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A PROGRAM FOR SOLVING SYSTEMS OF
HOMOGENEOUS LINEAR INEQUALITIES

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PROGRAM SUMMARY

Title of program: LIHOIN

Catalogue number:

Program obtainable from: CPC Program Library, Queen's University of Belfast,
N. Ireland.

Computer: CDC 7600, *Installation:* CERN, Geneva

Operating system: CDC scope

Programming languages used: FORTRAN IV

High speed store required: 721 words. *No. of bits in a word:* 60

Is the program overlaid? No

No. of magnetic tapes required: None

What other peripherals are used? Line Printer

No. of cards in combined program and test deck: 215

Card punching code: BCD

Keywords: Linear inequalities, Fourier expansions of positive functions, polyhedral cones.

Nature of the physical problem

The unitarity condition for physical scattering amplitudes implies the positivity of the imaginary parts of the partial waves. In an explicit construction of amplitudes this leads to inequality constraints for certain expansions.

Method of solution

The Motzkin-Burger rules are used to obtain iteratively the solution polyhedral cone of a system of homogeneous linear inequalities.

Restrictions on the complexity of the problem

It is assumed that the matrix of the system is of rank N , the number of unknowns. Similar to ill-conditioned systems of linear equations, a system of inequalities may also be ill-conditioned, and the program may fail in such cases. Note also that the number of intermediate vectors can become very large, and that some systems are inconsistent.

Running time

The running time is difficult to estimate. It depends not only on the dimensions of the system matrix, but also on its elements.

LONG WRITE-UP

1. THEORETICAL BACKGROUND

We consider a system of linear homogeneous inequalities

$$A\vec{x} \geq 0 \quad (1)$$

where $A = \{a_{mn}\}$ is a given $M \times N$ matrix with $M \geq N$ and $\text{rank}(A) = N$;

$\vec{x} = (x_1, \dots, x_N)^T$ is an N -dimensional vector. The theory of such systems is closely related to the theory of polyhedral cones. In fact, according to a theorem given by Minkowski [1], the set of all solutions of the system (1) consists of a convex polyhedral cone [2]. Let the basis vectors of this cone be $\vec{v}_i$, $i = 1, \dots, I$.

Then any solution $\vec{x}$ may be written in the form

$$\vec{x} = \sum_{i=1}^I \lambda_i \vec{v}_i \quad (2)$$

with non-negative coefficients λ_i , i.e. $\lambda_i \geq 0$. For finite M and N , I is also finite, in particular $I = N$ if $M = N$. A system of inequalities is called inconsistent if $\lambda_i = 0$ for all i . A particular inequality of the system (1) is called redundant if the solution is not changed by omitting it.

Although the theorem of Minkowski describes the solutions of the system (1) completely from a theoretical point of view, it does not answer the question how to obtain the basis vectors of the solution cone in practice. This problem has been solved by Motzkin [3] and Burger [4]. If the system (1) is solved by the cone (2), they describe how the solution cone of a new system which consists of (1) and the additional inequality

$$\vec{a} \cdot \vec{x} \geq 0 \quad (3)$$

is obtained; $\vec{a} = (a_1, \dots, a_N)$ is a given vector. To do this, take all pairs $(\vec{v}_i, \vec{v}_j)$ of basis vectors of the old cone (2) which meet the following criteria:

- i) The scalar products $\vec{a} \cdot \vec{v}_i$ and $\vec{a} \cdot \vec{v}_j$ differ in sign.
- ii) The vectors $\vec{v}_i$ and $\vec{v}_j$ annihilate $N - 2$ inequalities of the old system (1) simultaneously.

iii) There does not exist a third basis vector $\vec{v}_k$, $k \neq i, j$, which annihilates all those inequalities which are annihilated by $\vec{v}_i$ and $\vec{v}_j$.

From these pairs the new vectors

$$\vec{v} = |\vec{a} \cdot \vec{v}_i| \vec{v}_j + |\vec{a} \cdot \vec{v}_j| \vec{v}_i \quad (4)$$

are calculated. The basis vectors of the solution cone of the new system consisting of (1) and (3) include all vectors (4) together with those basis vectors $\vec{v}_i$ of the old cone for which $\vec{a} \cdot \vec{v}_i \geq 0$. By applying these rules iteratively, it is possible to solve arbitrarily large systems. To initiate the algorithm a subset of N linearly independent inequalities with coefficient matrix B is chosen from the system (1). The solution cone of this subsystem

$$B\vec{x} \geq 0 \quad (5)$$

is generated by the column vectors of the inverse matrix B^{-1} .

Some time ago, Chernikova [5] described an algorithm for solving systems of homogeneous linear inequalities and discussed the more general problem of systems of inhomogeneous linear inequalities. (See also the book by Tschernikow [6].) Although this algorithm works well in some cases, we have encountered problems of numerical instability in others. So we have developed a program which is directly based on the Motzkin-Burger procedure. It is described in Section 2. We have tested this program with many examples, and have found it to be numerically stable. An important test of any algorithm, namely the independence of the solution cone on the choice of the initial system (5), has been satisfied in all tested cases. In Section 3 we apply our program to the determination of the constraints on the coefficients of a positive Fourier series. This problem is closely related to our main application, i.e. constructing scattering amplitudes from the so-called axiomatic constraints. In this connection, the positivity constraint arises as part of the unitarity condition.

2. DESCRIPTION OF THE ALGORITHM

The calling sequence for the algorithm LIHOIN is

CALL LIHOIN (A, MA, M, N, MAXV, V, NV, NVEC, EPS, IOUT, W, IW).

The parameters have the following meaning.

DIMENSIONS

A	(MA, $\geq N$)	Two-dimensional array whose rows contain the coefficients of the inequalities, arranged in such a way that the upper left $N \times N$ corner has a non-vanishing determinant. Usually it is preferable to normalize the rows of A to unity before calling LIHOIN.
MA		The first dimension of A.
M		Number of inequalities.
N		Number of variables.
MAXV		The maximum number of basis vectors which may occur at any intermediate step (to be chosen sufficiently large, in any case $\geq N$).
V	(NV, MAXV)	Two-dimensional array whose columns contain, on return, the basis vectors of the solution cone.
NV		The first dimension of V.
NVEC		The number of basis vectors of the final cone.
EPS		A small parameter which discriminates small quantities against zero, chosen to take into account the accuracy of the machine used. For CDC 6000/7000 series computers an appropriate choice is $EPS = 10^{-13}$.
IOUT		IOUT = 0 gives no intermediate print-out. IOUT = 1 gives, for each iteration, the basis vectors of the respective cone, the matrix of scalar products and the index of the inequality taken into account in the next step.
W	(MAXV, $\geq M + 1$)	Two-dimensional array used as working space.
IW	(MA, 5)	Two-dimensional array whose columns serve as book-keepers for certain properties of the system during the iteration procedure.

We now describe the program in some detail, starting with the meaning of the book-keepers IW.

Column 1. The non-zero elements of this column are the indices of the redundant inequalities.

Column 2. Initially $IW(I,2) = I$. If the Ith inequality has been taken into account or becomes redundant, $IW(I,2)$ is set to zero.

Column 3. This column is used during the search for the new inequality. Its elements are set equal to the number of positive scalar products of the Ith inequality with the basis vectors of the old cone.

Column 4. The elements $IW(I,4)$ are used for determining the new basis vectors. If $IW(I,4) = 0$, the Ith inequality already taken into account has $N - 2$ zero scalar products in common with the basis vectors of the old cone under consideration.

Column 5. The non-zero elements of this column enumerate the inequalities which have already been considered.

The subsections of the program can be described as follows. (The headings of (a)-(g) agree with the comments preceding the respective sections of the FORTRAN code.)

a) Sets initial values for book-keeping.

The vectors $IW(I,1)$, $IW(I,2)$ and $IW(I,5)$ and the values NINC (the number of inequalities already taken into account) and NVEC (the number of basis vectors) are assigned their initial values N before the main iteration loop is entered.

b) Determines N basis vectors of the initial polyhedral cone.

The coefficient matrix of the first N inequalities is inverted. We have used the standard CDC MATRIX package, but any other suitable routine can be used. The columns of the inverse matrix are the N basis vectors of the first cone. They are stored in the first N columns of the matrix V .

c) *Computes matrix of scalar products.*

The scalar products of the NINC inequalities with the NVEC basis vectors of their solution cone are computed and stored in the upper left $NVEC \times NINC$ corner of the matrix W.

d) *Determines redundant inequalities and chooses new one.*

The scalar products of these inequalities which have not yet been considered with the basis vectors of the old solution cone are computed. The number of positive scalar products of inequality K is stored in $IW(K,3)$. If $IW(K,3) = 0$ for some value of K, the system is inconsistent, and an error exit is provided. If $IW(K,3) = NVEC$, inequality K is redundant; consequently $IW(K,1)$ and $IW(K,2)$ are assigned the values K and 0 respectively. The value KNEW gives the index of the inequality which is to be taken into account next. It is chosen by the requirement that the number NNEG of negative scalar products of the basis vectors of the old cone is maximal. If $NNEG = NVEC$, all inequalities have already been considered or are redundant.

e) *Computes vector of scalar products.*

The scalar products of the new inequality with the basis vectors of the old cone are computed and stored in the $(M+1)$ th column of W.

f) *Determines basis vectors for the new cone.*

The Motzkin-Burger rules are checked for the pair of basis vectors $(\vec{v}_i, \vec{v}_j)$ in the same order as described in Section 1. The number NT of common zeros of the scalar products of these vectors with the inequalities already taken into account is determined. The zeros of the first NINC elements of the fourth column of IW determine the inequalities to be tested for whether their scalar product with a third vector $v_k (k \neq i, j)$ are also zero. If the number of zeros MT so obtained is less than $N - 2$, a pair $\vec{v}_i, \vec{v}_j$ of basis vectors which meets the Motzkin-Burger criteria has been found. The linear combination (4) of these two vectors is calculated and entered in the

column with index NTVE of the matrix V. Finally the new vector is normalized.

g) *Eliminates vectors with negative scalar product.*

The column vectors of the matrix V with index between NVEC and NTVE which have a negative scalar product with the inequality KNEW are deleted, and V is rearranged. The values for NINC, NVEC and column 5 of IW are updated.

We conclude this section with some general remarks on how to use the sub-routine LIHOIN. It is advantageous to normalize the row vectors $\vec{a}_m$ of the coefficient matrix A to unity, in order to facilitate discrimination between zero and non-zero scalar products. For the same reason all basis vectors are normalized. The choice of the N inequalities which are used for determining the first cone is somewhat arbitrary. A favourable choice may considerably reduce the computation time and the storage space needed. This is due to the fact that the maximal number MAXV of basis vectors which may occur at an intermediate iteration depends very much on this choice. There seems to be no general criterion for an optimal selection. A rule which works in some cases is to take those N inequalities whose coefficient vectors $\vec{a}_m$ generate the largest volume. Better hints may usually be obtained from the nature of the problem from which the inequality system originates. The criterion according to which the new inequality in each iteration is chosen is not unique. Our rule, i.e. to take that inequality which has the largest number of negative scalar products with the basis vectors obtained in the preceding iteration has proved to be quite advantageous in many cases. Generally, the solution obtained should not depend on the choice of EPS and the N inequalities used for initiating the algorithm. Therefore, it is advisable to run the program several times for testing this independence.

The largest amount of storage space is needed for the matrix W. It may be avoided, at the cost of increasing considerably the computing time, by computing anew all scalar products at those places where they are needed. The necessary changes in the program are obvious.

3. ERROR EXITS

In the case where the system A is inconsistent, the error message
LIHOIN ... INCONSISTENT INEQUALITY (k)
is printed on logical unit 2, where (k) denotes the number of the inequality.
Control is returned to the main program.

In the case where MAXV is too small, the error message
LIHOIN ... MAXV TOO SMALL
is printed on logical unit 2. Control is returned to the main program.

4. A TEST EXAMPLE

We present a test example consisting of a system of $M = 8$ inequalities with
 $N = 5$ unknowns. The coefficients of A have been generated at random, and
truncated to 6 decimals. The solution polyhedral cone is generated by $I = 12$
basis vector.

Note that the seventh inequality is found to be redundant. It can therefore
be omitted without changing the result. Note also that reshuffling the inequali-
ties before calling LIHOIN may not show this redundancy, but would, of course, not
change the solution.

The coefficient matrix, the solution, and the vector IW(K,1) showing the
redundant equation are given in Table 1.

5. AN APPLICATION TO POSITIVE FOURIER EXPANSIONS

To illustrate the use of LIHOIN let us consider the constraints for the
coefficients of a positive Fourier series. In principle, these constraints are
known. For a finite Fourier series

$$f(x) = \sum_{n=1}^N c_n e^{inx} \quad (6)$$

with $c_n = \bar{c}_n$ to represent a positive function $f(x)$, it is necessary and sufficient
that the Toeplitz matrices $\{c_{\nu-\mu}\}$ ($\nu, \mu = 1, \dots, n$; $n = 1, \dots, N$) be positive

definite [7]. However, these conditions are rather difficult to apply in practical calculations. With our algorithm the constraints for the coefficients of a positive Fourier series are obtained in a convenient form for applications. Consider a finite Fourier sine series

$$F(\phi) = \sum_{n=1}^N c_n \sin n\phi \quad (7)$$

in the interval $0 \leq \phi \leq \pi$. Imposing the positivity condition at a finite number of arguments ϕ_m , $m = 1, \dots, M$ reduces this problem to a linear system of M homogeneous inequalities in N unknowns which reads

$$F(\phi_m) = \sum_{n=1}^N c_n \sin n\phi_m \geq 0. \quad (8)$$

To be able to represent the results geometrically, we restrict N to 3. The number of points ϕ_m is increased in several steps. For $M = 5$ and $M = 13$ the intersection of the resulting polyhedral cone with the plane $c_1 = 1$ is shown in Fig. 1. The curve represents the limiting cone which is obtained as M goes to infinity. In this special case it turns out that the number I of basis vectors $\vec{v}_i$ equals the number M of inequalities. The iteration was started with the three equations for $m = 0$, $m = [M/2] + 1$, and $m = M$. (Note that (8) has to be divided by $\sin \phi_m$ for $\phi = 0$ and $\phi = \pi$.)

To get a better idea of the numerical accuracy of the approximations obtained for the exact cone, we have calculated, for several values of M , the minimum of $F(\phi)$ in the interval $0 \leq \phi \leq \pi$, for $0 \leq \lambda_i \leq 1$ using a simplex algorithm. The result is given in Table 2. In applications it may be inconvenient to use all the vectors v_i obtained for approximating the exact non-polyhedral cone. Usually one wants to cover with a fixed number of variables a portion which is as large as possible of the exact cone. If the number of variables λ_i is restricted to N , this is most easily achieved by solving system (8) with a large value for M and choosing those N basis vectors of the resulting cone which span the largest volume. For the case under consideration these three vectors are

$$\vec{v}_1 = \begin{pmatrix} 0.948683 \\ 0.000000 \\ -0.316228 \end{pmatrix}, \quad \vec{v}_{2,3} = \begin{pmatrix} 0.653716 \\ \pm 0.617003 \\ 0.438136 \end{pmatrix}. \quad (9)$$

They have been obtained from system (8) with $M = 49$.

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Table 1

Test example

THE MATRIX OF COEFFICIENTS

1	.799217	.864099	.160325	.632229	.052457
2	.466098	.375965	.596879	.271234	.072917
3	.791670	.887416	.650871	.039460	.048344
4	.367939	.400156	.476228	.457051	.462931
5	.845622	.513245	.942230	.099140	.553517
6	.341419	.891279	.227223	.340023	.059538
7	.800058	.926708	.638934	.819688	.514403
8	.464460	.330900	.811808	.546348	.262904

THE BASIS VECTORS FOR THE SOLUTION CONE

1	-.732291	-.154490	-.248447	.060309	-.676429
2	.535582	-.180492	-.210782	-.024277	.684953
3	.129391	.600374	.601146	-.108070	.132250
4	.129470	.346207	.494253	-.097563	-.130917
5	.378679	-.680623	-.536827	.987206	.196604
1	-.594082	.684221	.769376	.722702	.719049
2	.690988	-.205363	-.274743	-.272360	-.260192
3	.280985	-.585115	-.463982	-.533018	-.552519
4	-.250853	.376671	.289682	.345509	.307486
5	-.166498	.073672	-.182709	-.006095	.124277
1	.543936	.282578			
2	.047810	.045654			
3	.161602	-.145844			
4	-.806938	-.451687			
5	.156796	.832330			

THE VECTOR SHOWING THE REDUNDANT INEQUALITIES

0 0 0 0 0 0 7 0

Table 2

M(=I)	5	13	25	49
$\phi_m (m=0, \dots, M-1)$	$0, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \pi$	$\frac{m\pi}{12}$	$\frac{m\pi}{24}$	$\frac{m\pi}{48}$
$F_{\min}(\phi)$	-0.139	-0.0441	-0.0114	-0.00289

Fig. 1 The two polygons and the curve are the intersection of the solution cone of system (8) for $M = 5$, $M = 13$, and $M \rightarrow \infty$, with the plane $c_1 = 1$. The triangle represents the largest polyhedron cone with three faces which may be inscribed inside the solution cone for $M \rightarrow \infty$.