Ref.TH.2453-CERN

SUPERFIELD LAGRANGIAN FOR SUPERGRAVITY

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ABSTRACT

We show that the action for supergravity in superspace is the integral of the determinant of the supervierbein. Our method allows the construction of actions describing the coupling to matter supermultiplets. As an example we give the coupling to the massless vector supermultiplet.

Ref.TH.2453-CERN
16 January 1978

In two previous papers [1, 2] a formulation of supergravity was given, based on the differential geometry of superspace^{*}. Starting from a general affine superspace the geometry was determined by specializing the tangent space group to be an ordinary (but x and θ dependent) Lorentz group and by imposing certain conditions on the components of the supertorsion. These conditions can be separated into kinematic and mass-shell conditions. The kinematic conditions on the supertorsion complete the determination of the geometry and permit to express all components of the superconnection in terms of the supervierbein. (They are the analogue of the requirement one usually imposes in an ordinary Riemann space, that all components of the torsion should vanish.) The mass-shell conditions, or equations of motion, complete the dynamics of the theory and make it equivalent to supergravity, as formulated in ordinary space-time [4, 5]. Until now these equations of motion were simply imposed as additional restrictions on the supertorsion. In the present paper we solve the outstanding problem of the theory and give a Lagrangian from which the equations of motion can be derived. The action turns out to be just the superspace integral of the determinant [6] of the supervierbein. Its variation must be calculated by varying the supervierbein subject to the restriction imposed upon it by the kinematic conditions on the supertorsion. The coupling to matter supermultiplets is easily constructed within the present framework. As an example we give, at the end, the coupling to a massless vector supermultiplet as described by a real scalar superfield.

As mentioned above, the action is just the superspace integral of the determinant of the supervierbein. Therefore we wish to calculate the variation

$$\delta \int E \, dx \, d\theta = \int E \, H_A^A (-1)^a \, dx \, d\theta, \quad (1)$$

* A more detailed description can be found in ref. [3].

where

$$E = \det E_M^A, \quad H_A^B = E_A^M \delta E_M^B, \quad (2)$$

subject to the restrictions imposed on the variations H_A^B by the kinematic conditions on the torsion. Our method will consist in finding the equations satisfied by these variations and in solving them by giving the variations themselves in terms of two superfields $\mathcal{V}_a$ and U and their derivatives (a is an ordinary vector index). Since these two superfields are arbitrary, inserting the expression for H_A^B into (1) and performing some integrations by parts one obtains the equations of motion. In order to obtain the equations for H_A^B we observe that, in general, an infinitesimal variation of the torsion T_{BC}^A is given by

$$\begin{aligned} \delta T_{BC}^A &= \mathcal{D}_B H_C^A - (-1)^{bc} \mathcal{D}_C H_B^A \\ &+ T_{BC}^D H_D^A - H_B^D T_{DC}^A + (-1)^{bc} H_C^D T_{DB}^A \\ &+ \Omega_{CB}^A - (-1)^{bc} \Omega_{BC}^A, \end{aligned} \quad (3)$$

where the $\mathcal{D}_B$ are covariant derivatives and

$$\Omega_{CB}^A = E_C^M \delta \Phi_{MB}^A \quad (4)$$

expresses the variation of the connection Φ_{MB}^A . We can now impose on the torsion in (3) and on its variation the kinematic conditions. We recall that, in two-component spinor notation, they state the vanishing of all components of the torsion except for

$$T_{\alpha\dot{\beta}}^{\phantom{\alpha\dot{\beta}}c} = T_{\dot{\beta}\alpha}^{\phantom{\dot{\beta}\alpha}c} = 2i(\sigma^c)_{\alpha\dot{\beta}} \quad , \quad (5)$$

and for T_{ab}^{γ}, $T_{ab}^{\gamma} = -T_{ba}^{\gamma}$, $T_{ab}^{\dot{\gamma}} = -T_{ba}^{\dot{\gamma}}$ which are left undetermined, together with their complex conjugates. Observe that the resulting equations are invariant under the transformations

$$\delta H_A^{B} = \mathcal{D}_A \xi^B + \xi^C T_{CA}^{B} \quad (6)$$

$$\delta \Omega_{AB}^{C} = \xi^D R_{DA,B}^{C}$$

and

$$\delta H_A^{B} = X_A^{B}$$

$$\delta \Omega_{AB}^{C} = -\mathcal{D}_A X_B^{C} \quad , \quad (7)$$

which are, respectively, the variational version of general coordinate transformations and of Lorentz transformations. This can be checked directly, for (6) by using the Bianchi identities involving the torsion and the curvature tensor $R_{DA,B}^{C}$. The tangent group choice implies that the matrix X_A^{B} is an infinitesimal Lorentz transformation X_a^{b}, X_{α}^{β}, $X_{\dot{\alpha}}^{\phantom{\dot{\alpha}}\dot{\beta}}$, the same transformation operating on vectors and spinors.

The same restriction applies to the curvature and connection coefficients, which belong to the algebra of the tangent space group. In solving the equations (3) for H_A^B and for Ω_{AB}^C , one can use the invariances (6) and (7) and go to a special gauge where the solution has a simpler form.

As explained in ref. [2], the kinematic conditions on the torsion, combined with the tangent group restrictions and with the Bianchi identities, allow one to express all components of the curvature and of the torsion in terms of the three superfields $W_{\alpha\beta\gamma}$, $G_{\alpha\dot{\beta}}$ and R and of their complex conjugates ($G_{\alpha\dot{\beta}}$ is hermitean). For instance

$$R_{\alpha\beta,\gamma\delta} = -4 (\varepsilon_{\alpha\gamma} \varepsilon_{\beta\delta} + \varepsilon_{\alpha\delta} \varepsilon_{\beta\gamma}) R^*$$

$$R_{\alpha\beta,\dot{\gamma}\dot{\delta}} = 0$$

$$T_{\alpha,\beta\dot{\beta},\gamma} = \frac{i}{4} (\varepsilon_{\beta\gamma} G_{\alpha\dot{\beta}} - 3 \varepsilon_{\alpha\gamma} G_{\beta\dot{\beta}} - 3 \varepsilon_{\alpha\beta} G_{\gamma\dot{\beta}})$$

$$T_{\dot{\alpha},\beta\dot{\beta},\dot{\gamma}} = \frac{i}{4} (\varepsilon_{\beta\dot{\gamma}} G_{\alpha\dot{\beta}} - 3 \varepsilon_{\dot{\alpha}\dot{\gamma}} G_{\beta\dot{\beta}} - 3 \varepsilon_{\dot{\alpha}\dot{\beta}} G_{\beta\dot{\gamma}}) \quad (8)$$

$$T_{\alpha,\beta\dot{\beta},\dot{\gamma}} = -2i \varepsilon_{\alpha\beta} \varepsilon_{\dot{\beta}\dot{\gamma}} R^*,$$

where

$$\mathcal{D}_{\alpha} R^* = 0 \quad (9)$$

In a general affine superspace the identity

$$\int E \left(\mathcal{D}_A \xi^A + \xi^C T_{CA}^A \right) (-1)^a dx d\theta = 0 \quad (10)$$

expresses the invariance of the integral of the determinant of the supervierbein under general coordinate transformations. Our kinematic restrictions imply that the second term in (10) vanishes and simplify the equation to

$$\int E \mathcal{D}_A \xi^A (-1)^a dx d\theta = 0 \quad , \quad (11)$$

a very useful formula which permits systematic use of integration by parts.

We shall not give here the solution for all components of H_A^B , but only the trace which appears in (1). One finds $(V_{\alpha\dot{\alpha}} = (\sigma^b)_{\alpha\dot{\alpha}} V_b)$

$$H_A^A (-1)^a = -\frac{1}{4} [\mathcal{D}_\alpha, \mathcal{D}_{\dot{\alpha}}] V^{\alpha\dot{\alpha}} + V^b (T_{b\alpha}^\alpha + T_{b\dot{\alpha}}^{\dot{\alpha}}) \quad (12)$$

$$+ (\mathcal{D}_{\dot{\sigma}} \mathcal{D}^{\dot{\sigma}} - \not{g} R) U + (\mathcal{D}^\sigma \mathcal{D}_\sigma - \not{g} R^*) U^* .$$

Inserting this into (1), all derivative terms drop out by partial integration and one has the final result

$$\begin{aligned} & \delta \int E dx d\theta \\ & = \int E \left\{ V^b (T_{b\alpha}{}^\alpha + T_{b\dot{\alpha}}{}^{\dot{\alpha}}) - \delta R U - \delta R^* U^* \right\} dx d\theta \end{aligned} \quad (13)$$

The corresponding equations of motion are

$$G_{\alpha\dot{\beta}} = 0 \quad , \quad R = 0 \quad , \quad (14)$$

where we have used (8). It is also possible to see that the choice of gauge made to obtain the solution in the form (12) still leaves a certain gauge arbitrariness which can be used to transform the superfield U to zero. The action then takes a form perfectly analogous to that of the linearized approximation [7].

We now show briefly how one describes the coupling of supergravity to a vector supermultiplet^{*}. In analogy with the case of a flat superspace, we described the vector supermultiplet by a real scalar superfield V and a supersymmetric gauge transformation by

$$V \rightarrow V + i\Lambda - i\Lambda^* \quad , \quad (15)$$

where the superfield Λ is covariantly chiral

$$\mathcal{D}_{\dot{\alpha}} \Lambda = 0 \quad . \quad (16)$$

* The coupling has been given already in terms of ordinary fields in refs. [8, 9].

The gauge invariant superfield which contains the electromagnetic field is defined as

$$W_\alpha = (\mathcal{D}_{\dot{\alpha}} \mathcal{D}^{\dot{\alpha}} - \not{R}) \mathcal{D}_\alpha V . \quad (17)$$

The extra term, containing R , with respect to the analogous flat superspace definition is dictated by gauge invariance. Indeed, it is easy to verify that W_α is invariant under (15), if one makes use of the commutation relations

$$\{ \mathcal{D}_\alpha, \mathcal{D}_{\dot{\alpha}} \} \Lambda = -2i \mathcal{D}_{\alpha\dot{\alpha}} \Lambda \quad (18)$$

and

$$[\mathcal{D}_{\dot{\alpha}}, \mathcal{D}_{\beta\dot{\beta}}] \Lambda = -2i \varepsilon_{\dot{\alpha}\dot{\beta}} R \mathcal{D}_\beta \Lambda , \quad (19)$$

which follow from the general commutation relations between covariant derivatives [2] when one imposes our kinematic conditions. In a very similar way one verifies that

$$\mathcal{D}_{\dot{\alpha}} W_\beta = 0 . \quad (20)$$

It is now easy to write a Lagrangian which has all the desired invariances, including the gauge invariance (15). It is

$$L = E \left(W^\alpha \mathcal{D}_\alpha V + \overline{W}_{\dot{\alpha}} \mathcal{D}^{\dot{\alpha}} V \right) . \quad (21)$$

This Lagrangian is real and it changes by a total derivative under (15).
The corresponding equation of motion is

$$\mathcal{D}^\alpha W_\alpha + \mathcal{D}_\alpha \overline{W}^{\dot{\alpha}} = 0 \quad , \quad (22)$$

which reduces to the usual one in a flat superspace.

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