Dimensionality in the statistical bootstrap model

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We obtain the relation between critical behavior and dimensionality of statistical bootstrap systems, and we show that it is the same as that found in the dual resonance model.

The statistical bootstrap model and the dual resonance model both obtain for multihadron systems an asymptotic density of states of the form

$$\sigma(M^2) = c M^2 e^{bM^2}$$,

(1)

where $M$ denotes the c.m. energy of the system and $a$, $b$, and $c$ are constants. In a statistical treatment of hadronic matter, $b$ becomes an inverse critical temperature, $T_c = b^{-1}$; the exponential growth of $\sigma(M^2)$ channels additional energy into new states rather than into kinetic degrees of freedom, and hence leads to the temperature bound $T_c$. The details of the critical behavior (ultimate temperature vs. phase transition, critical exponents and their inequalities, etc.) are determined by the value of the constant $a$.\textsuperscript{2-3}

In the dual resonance model, the dimensionality $d$ of the space of the associated set of annihilation and creation operators determines\textsuperscript{3} $a$,

$$a = -(d + 1)/2.$$  

(2)

Thus, $d = 4$ leads to the original bootstrap value while $d = 5$ yields the value $a = -3$ of the strong bootstrap condition.\textsuperscript{5} The critical temperature $b^{-1}$, though in scale fixed by the slope $\alpha'$ of the Regge trajectories, has the dimensional dependence\textsuperscript{5}

$$b = 2\pi(d\alpha'/6)^{1/2}.$$  

(3)

Together with $\alpha' = 1$ GeV$^{-2}$, $d = 4$ and $d = 5$ give the critical temperatures 195 and 174 MeV, respectively, in fair agreement with the 180 MeV needed in the statistical bootstrap interpretation of hadronic transverse-momentum spectra.

The aim of this work is to point out that the statistical bootstrap model, if generalized to an arbitrary number of space dimensions, leads to an analogous dimensionality dependence as that shown in Eq. (2) and Eq. (3) by the dual resonance model. We shall also comment briefly on the formal basis of this similarity, provided by partition theorems.

The density of states of a multihadron system in one time and $\nu - 1$ space dimensions is given by

$$\sigma(p^2) = \delta^{(\nu)}(p^2) + \sum_{n=1}^{\infty} \frac{B^n}{N^n} \int \prod_{i=1}^{n} [d^\nu p_i \rho(p_i^2)]$$

$$\times \delta^{(\nu)}\left(\sum_{i=1}^{n} p_i - P\right),$$

(4)

where $P$ denotes the overall $\nu$ momentum of the system, and $\rho(p^2)$ describes the resonance excitation spectrum. The bootstrap condition fixes $\rho(p^2)$ by\textsuperscript{7}

$$\rho(p^2) = \delta(p^2 - \mu^2) + \frac{1}{B} \left[ \sigma(p^2) - \delta^{(\nu)}(p^2) - B\rho(p^2) \right].$$

(5)

In Eqs. (4) and (5),

$$p^2 = p_0^2 - \sum_{i=1}^{\nu-1} p_i^2,$$

(6)

$B$ is a $(\nu - 2)$-dimensional space volume, and $\mu$ is the mass of the basic hadron ("pion"). For $\nu = 4$, we recover the usual bootstrap system: $B = 2\mu V$ then yields in the nonrelativistic limit the connection to a three-dimensional coordinate-space volume $V$.

The partition function of the system

$$Z(\beta) = \int d^\nu p e^{-\beta p^2} e^{(p^2 - \mu^2)}$$

(7)

is by Laplace transform of Eq. (5) determined through

$$Z(\beta) = 1 - B\varphi(\beta) + 2 \ln Z(\beta),$$

(8)

where

$$\varphi(\beta) = \int d\rho p e^{-\beta\rho(p^2)} = \frac{1}{B} \ln Z(\beta)$$

(9)

and

$$\varphi_0(\beta) = \int d\rho p e^{-\beta\rho(p^2 - \mu^2)}$$

(10)

denote resonance and single-pion generating functions, respectively.

As in the four-dimensional case,\textsuperscript{8} we encounter
a critical point for
\[ B \varphi_0(\beta_c) = 2 \ln 2 - 1, \tag{11} \]
since \( aZ/a\beta \) then diverges, and also here \( Z(\beta) \) has
a square-root branch point
\[ Z(\beta) = Z(\beta_\nu) - \left\{ 2[\varphi_0(\beta) - \varphi_0(\beta_\nu)]^{1/2}/(\beta^2 \varphi_0/\alpha Z^2)_{\beta=\beta_\nu} \right\}^{1/2} \tag{12} \]

by going into the \( \beta \) c.m. system and performing the
energy integration. Changing to spherical coordina-
tes, we have
\[ \varphi_0(\beta) = S_{n-1} \int_0^\infty dp p^{n-2} \times \exp \left[ -\beta(p^2 + \mu^2)^{1/2} / 2(p^2 + \mu^2)^{1/2} \right] , \tag{14} \]
where
\[ S_{n-1} = 2 \pi^{\nu_2 - 1/2} / \Gamma(\nu - 1)/2 \tag{15} \]
is the surface of a sphere in \( \nu - 1 \) space dimen-
sions. Hence, for \( \mu = 0 \) we have
\[ \varphi_0(\beta) = \beta^{\nu_2 - 1/2} \mu^{\nu_2 - 1/2} \Gamma(\nu - 2)/\Gamma(\nu - 1)/2 \tag{16} \]
as a single-pion generating function. From Eq.
(11), this gives
\[ Z(\beta) = S_{n-1} \int_0^\infty dM \sigma(M^2) \int_0^\infty dP P^{n-2} \exp \left[ -\beta(P^2 + M^2)^{1/2} / (P^2 + M^2)^{1/2} \right] \tag{20} \]

and hence
\[ Z(\beta) = S_{n-1} \int_0^\infty dM M^{\nu_2-1} \sigma(M^2) \int_0^\infty dy (y^2 - 1)^{\nu_2 - 3/2} e^{-\delta y} \times \left[ 2(2\pi/\beta)^{\nu_2 - 1/2} \int_0^\infty dM M^2 \sigma(M^2) K_{\nu_2 - 3/2}^2(\beta M) \right] \]
\[ = (2\pi/\beta)^{\nu_2 - 1/2} \int_0^\infty dM M^2 \sigma(M^2) K_{\nu_2 - 3/2}^2(\beta M) \times \int_0^\infty dx x^{\nu_2 - 3/2} e^{-x} \times \beta_0 = R \].

TABLE I. Dimensional dependence of the critical
temperature in the statistical bootstrap model.

<table>
<thead>
<tr>
<th>( \nu )</th>
<th>( \beta_c )</th>
<th>( \beta_0^{2\nu}/\beta_c )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>1.21 R</td>
<td>2.18</td>
</tr>
<tr>
<td>4</td>
<td>1.56 R</td>
<td>1.95</td>
</tr>
<tr>
<td>5</td>
<td>1.96 R</td>
<td>1.73</td>
</tr>
<tr>
<td>6</td>
<td>2.35 R</td>
<td>1.58</td>
</tr>
<tr>
<td>7</td>
<td>2.70 R</td>
<td>1.49</td>
</tr>
</tbody>
</table>

at \( \beta = \beta_c \). The formal singularity pattern of
the bootstrap system thus remains unchanged. How-
ever, the \( \beta \) dependence of \( \varphi_0(\beta) \) is dependant on
\( \nu \), and hence the critical temperature \( \beta_c \), in Eq. (11)
also becomes \( \nu \) dependent. Similarly, the \( \nu \) de-
pendence of Eqs. (7) and (9) will be seen to affect \( \alpha \) in
Eq. (1).

The integral (10) can be rewritten as
\[ \beta_0 = R(2 \ln 2 - 1) \pi^{\nu_2 - 1/2} \Gamma(\nu - 2)/\Gamma(\nu - 1)/2 \times \beta_0^{2\nu} \tag{17} \]
as the critical temperature, where we have in-
troduced \( R \),
\[ B = R^{\nu_2 - 2} \tag{18} \]
as the basic scale (length) parameter. For large \( \nu \), Eq. (17)
leads to
\[ \beta_0^{2\nu} = (2\pi/e)^{1/2} R \sqrt{\nu} \tag{19} \]
in accord with the dual resonance form (3). The
critical temperatures for some particular \( \nu \) values are
shown in Table I together with their derivation from
the asymptotic form (19).

Let us now consider the power factor \( M^2 \) in the
level density (1). The partition function (7) be-
comes in spherical coordinates

\[ Z(\beta) = c(2\pi/\beta)^{\nu_2 - 1/2} \int_0^\infty dM M^{2\nu - 1/2} e^{-\delta M} \]
\[ = c(2\pi/\beta)^{\nu_2 - 1/2} (\beta - \delta)^{\nu_2 - 1/2} \int_0^\infty dx x^{\nu_2 - 3/2} e^{-x} \times \beta_0 \]

By Eq. (12), \( Z(\beta) \) has at \( \beta = \beta_c = b \) a square-root
branch point in \( \varphi_0(\beta) \) and hence, by Eq. (16), also
in \( \beta \)
\[ Z(\beta) = Z(\beta_\nu) - \text{const} \times (\beta - \beta_\nu)^{\nu_2/2} \times (\beta - \beta_0)^{\nu_2/2} \tag{23} \]
Together with Eq. (22), this implies
\[ a = -(\nu/2 + 1) \tag{24} \]
for the dimensional dependence of the "critical exponent" \( \alpha \); for \( \nu = 4 \), we recover, of course, the
familiar \( a = -3 \).
From Eqs. (19) and (23), we thus see that the statistical bootstrap model leads to essentially the same dimensionality dependence of critical temperature and exponent as the dual resonance model in Eqs. (2) and (3).

The similarity between statistical bootstrap and dual resonance models is of a large degree due to similar underlying partition features. The level density in the one-dimensional dual resonance model is just the number of root systems of the equation

\[ \sum_{\nu=1}^{N} r_{\nu} \lambda_{\nu} = N, \quad r, \lambda, N - 1 \gg 0 \]  

(25)

where \( N = M^{3} \) is the squared c.m. energy of the system.\(^{9}\) The asymptotic solution of this problem is obtained by “inversion” of the generating function

\[ Z_{1}(x) = \sum_{N=1}^{\infty} \sigma_{1}(N)x^{N} = \prod_{k=1}^{\infty} \frac{1}{1-x^{N}} \]  

(26)

and leads to the form (1).\(^{10}\) The \( d \)-dimensional case

\[ \sum_{\nu=1}^{N} \sum_{\nu_{1}=1}^{\infty} r_{\nu_{1}} \lambda_{\nu} = N \]  

(27)

yields\(^{8}\)

\[ Z_{d}(x) = [Z_{1}(x)]^{d}, \]  

(28)

so that the singularity structure of \( Z_{d}(x) \) is essentially maintained. The statistical bootstrap model is also basically a partition problem\(^{11,12}\). Here \( \sigma(M) \) is the number of partitions obtained by dividing \( M \) elements into \( a_{1} \) subsets of one, \( a_{2} \) subsets of two, ..., \( a_{k} \) subsets of \( k (\leq N) \) elements, then subdividing each \( r (\leq k) \) element subset again in the same fashion, and iterating on until no further subdivision is possible. The singularity structure of the corresponding partition function is determined by Eq. (8), independent of dimension; only through the inverse Laplace transform

\[ \sigma(M) \sim \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} d\beta e^{\beta \tilde{a}_{\nu} \tilde{a}_{\mu}} (\beta - \beta_{c})^{1/2} \]  

(29)

does \( \nu \) enter.

The basic difference between the two approaches is independent of dimension: the linearly exponential growth in \( M \) for the level density is obtained in the dual resonance model by equidistribution of states (integral values) of squared energies,\(^{13}\) in the statistical bootstrap model by iterated partitions.\(^{8}\)

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