussions, and C. Sachrajda for reading the manuscript.

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FIGURE CAPTIONS

- Fig. 1 : The two lowest order contributions to $F(\delta)$.
- Fig. 2 : a)b)c): Ladder diagrams describing $\gamma^* \rightarrow q \dots \bar{q} \dots g \dots$
- Fig. 3 : Dressed ladder diagrams: (a) before, and (b) after applying Ward identities.
- Fig. 4 : Plots of $F_{LL}(y) = \sum_{N=1}^3 F_{LL}^{(N)}(y)$ and $F_{LL}^{(1)}(y)$, using $\alpha_s(E) \approx 0.15$, $y = (1 - \cos \delta)/2$.

FIG.1

FIG.2a

FIG. 2b

FIG. 3a

FIG. 3b

FIG.4