Ref. TH.2680-CERN

AN EFFECTIVE LAGRANGIAN WITH NO U(1) PROBLEM
IN CP^{n-1} MODELS AND QCD

P. Di Vecchia

CERN -- Geneva

A B S T R A C T

We write an effective Lagrangian which gives two point Green's functions satisfying the anomalous Ward identities for the U(1) axial vector current with a singlet particle that has a non-vanishing mass in the chiral limit. We show that the mechanism that has been postulated by Witten and Veneziano for solving the U(1) problem in the framework of the $1/n$ expansion in QCD is fully active in the two dimensional CP^{n-1} model where the $1/n$ expansion can be explicitly done.

Ref. TH.2680-CERN

28 May 1979

The U(1) problem is one of the long standing problems in hadron physics¹⁾. Recently in two very interesting papers by Witten²⁾ and Veneziano³⁾ it has been shown that under certain assumptions in the framework of the $1/n$ expansion of QCD the singlet acquires a non vanishing mass in the chiral limit. Moreover the existence of a singlet with a non vanishing mass is also needed in order to eliminate the dependence on the θ angle in massless QCD²⁾. An algebraic saturation of the anomalous Ward identities without a Goldstone boson in the singlet channel has been also achieved³⁾. Finally low energy theorems for the emission of the singlet have been derived in the limit $n \rightarrow \infty$ ^{2), 3)}.

In this letter we write down an effective Lagrangian giving two point Green's functions that satisfy the anomalous Ward Identities without a massless singlet. We then analyse in some detail the U(1) problem in CP^{n-1} models where the $1/n$ expansion has been explicitly performed^{4), 5)} and we show that this problem is solved in these models in exactly the same way as it has been postulated for QCD^{2), 3)}.

It is well known that in QCD the singlet axial vector current has an anomaly proportional to the density of topological charge. One finds that the divergence of the axial vector current is given by^{*}):

$$\partial_\mu J_\mu^L = 2iL q(x) + D_L(x) \quad (1)$$

where

$$q(x) = \frac{g^2}{32\pi^2} F_{\mu\nu}(x) \tilde{F}_{\mu\nu}(x) \quad (2)$$

and D_L is a soft term that is vanishing when the quark masses are zero. L is the number of light quark flavours (assumed to have a common mass $m \rightarrow 0$). The anomaly term can be written as the divergence of the following current:

$$K_\mu = 2L \frac{g^2}{16\pi^2} \epsilon_{\mu\alpha\beta\gamma} A_\alpha^\alpha (\partial^\beta A_\alpha^\gamma + \frac{1}{3} g f_{abc} A_b^\beta A_c^\gamma) \quad (3)$$

One gets:

$$\partial_\mu K_\mu = 2L q(x) \quad (4)$$

^{*}) We follow the notations of Ref. 1) except that we work in Euclidean space [Eq. (3) differs for a factor $2L$].

The absence of a massless singlet implies the following Ward identities:

$$\int d^4x \langle 2iLq(x) D_L(0) \rangle = -\int d^4x \langle D_L(x) D_L(0) \rangle + 4 \langle \Delta_L \rangle \quad (5)$$

$$\int d^4x \langle D_L(x) 2iLq(0) \rangle = -\int d^4x \langle 2iLq(x) 2iLq(0) \rangle \quad (6)$$

The satisfaction of these Ward identities is a necessary condition for the resolution of the U(1) problem¹⁾. Introducing an axial four-vector ghost Veneziano has proved that Eqs. (5) and (6) are satisfied.

It is possible to write down an effective Lagrangian involving the fields appearing in the Ward Identities that summarizes Veneziano's result. One gets the following effective Lagrangian:

$$\mathcal{L} = \frac{1}{2F_S^2(m_S^2 - m_{NS}^2)} (\partial_\mu k_\mu)^2 + \frac{1}{2} [(\partial_\mu S)^2 + m_{NS}^2 S^2] + \frac{i}{F_S} S \partial_\mu k_\mu \quad (7)$$

where S is the singlet field which is related to D_L by the following relation:

$$D_L(x) = m_{NS}^2 F_S S(x) \quad (8)$$

$m_S[m_{NS}]$ is the mass of the singlet (non singlet) and in the case of $L = 2$ m_{NS} is equal to the pion mass. F_S is the decay constant of the singlet defined by the following relation:

$$\langle S | J_{\mu 5}^L | 0 \rangle = -i q_\mu F_S \quad (9)$$

From Lagrangian (7) one can easily compute the two point Green's functions.

One gets:

$$\langle S(x) S(y) \rangle = \int \frac{d^4p}{(2\pi)^4} e^{ip(x-y)} \frac{1}{p^2 + m_S^2} \quad (10)$$

$$\langle S(x) k_\mu(y) \rangle = \int \frac{d^4p}{(2\pi)^4} e^{ip(x-y)} \frac{F_S(m_S^2 - m_{NS}^2)}{p^2 + m_S^2} \frac{p_\mu}{p^2} \quad (11)$$

$$\langle k_\mu(x) k_\nu(y) \rangle = \int \frac{d^4p}{(2\pi)^4} e^{ip(x-y)} \frac{p^2 + m_{NS}^2}{p^2 + m_S^2} F_S^2(m_S^2 - m_{NS}^2) \frac{p_\mu p_\nu}{p^2} \quad (12)$$

It is easy to check that those Green's functions saturate the Ward identities (5) and (6). The vacuum expectation value of Δ_L can be computed from the Ward identity for the non-singlet current:

$$\int d^4x \langle \partial_\mu J_{\mu 5}^a(x) \partial_\nu J_{\nu 5}^b(0) \rangle = \delta^{ab} \langle \Delta_2 \rangle \quad (13)$$

which gives:

$$\langle \Delta_2 \rangle = m_{NS}^2 F_{NS}^2 \quad (14)$$

after having used the SU(L) symmetry that implies:

$$\langle \Delta_L \rangle = \frac{L}{2} \langle \Delta_2 \rangle = \frac{L}{2} m_{NS}^2 F_{NS}^2 \quad (15)$$

Notice that the propagators (10), (11) and (12) have a pole at the mass of the singlet. Those involving K_μ have also a pole at $p^2 = 0$ which is a consequence of a modified Kogut-Susskind mechanism⁶⁾ as it has been shown in Ref. 3). This pole at $p^2 = 0$ is, however, only appearing in Green's functions involving fields which are gauge variant like K_μ . $\partial_\mu K_\mu$ is gauge invariant and Green's functions involving it do not have any pole at $p^2 = 0$. Finally as it has been shown in Refs. 2) and 3), consistency in the framework of the $1/n$ expansion requires that the parameters F_S and $m_S^2 - m_{NS}^2$ appearing in Eq. (7) must be related to $\int d^4x \langle q(x) q(0) \rangle$ of the pure Yang-Mills theory. The precise relation which can be immediately obtained from Eqs. (4) and (12) is the following:

$$\int d^4x \langle q(x) q(0) \rangle_{\text{Yang-Mills}} = \frac{F_S^2}{4L^2} (m_S^2 - m_{NS}^2) \quad (16)$$

In the following we review some of the relevant properties of the CP^{n-1} models and we show that the U(1) problem, that was already analysed in Refs. (4) and (5), is solved in these models in exactly the same way as it has been postulated in QCD in Refs. 2) and 3).

The CP^{n-1} model with quarks is defined by the following Lagrangian:

$$\begin{aligned} \mathcal{L} = & \overline{D}_\mu \tilde{z} D_\mu \tilde{z} + \overline{\Psi} (\not{D} - M_B) \Psi + \frac{e^2}{2n} (\overline{\Psi} \gamma_\mu \Psi)^2 \\ & - \frac{g_V}{2n} [(\overline{\Psi} \tau^i \Psi)^2 + (\overline{\Psi} \tau^i \gamma_5 \Psi)^2] \end{aligned} \quad (17)$$

where $D_\mu = \partial_\mu + (i/\sqrt{n}) \lambda_\mu$ when acting on z and $D_\mu = \partial_\mu + (i/\sqrt{n}) e \lambda_\mu$ when acting on ψ . One has in addition also the constraint:

$$|z|^2 = n/2f \quad (18)$$

We follow the notations of Ref. 4) but we do not give any "colour" index to the fermions and therefore the constraint $\bar{z} \cdot \psi = 0$ is not imposed. The field z^α has a "colour" index α that runs from 1 to n , while the quark field ψ^a has only a flavour index a that goes from 1 to L . In this way as in QCD a fermion loop is down by a factor L/n with respect to a boson loop in the limit $L, n \rightarrow \infty$ and L/n fixed and small.

The interest for the CP^{n-1} model in two dimensions stems from the fact that it has many properties in common with QCD in four dimensions. We list some of them:

- i) Lagrangian (17) contains no dimensional coupling constant. It is conformal invariant at the classical level.
- ii) Asymptotic freedom and dimensional transmutation.
- iii) Non-trivial topology for any value of n .

In addition one can prove that there is a spontaneous mass generation for both the parton field z and the quark field ψ and that chiral symmetry is spontaneously broken. The vacuum expectation value of $\bar{\psi}\psi$ is given by:

$$\langle \bar{\psi}\psi \rangle = LM \quad (19)$$

where M is the spontaneously generated quark mass.

The properties of the CP^{n-1} model can be studied by expanding the Green's functions for large values of n and L . This expansion has been already performed in Refs. (4) and (5). One gets an effective Lagrangian involving some composite fields made up with the fundamental z and ψ fields. In the low energy region the effective Lagrangian becomes the following:

$$\begin{aligned} \mathcal{L} = & \alpha F^2 + \frac{1}{2} [(\partial_\mu S)^2 + m_{NS}^2 S^2] + \\ & + \frac{1}{2} [(\partial_\mu \pi^i)^2 + m_{NS}^2 (\pi^i)^2] + ie F \cdot S \sqrt{\frac{L}{\pi n}} \end{aligned} \quad (20)$$

where $F = \epsilon_{\mu\nu} \partial_\mu \lambda_\nu$, S is the singlet and π^i is the non-singlet. The constant α is given by:

$$\alpha = \frac{1 + 2 \frac{L}{n} e^2 \frac{m^2}{M^2}}{24\pi m^2} \quad (21)$$

$m^2 (M^2)$ is the mass spontaneously generated for the $z(\psi)$ field. As it has been shown in Ref. 4) the low energy effective Lagrangian contains also other fields, that we omit here because they are not relevant for the analysis of the U(1) problem. As in the case of QCD the divergence of the U(1) axial vector current has an anomaly that is proportional to the density of topological charge. One gets that:

$$\partial_\mu J_{\mu 5}^L = 2ieL q(x) + m_{NS}^2 F_S S \quad (22)$$

where

$$q(x) = \frac{1}{2\pi\sqrt{n}} F \quad (23)$$

In terms of the following field:

$$K_\mu = 2eL \frac{1}{2\pi\sqrt{n}} \epsilon_{\mu\nu} \lambda_\nu \quad (24)$$

the effective Lagrangian (20) takes exactly the same form as (7) except for the "pion" term which has been omitted in Eq. (7). In addition one gets an explicit expression for the parameters F_S and $m_S^2 - m_{NS}^2$ in terms of the fundamental parameters of the theory. They are given by:

$$F_S = \sqrt{\frac{L}{\pi}} \quad ; \quad m_S^2 - m_{NS}^2 = \frac{12e^2 m^2 L}{n} \left[1 + \frac{2L}{n} e^2 \frac{m^2}{M^2} \right]^{-1} \quad (25)$$

Remember that in two dimensions F_S is dimensionless.

One gets therefore the same two point functions as Eqs. (10), (11) and (12) and they satisfy of course the anomalous Ward identities with a singlet, that is not massless in the chiral limit. Therefore the mechanism for avoiding a Goldstone boson in the singlet channel is quite the same in CP^{n-1} model as in QCD.

Also the relation (16), which has been postulated in QCD for satisfying the Ward identities and for eliminating the dependence on the θ angle in the massless theory, is valid in CP^{n-1} model. One finds that

$$\int d^2x \langle q(x) q(0) \rangle_{\text{Without quarks}} = \frac{3m^2}{n\pi} = \frac{F_S^2}{4L^2e^2} (m_S^2 - m_{NS}^2) \quad (26)$$

in the limit $L/n \ll 1$. This is a relation that gives the product $F_S^2(m_S^2 - m_{NS}^2)$ in terms of the theory without quarks. Another interesting relation that can be obtained from Eq. (26) is the following:

$$\frac{\int \langle q(x) q(0) \rangle_{\text{With quarks}} d^2x}{\int \langle q(x) q(0) \rangle_{\text{Without quarks}} d^2x} = \frac{m_{NS}^2}{m_S^2} \quad (27)$$

For the sake of completeness it would be nice to check the saturation of the anomalous Ward identities for the general N -point Green's function and in particular to check the θ dependence of certain Green's functions as $\langle \bar{\psi}\psi \rangle_\theta$ and $\langle F \rangle_\theta$. Work in this direction is in progress and we hope to report on this problem in a future publication.

ACKNOWLEDGEMENTS

We thank G. Veneziano for telling us his results prior to publication and for many discussions that strongly stimulated the present analysis.

REFERENCES

- 1) For a review of the difficulties in solving the U(1) problem see: R. Crewther, Status of the U(1) problem, CERN preprint TH 2546 (1978). See also R. Crewther, Phys. Letters 70B (1977) 349.
- 2) E. Witten, Harvard University preprint (1979).
- 3) G. Veneziano, CERN preprint TH. 2651 (1979).
- 4) A. D'Adda, P. Di Vecchia and M. Lüscher, Nucl. Phys. B146 (1978) 63 and DESY preprint 78/75 (1978).
- 5) E. Witten, Nucl. Physics B149 (1979) 285.
- 6) J. Kogut and L. Susskind, Phys. Rev. D11 (1975) 3594; see also S. Weinberg. Phys. Rev. D11 (1975) 3583.