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THE COMPONENT FORMALISM FOLLOWS FROM THE

SUPERSPACE FORMULATION OF SUPERGRAVITY

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ABSTRACT

We identify the transformations of local supersymmetry as particular combinations of general co-ordinate transformations and local Lorentz transformations in superspace. Component fields for supergravity are defined as the values at $\theta = 0$ of the supervielbein and (for the auxiliary fields) of certain components of the supertorsion. For chiral and general multiplets they are defined as the values at $\theta = 0$ of (tangent space) spinorial covariant derivatives. We derive the transformation properties for field multiplets and for chiral density multiplets. The same method can be used for general densities. The Lagrangian of supergravity in terms of component fields is obtained.

Recently, a minimal set of auxiliary fields for simple supergravity has been found and a tensor calculus developed [1-6]. Earlier results obtained in the linearized approximation [7,8] were thus generalized to the full non-linear theory. Clearly, the component formalism with auxiliary fields is contained in the superspace formulation of supergravity [9-13]. There are two ways to extract it. The first is to solve the superspace constraints on the torsion by choosing an appropriate gauge in superspace [13-15]. The second, which is simpler and more elegant, consists in using the constraints, without actually solving them, through repeated use of the Bianchi identities. As we explain below, one can derive in this way all known result of the component formalism*. In particular, the explicit component form of the supergravity Lagrangian including matter coupling can be easily worked out. Some new component results are also obtained (transformation properties of density multiplets). We use the notation and results of ref. [9-13, 16] and, for the component formalism, we adopt as much as possible the notation of ref. [4-6], since those authors have kept close to our own earlier notation.

For any superfield V (with any number of tangent space vector or spinor indices, which we do not indicate explicitly) we consider the transformation

$$\delta V = \xi^A \partial_A V, \quad (1)$$

where the covariant derivative is defined in [9]. This can be interpreted as the combination of a general co-ordinate transformation and of a local Lorentz transformation of parameters

* A different, and larger, set of auxiliary fields was introduced earlier by Breitenlohner [17]. It corresponds to a different choice of constraints in superspace [9, 18]. These different constraints appear to be too restrictive when supergravity is coupled to non conformally invariant supermatter.

$$\xi^M = \xi^A E_A^M, \quad L_{ab} = -\xi^M \phi_{M,ab}, \quad (2)$$

where E_A^M is the inverse supervielbein and $\phi_{M,ab}$ the superconnection. These parameters are field dependent, since we take ξ^A to be field independent. The commutator of two transformations (1) can be obtained using

$$\delta_2 \delta_1 V = \xi_1^A \xi_2^B \mathcal{D}_B \mathcal{D}_A V \quad (3)$$

and the commutator of two covariant derivatives. One finds

$$[\delta_2, \delta_1] V = -\xi_1^A \xi_2^B \left(\frac{1}{2} R_{BA,cd} \mathcal{M}^{cd} + T_{BA}^C \mathcal{D}_C \right) V \quad (4)$$

This is the combination of a field dependent transformation like (1) and of a local Lorentz transformation ($\mathcal{M}^{cd}$ is the infinitesimal Lorentz operator). For the supervielbein itself (2) give

$$\delta E_M^A = \mathcal{D}_M \xi^A + \xi^B T_{BM}^A, \quad (5)$$

where T_{BC}^A is the supertorsion. We shall work in the gauge [9] in which, for $\theta = 0$, the components of the supervielbein take the values

$$E_m^a = e_m^a \quad E_m^\alpha = \frac{1}{2} \psi_m^\alpha \quad (6)$$

$$E_\mu^a = 0 \quad E_\mu^\alpha = \delta_\mu^\alpha \quad (7)$$

and $\phi_{\mu,ab}$ vanishes, while $\phi_{m,ab}$ equals the co-ordinate space

connection^{*}. This still leaves a considerable gauge arbitrariness which can be used to give a convenient form to the higher θ coefficients of the supervielbein, but we shall not need these coefficients here. For a supersymmetry transformation we take $\xi^a = 0$ and $\xi^\alpha = \zeta^\alpha(x)$ for $\theta = 0$. Using the constraints on the torsion, in particular

$$T_{\alpha\beta,}^c = 2i(\gamma^c)_{\alpha\beta}, \quad (8)$$

and setting $\theta = 0$, it is easy to see that (5) will preserve the gauge conditions (7) provided

$$\bar{\zeta}^a = -2i\bar{\zeta}\gamma^a\theta. \quad (9)$$

On the other hand, one finds

$$\delta e_m^a = \frac{1}{2}\psi_m^\alpha \bar{\zeta}^\beta 2i(\gamma^a)_{\alpha\beta} = i\bar{\zeta}\gamma^a\psi_m \quad (10)$$

and

$$\frac{1}{2}\delta\psi_m^\alpha = D_m \bar{\zeta}^\alpha + e_m^c \bar{\zeta}^\beta T_{\beta c,}^\alpha, \quad (11)$$

where D_m is the co-ordinate space covariant derivative. The torsion components $T_{\beta c,}^\alpha$ are expressible [12, 13] in terms of the superfields G_a and R (which is chiral). We identify their values at $\theta = 0$ with

* By the constraint $T_{ab,}^c = 0$, $\phi_{m,ab}$ contains the contribution from the spin $3/2$ field.

the auxiliary fields *

$$G_a = -\frac{1}{3}b_a, \quad R = -\frac{1}{6}M, \quad (\theta=0). \quad (12)$$

From now on we shall use two-component spinor notation. One finds

$$\delta\psi_{m\alpha} = 2\mathcal{D}_m\zeta_\alpha - i b_m\zeta_\alpha - \frac{i}{3}(\sigma_m\bar{\sigma}_n\zeta)_\alpha b^n - \frac{i}{3}(\sigma_m\bar{\zeta})_\alpha M \quad (13)$$

Equations (10) and (13) agree with the transformation laws postulated in [1, 4]. We postpone for a moment discussion of the transformation laws for the auxiliary fields.

The components of a chiral superfield ϕ

$$\mathcal{D}_\alpha\phi = 0 \quad (14)$$

are defined as the values for $\theta = 0$ of certain spinorial tangent space derivatives

$$A = \phi, \quad \chi_\alpha = \mathcal{D}_\alpha\phi, \quad F = -\frac{1}{4}\mathcal{D}^\alpha\mathcal{D}_\alpha\phi, \quad (\theta=0). \quad (15)$$

In curved superspace this definition is more convenient than that in terms of the θ expansion and gives component fields with well defined Lorentz character. These components are complete in the sense that their knowledge determines the chiral superfield. By means of the known commutation relations, higher spinorial derivatives can be reduced to the components above and to space derivatives of them. Applying (1) and

* The complex field M equals $\frac{1}{2}(M + iN)$ of ref. [4].

setting $\theta = 0$ one obtains, from (9) and (14),

$$\delta A = \xi^\alpha \partial_\alpha \phi = \xi^\alpha \chi_\alpha \quad . \quad (16)$$

Similarly

$$\begin{aligned} \delta \chi_\alpha &= (\xi^\beta \partial_\beta - \xi^{\dot{\beta}} \partial_{\dot{\beta}}) \partial_\alpha \phi \\ &= -\frac{1}{2} \xi_\alpha \partial^\gamma \partial_\gamma \phi + 2i \xi^{\dot{\beta}} \partial_{\alpha\dot{\beta}} \phi \quad , \end{aligned} \quad (17)$$

where we have used (14) and the commutation relations [10, 12] among the covariant derivatives. Now we observe that, for any component, the vectorial tangent space covariant derivative in superspace becomes, for $\theta = 0$, the supercovariant derivative of the component formalism^{*}.

Therefore (17) gives

$$\delta \chi_\alpha = 2 \xi_\alpha F + 2i \xi^{\dot{\beta}} \hat{D}_{\alpha\dot{\beta}} A \quad (18)$$

$$\hat{D}_a A = \mathcal{D}_a A - \frac{1}{2} \psi_a^\alpha \chi_\alpha \quad (19)$$

^{*} For instance

$$\begin{aligned} \partial_a \phi &= E_a^m \partial_m \phi + E_a^\mu \partial_\mu \phi \\ \partial_\alpha \phi &= E_\alpha^m \partial_m \phi + E_\alpha^\mu \partial_\mu \phi \end{aligned}$$

For $\theta = 0$, one finds $\partial_\alpha \phi = \delta_\alpha^\mu \partial_\mu \phi$ and

$$\partial_a \phi = e_a^m \mathcal{D}_m \phi - \frac{1}{2} \psi_a^\alpha \partial_\alpha \phi = \mathcal{D}_a A - \frac{1}{2} \psi_a^\alpha \chi_\alpha \quad .$$

Similarly

$$-4 \delta F = (\xi^\alpha \mathcal{D}_\alpha - \xi^{\dot{\alpha}} \mathcal{D}_{\dot{\alpha}}) \mathcal{D}^\beta \mathcal{D}_\beta \phi \quad . \quad (20)$$

Here the commutation relation among the covariant derivatives introduce curvative and torsion terms expressible in G_a and R^*

$$\delta F = -2 \xi^\alpha R^* \mathcal{D}_\alpha \phi + \xi^{\dot{\alpha}} (i \mathcal{D}_{\alpha \dot{\alpha}} \mathcal{D}^\alpha \phi + \frac{1}{2} G_{\alpha \dot{\alpha}} \mathcal{D}^\alpha \phi) \quad . \quad (21)$$

For $\theta = 0$ this gives

$$\delta F = \frac{1}{3} M^* \xi^\alpha \chi_\alpha - \frac{1}{6} \xi^{\dot{\alpha}} b_{\alpha \dot{\alpha}} \chi^\alpha + i \xi^{\dot{\alpha}} \hat{\mathcal{D}}_{\alpha \dot{\alpha}} \chi^\alpha \quad (22)$$

$$\hat{\mathcal{D}}_a \chi_\alpha = \mathcal{D}_a \mathcal{D}_\alpha \phi = \mathcal{D}_a \chi_\alpha - \psi_{a\alpha} F - i \psi_a^{\dot{\beta}} \hat{\mathcal{D}}_{\alpha \dot{\beta}} A \quad . \quad (23)$$

Equations (16), (18) and (22) agree with ref. [2]. Observe that the terms with the auxiliary fields appear naturally and are not part of the supercovariant derivative. The components of the chiral multiplet [13, 20]

$$\Xi = (\mathcal{D}_{\dot{\alpha}} \mathcal{D}^{\dot{\alpha}} - \not{\partial} R) \phi^*, \quad \mathcal{D}_{\dot{\alpha}} \phi = 0, \quad \mathcal{D}_{\dot{\alpha}} \Xi = 0 \quad , \quad (24)$$

can be calculated by the same technique in terms of those of ϕ . One uses the definitions (15) on Ξ and commutes through the covariant derivatives. This is, up to a factor, the "kinetic multiplet" of ref. [3].

For a "vector" supermultiplet, described by a general real superfield V one defines the components as the values for $\theta = 0$ of the following expressions (and complex conjugates when applicable)

$$\begin{aligned}
 C &= V \quad , \\
 \varphi_\alpha &= -i \mathcal{D}_\alpha V \quad , \\
 H &= \frac{i}{4} \mathcal{D}^\alpha \mathcal{D}_\alpha V \quad , \\
 v_{\alpha\dot{\alpha}} &= -\frac{1}{2} [\mathcal{D}_\alpha, \mathcal{D}_{\dot{\alpha}}] V \quad , \\
 \lambda_\alpha &= -\frac{i}{4} W_\alpha \quad , \\
 \mathcal{D} &= \frac{1}{\rho} \mathcal{D}^\alpha W_\alpha \equiv \frac{1}{\rho} \mathcal{D}_{\dot{\alpha}} W^{\dot{\alpha}} \quad , (\theta = 0) \quad ,
 \end{aligned} \tag{25}$$

where

$$W_\alpha = (\mathcal{D}_{\dot{\alpha}} \mathcal{D}^{\dot{\alpha}} - \rho R) \mathcal{D}_\alpha V \quad , \quad \mathcal{D}_{\dot{\alpha}} W_\alpha = 0 \quad . \tag{26}$$

The reason for the particular definition of the components λ and $\mathcal{D}$ is that they are invariant [13] under the gauge transformation

$$V \rightarrow V + i(\Lambda - \Lambda^*) \quad , \quad \mathcal{D}_{\dot{\alpha}} \Lambda = 0 \quad . \tag{27}$$

Observe that $\mathcal{D}$ is real. It is easy to see, using again (1) and the above technique that the components here defined transform as postulated

in ref. [5]. One could choose a different definition, for instance one could choose to antisymmetrize the products of covariant spinorial derivatives. This would change only the components λ_α and D by amounts which can be easily calculated and would correspond to defining the components by expansion in θ . Indeed, one can choose a gauge in which all additional terms coming from the use of covariant, rather than ordinary, derivatives, vanish for $\theta = 0$ due to the antisymmetrization.

We can now return to the transformation laws of the auxiliary fields. We give the example of M . Since M is the lowest component of the chiral superfield $-3R$ it will transform as in (16) into the next component $\eta_\alpha = -3\mathcal{D}_\alpha R$. This is easily calculated by using the identity

$$-3\mathcal{D}_\alpha R = (\sigma^a \bar{\sigma}^b)_\alpha{}^\tau T_{ab,\tau} \quad , \quad (28)$$

which follows from the Bianchi identities (see e.g. ref. [19] formula (4)).

Now

$$T_{ab,\tau} = E_b{}^N E_a{}^M T_{MN,\tau} \quad (29)$$

and

$$T_{MN,\tau} = E_N{}^B E_M{}^A T_{AB,\tau} (-1)^{m(n+b)} \quad (30)$$

Using the constraints on the torsion this last formula shows that $T_{\mu\nu,\tau}$ and $T_{\dot{\mu}\dot{\nu},\tau}$ vanish for $\theta = 0$ and that $T_{m\nu,\tau} = -T_{\nu m,\tau}$ is reducible to $T_{a\beta,\tau}$ and therefore to G_b and R . After some calculation one finds, setting $\theta = 0$,

$$\delta M = \zeta^\alpha \eta_\alpha \quad ,$$

$$\eta_\alpha = \frac{1}{2} (\sigma^a \bar{\sigma}^b \psi_{ab})_\alpha - i(\sigma \cdot \bar{\psi})_\alpha M + \frac{i}{2} (b \cdot \psi)_\alpha \quad (31)$$

The case of b_a can be treated in analogous fashion (now one must calculate $\mathcal{D}_\alpha G_b$) and also gives a result in agreement with [1] and [2]. By the same technique one can calculate the last component of R and the other components of G_b .

By means of the components (15) of the chiral superfield ϕ one can define a superfield ϕ_1 which depends only on θ^α and not on $\bar{\theta}^{\dot{\alpha}}$

$$\phi_1(x, \theta) = A + \theta^\alpha \chi_\alpha + \theta^\alpha \theta_\alpha F \quad (32)$$

and the transformation properties (16), (18) and (22) can be written compactly in the form

$$\delta \phi_1 = \xi^M \partial_M \phi_1 \quad (33)$$

(the indices μ and α are now identified). The superfield ϕ_1 could be obtained from ϕ by choosing a special gauge in superspace (chiral gauge) but we prefer here to define it abstractly. The field dependent parameters ξ^M can be calculated by comparing different powers of θ . One finds (see also ref. [20])

$$\begin{aligned} \xi^m &= 2i \theta \sigma^m \bar{\zeta} + \theta \theta \bar{\psi}_m \bar{\sigma}^m \sigma^n \bar{\zeta} \\ \xi^\mu &= \zeta^\mu - i \theta \sigma^m \bar{\zeta} \psi_m^\mu + \theta \theta \bar{\zeta}_2^\mu \end{aligned} \quad (34)$$

$$\begin{aligned} \bar{\zeta}_2^\mu &= \frac{1}{3} M^* \zeta^\mu + \frac{1}{6} (b, \sigma \bar{\zeta})^\mu - i \phi_{m,}{}^{\mu\alpha} (\sigma^m \bar{\zeta})_\alpha \\ &\quad + \frac{1}{2} \psi_m^\mu (\bar{\psi}_m \bar{\sigma}^n \sigma^m \bar{\zeta}) \end{aligned}$$

and, of course, $\xi^{\dot{u}}$ can be taken to vanish. Now it is natural to define a chiral density superfield

$$\Delta = a + \theta^\alpha \rho_\alpha + \theta^\alpha \theta_\alpha f \quad (35)$$

by the transformation law

$$\delta \Delta = \partial_M (\xi^M \Delta) (-1)^m = \xi^M \partial_M \Delta + \partial_M \xi^M (-1)^m \Delta \quad (36)$$

The closure of the algebra in the same way as for fields is automatically ensured. The factor $\partial_M \xi^M (-1)^m$ can be calculated from (1). Expanding (36) in powers of θ one finds

$$\delta a = \zeta \rho - i a \psi \cdot \sigma \bar{\zeta} \quad , \quad (37)$$

$$\begin{aligned} \delta \rho_\alpha = & 2 \zeta_\alpha f + 2i D_m [a (\sigma^m \bar{\zeta})_\alpha] - i (\sigma^m \bar{\zeta})_\alpha \psi_m \rho \\ & + \frac{2}{3} \zeta_\alpha M + \frac{1}{3} a (\sigma \cdot b \bar{\zeta})_\alpha - \psi_{m\alpha} \bar{\psi}_m \bar{\sigma}^n \sigma^m \bar{\zeta} \end{aligned} \quad (38)$$

and

$$\delta f = \partial_m (a \bar{\psi}_m \bar{\sigma}^m \sigma^n \bar{\zeta} - i \rho \sigma^m \bar{\zeta}) \quad (39)$$

Appropriately the f component changes by a total derivative. Observe that we could have written (36) in terms of covariant superspace derivatives,

however, the calculation would have then required knowledge of the θ expansion of the supervielbein, which we were able to avoid consistently in this paper. One knows from superspace arguments [14] that there is a very special chiral density multiplet $\mathcal{E}$ which has first component proportional to the determinant of the ordinary vierbein. The other components can be calculated by comparing (37) to (39) with the explicit variation from the transformation laws of the supergravity fields. For instance, set, for $\mathcal{E}$,

$$a = \frac{e}{2} \quad . \quad (40)$$

Then

$$\delta a = \zeta \rho - i a \psi \cdot \sigma \bar{\zeta} = \frac{ie}{2} (\zeta \sigma \cdot \bar{\psi} - \psi \cdot \sigma \bar{\zeta}) \quad . \quad (41)$$

Therefore

$$\rho = i \frac{e}{2} \sigma \cdot \bar{\psi} \quad . \quad (42)$$

In the same way one finds

$$f = - \frac{e}{2} M - \frac{e}{\not{x}} \bar{\psi}_m (\bar{\sigma}^m \sigma^n - \bar{\sigma}^n \sigma^m) \psi_n \quad . \quad (43)$$

Inserting a and ρ from (40) and (42) into (39) one finds for this f

$$\delta f = - \frac{1}{2} \partial_m \left[e \bar{\psi}_m (\bar{\sigma}^n \sigma^m - \bar{\sigma}^m \sigma^n) \bar{\zeta} \right] \quad , \quad (44)$$

which can also be verified directly. Multiplication of a chiral superfield with the density multiplet $\mathcal{E}$ gives another density multiplet. The last component of the product gives the density formula of ref. [2]. One knows

from superspace arguments [13] that the last component of $\mathcal{E} R$ is the supergravity Lagrangian (up to a factor). The above formulas permit explicit calculation of it.

The method applied here to chiral densities can also be used for general ("vector") densities. First, one finds the relation between the components (25) of the real superfield V and those defined by θ expansion, or by antisymmetrization of the covariant spinorial derivatives. Using (1) in the form

$$\delta V = \xi^M \partial_M V, \quad (45)$$

one can then calculate the field dependent parameters ξ^M . The density transformation law has the form

$$\delta D = \partial_M (\xi^M D) (-1)^m. \quad (46)$$

Comparing the transformation for a general density multiplet with that explicitly calculated, one finds the components of the density $E = \det E_M^A$ whose lowest component is e . In this way one derives the "vector" density formula of ref. [5]. One knows from superspace arguments [12, 13] that the last component of E is the supergravity Lagrangian; its explicit component form can therefore also be derived by this method.

"Well, in *our* country", said Alice, "you'd generally get to somewhere else - if you ran very fast for a long time as we've been doing."

"A slow sort of country!" said the Queen. "Now, *here*, you see, it takes all the running *you* can do, to keep in the same place. If you want to get somewhere else, you must run at least twice as fast as that." [L. Carroll, Through the Looking Glass (1872)].

REFERENCES

- [1] S. Ferrara and P. van Nieuwenhuizen, Phys.Lett. 74 B (1978) 333.
- [2] S. Ferrara and P. van Nieuwenhuizen, Phys.Lett. 76 B (1978) 404.
- [3] S. Ferrara and P. van Nieuwenhuizen, Ecole Normale preprint LPTENS 78/14.
- [4] K.S. Stelle and P.C. West, Phys.Lett. 74 B (1978) 330.
- [5] K.S. Stelle and P.C. West, Imperial College preprint ITCP/77-78/15.
- [6] K.S. Stelle and P.C. West, Imperial College preprint ITCP/77-78/24.
- [7] V. Ogievetsky and E. Sokatchev, Nucl.Phys. B 124 (1977) 309.
- [8] S. Ferrara and B. Zumino, Nucl.Phys. B 134 (1978) 301.
- [9] J. Wess and B. Zumino, Phys.Lett. 66 B (1977) 361.
- [10] R. Grimm, J. Wess and B. Zumino, Phys.Lett. 73 B (1978) 415.
- [11] J. Wess, Supersymmetry - Supergravity, Lectures given at the VIII G.I.F.T. Seminar (Salamanca, 1977), Karlsruhe University preprint, to be published in Springer Tracts.
- [12] J. Wess and B. Zumino, Phys.Lett. 74 B (1978) 51.
- [13] B. Zumino, Cargèse Summer School Lectures (July 1978), to be published in the Proceedings, Plenum Press.
- [14] W. Siegel, Harvard University preprint, to be published in Nucl.Phys.
- [15] R. Grimm, Karlsruhe University Ph.D. Thesis.

- [16] S. Ferrara, J. Wess and B. Zumino, Phys.Lett. 51 B (1974) 239.
- [17] P. Breitenlohner, Phys.Lett. 67 B (1977); Nucl.Phys. B 124 (1977) 500.
- [18] L. Brink, M. Gell-Mann, P. Ramond, J.H. Schwarz, Phys.Lett. 74 B (1978) 336.
- [19] P.S. Howe and R.W. Tucker, CERN preprint TH.2524, to be published in Phys.Lett.
- [20] M. Roček and U. Lindström, Components of Superspace, submitted to Phys.Lett.