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ON THE DIRECTIONS OF SPONTANEOUS SYMMETRY BREAKING
IN $SU(n)$ GAUGE THEORIES

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A B S T R A C T

The $SU(n)$ representations containing a vector stabilized by a maximal $SU(n)$ subgroup or by a subgroup of rank $(n-1)$ are characterized. Large use is made of the Gel'fand-Zetlin basis. The relevance of these results for selecting Higgs representations in $SU(n)$ gauge models is emphasized.

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1. - INTRODUCTION

From the long list of unified gauge theory models which have been proposed these last few years, only few of them seem to survive ¹⁾ after the discovery last June at SLAC of a large violation of parity in polarized e^- deuteron scattering. Actually the best candidate in unified weak and electromagnetic gauge theories remains today the "old" $SU(2) \times U(1)$ model first proposed by Weinberg and Salam ²⁾, while including strong interactions, the favoured groups are now $SU(3)_C \times SU(2) \times U(1)$ and $SU(5)$ ^{3), 4)} for a grand unification theory. However it is not at all excluded that the weak and electromagnetic interactions are unified in a gauge group bigger than $SU(2) \times U(1)$ ⁵⁾, and that the grand unified theory group must consequently be of rank bigger than four.

Among the difficulties a gauge model builder is faced with, a crucial one is the choice of the Higgs representations, i.e., the representation R_G of the gauge group G which will contain the Higgs scalar field. Aside from the physical implications, the chosen Higgs representation must satisfy stringent conditions. As the gauge group G is supposed spontaneously broken down to the subgroup S , there must exist in R_G a ϕ_0 component which develops a vacuum expectation value $\langle \phi_0 \rangle$ invariant under (and only under) the subgroup S . Moreover there must be a G invariant quartic polynomial potential whose absolute minimum occurs at $\langle \phi_0 \rangle$.

Actually the first condition can be formulated in a slightly different way : let G be a compact Lie group, S a subgroup of G , what are the representations R_G of G containing a vector ϕ_0 the stabilizer - or little group - of which is exactly S ? We recall that the stabilizer or little group of a vector ϕ_0 in R_G is defined as the biggest subgroup which leaves this vector invariant : $\text{Stab}(\phi_0) = \{g \in G | g(\phi_0) = \phi_0\}$. Let us also define by $\mathcal{L}(R_G)$ the set of all the little groups of the vectors in R_G : $\mathcal{L}(R_G) = \{S \subset G | \exists \phi_0 \in R_G \text{ with } \text{Stab}(\phi_0) = S\}$.

We know from a theorem by Mostow ⁶⁾ that such a question has always an answer, i.e., that for any couple of compact groups G, S with $G \supset S$, there exists a, in general reducible, representation R_G with $\phi_0 \in R_G$ and $\text{Stab}(\phi_0) = S$. Unfortunately the proof of this theorem does not

bring any help in the construction of such a representation. It is clear that a complete answer to this question would solve the interesting but difficult problem of the classification of all the orbits (and strata) ⁷⁾ of any representation of a compact group.

Restricting to the case of $G = \text{SU}(n)$ - or $\text{U}(n)$ - it is possible, using simple methods, to answer in part this question, at least to deduce in certain cases some necessary and sufficient conditions for any irreducible representation to contain a vector stabilized by a specific Lie subgroup S of $\text{SU}(n)$ [or $\text{U}(n)$]. Using mainly the framework of the Gel'fand-Zetlin basis, we characterize hereafter all the $\text{SU}(n)$ irreducible representations $R_{\text{SU}(n)}$ such that $\mathcal{S}(R_{\text{SU}(n)})$ contains a maximal or/and a rank $(n-1)$ Lie subgroup of $\text{SU}(n)$. It has to be noted that our results are formulated in a remarkably simple way. Besides the usefulness of these properties for selecting Higgs representations in $\text{SU}(n)$ gauge models, this paper can be considered as a first step in the above-mentioned classification of orbits. Of course, the generalization of this study which deals with irreducible representations is in principle easy since any $\text{SU}(n)$ representation R is a finite sum of irreducible ones : $R = \bigoplus_{i=1}^n R_i$, and therefore the stabilizer $S = \text{Stab}(\phi)$ of a vector $\phi = (\phi_1, \dots, \phi_n) \in R$ is the intersection $S = \bigcap_{i=1}^n S_i$ of the stabilizers $S_i = \text{Stab}(\phi_i)$ with $\phi_i \in R_i$.

A first approach to this problem has already been tried by Lieberman ⁸⁾ who studied the case where G is a classical Lie group and S a $\text{U}(1)$ subgroup. In the same spirit, the isotropy subgroups of $\text{O}(3)$, discrete subgroups included, have been classified by Ovrut ⁹⁾. Moreover, Li ¹⁰⁾, constructing the possible Higgs potentials V corresponding to certain irreducible representations (namely fundamental, adjoint, and second rank tensor) of a classical compact Lie group, deduced in each case the associate S subgroup stabilizing the minimum of V .

In Section 2 we briefly recall the main properties of the Gel'fand-Zetlin basis for an $\text{SU}(n)$ representation. In Section 3 we characterize the irreducible representations $R_{\text{SU}(n)}$ such that $\mathcal{S}(R_{\text{SU}(n)})$ contains an $\text{SU}(n)$ subgroup of rank $(n-1)$, while Section 4 is devoted to the case of the maximal $\text{SU}(n)$ subgroups $\text{SO}(n)$, $\text{Sp}(n)$ and $\text{SU}(r) \times \text{SU}(s)$ with $r \cdot s = n$. In Section 5 all the continuous stabilizers for any irreducible $\text{SU}(3)$ representations are classified. A brief illustration of our results for the $\text{SU}(5)$ and $\text{SU}(6)$ gauge groups is given in Section 6. Finally, the Appendix is devoted to the proof of the results presented in Section 3.

2. - THE GEL'FAND-ZETLIN BASIS FOR AN $SU(n)$ REPRESENTATION

Hereafter we make a large use of the Gel'fand basis for an irreducible representation of $SU(n)$. So let us start by briefly recalling its definition and first properties ¹¹⁾⁻¹³⁾.

Any irreducible representation of $SU(n)$ may be defined by its Young symmetry pattern characterized by the ordered set of $(n-1)$ integers $m_{1n}, m_{2n}, \dots, m_{n-1,n}$ ($m_{n,n}$ being zero because of the unimodular restriction) satisfying : $m_{1n} \geq m_{2n} \geq \dots \geq m_{n-1,n} \geq 0$. Expressions of the Casimir operator eigenvalues for $SU(n)$ in terms of these integers m_{in} , $i=1,2,\dots,n-1$, can be found in Ref. 12). Let us recall that the dimension of the representation $(m_{1n}, m_{2n}, \dots, m_{n-1,n})$ of $SU(n)$, i.e., the representation of the Young tableau which contains m_{1n} boxes in the first row, m_{2n} boxes in the second row and $m_{n-1,n}$ boxes in the $(n-1)$ th row can be obtained by the formula

$$N = \frac{(\lambda_1+1)(\lambda_2+1) \dots (\lambda_{n-1}+1)(\lambda_1+\lambda_2+2) \dots (\lambda_{n-2}+\lambda_{n-1}+2) \dots (\lambda_1+\lambda_{n-1}+n-1)}{1! 2! \dots (n-1)!} \quad (2.1)$$

where

$$\lambda_1 = m_{1n} - m_{2n} ; \lambda_2 = m_{2n} - m_{3n} ; \dots ; \lambda_{n-1} = m_{n-1,n}.$$

As is well known, one can get the states of the corresponding representation by considering all possible lexical Young tableaux which are obtained by filling the Young pattern with the integers one to n with repetitions but in lexical order : these states are designated as Weyl basis states. Now, it is possible to make a one-to-one correspondence between a Weyl basis and a Gel'fand basis defined by the orthonormal states (m) represented by the triangular pattern

$$(m) = \begin{pmatrix} m_{1n} & m_{2n} & \dots & m_{n-1,n} & 0 \\ & m_{1,n-1} & \dots & m_{n-1,n-1} & \\ & & \dots & & \\ & & m_{12} & m_{22} & \\ & & & m_{11} & \end{pmatrix} \quad (2.2)$$

with

$$m_{i,j+1} \geq m_{i,j} \geq m_{i+1,j+1} \geq 0. \quad (2.3)$$

This condition (2.3) comes from the requirement that the Weyl basis tableau is lexical.

As emphasized in Refs. 11)-13), the Gel'fand basis for an irreducible representation of $SU(n)$ makes transparent the canonical decomposition $SU(n) \supset SU(n-1) \times U(1) \supset SU(n-2) \times U(1) \times U(1) \dots$ (cf. the branching law theorem of Weyl) which allows to solve in particular the state labelling problem. Indeed, the $SU(n-1)$ representations appearing in the $SU(n)$ representation $(m_{1n}, m_{2n}, \dots, m_{n-1,n})$ are fully characterized from the second row, i.e., $(m_{1,n-1} - m_{n-1,n-1}, \dots, m_{n-2,n-1} - m_{n-1,n-1})$ and the corresponding states by the above triangle of (2.2) in which the first row has been thrown away. In the same way, the representations of the $SU(n-p)$ subgroup with $p = 2, \dots, n-2$ in the canonical chain, are described by the triangle obtained from (2.2) by taking away the p first rows.

In order to obtain the action of the group $SU(n)$ on its representation $(m_{1n}, \dots, m_{n-1,n})$ it is convenient to introduce the so-called Weyl infinitesimal generators $E_{ij} = E_{ji}^+$, $i, j = 1, 2, \dots, n$, of the unitary group, which satisfy the commutation relations :

$$[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{il} E_{kj} ; \quad i, j, k, l = 1, 2, \dots, n \quad (2.4)$$

The Weyl generators E'_{ij} for $SU(n)$ will be deduced from those of $U(n)$:

$$E'_{ij} = E_{ij} - \delta_{ij} \frac{1}{n} \sum_{k=1}^n E_{kk} \quad (2.5)$$

The Gel'fand basis is chosen such that :

$$E'_{jj} | (m) \rangle = \sum_{i \leq j} (m_{ij} - m_{i,j-1}) | (m) \rangle \quad (2.6)$$

As for the non-diagonal generators E_{ij} ($i \neq j$), they act on the basis vector as follows :

$$E_{ij} |(m)\rangle = \sum_{(m')} \langle (m') | E_{ij} | (m) \rangle |(m')\rangle \quad (2.7)$$

with the coefficients $\langle (m') | E_{ij} | (m) \rangle$ vanishing unless the labels of the state (m) are the same as those of the state (m') in all rows except rows $i, i+1, \dots, j-1$ if $i < j$ (or $j, j+1, \dots, i-1$ if $j < i$). In these rows, the allowed final labels are those which can be obtained from the initial ones by adding 1 (if $i < j$) or subtracting 1 (if $j < i$) in all possible positions. As an example which will be useful in the following let us write down explicitly the action of the "raising" operator $E_{i,i+1}$ on the basis vector (m) :

$$E_{i,i+1} \left| \begin{pmatrix} m_{1n} & \dots & m_{n-1,n} & 0 \\ \vdots & & & \\ m_{1i} & \dots & m_{ii} \\ \vdots & & \\ m_{11} \end{pmatrix} \right\rangle = \sum_{\tau=1}^i \left[(-1)^{\frac{i+1}{2}} \frac{\prod_{\delta=1}^{i+1} (m_{\delta,i+1} - m_{\delta,i} - \delta + \tau)}{\prod_{\substack{\delta=1 \\ \delta \neq \tau}}^i (m_{\delta,i} - m_{\tau,i} - \delta + \tau - 1)} \right] \times$$

$$\left[\frac{\prod_{\delta=1}^{i-1} (m_{\delta,i-1} - m_{\tau,i} - \delta + \tau - 1)}{\prod_{\substack{\delta=1 \\ \delta \neq \tau}}^i (m_{\delta,i} - m_{\tau,i} - \delta + \tau)} \right]^{\frac{1}{2}} \left| \begin{pmatrix} m_{1n} & \dots & m_{n-1,n} & 0 \\ \vdots & & & \\ m_{1i} & \dots & m_{\tau,i}+1 & \dots & m_{ii} \\ \vdots & & & \\ m_{11} \end{pmatrix} \right\rangle \quad (2.8)$$

3. - LITTLE GROUPS OF RANK (n-1) IN SU(n) REPRESENTATIONS

As previously mentioned, the Gel'fand-Zetlin basis for $SU(n)$ is directly related to the canonical decomposition

$$SU(n) \supset SU(n-1) \times U(1) \supset \dots \supset SU(n-k) \times \prod_{j=1}^k U(1)_j \supset \dots \supset \prod_{j=1}^{n-1} U(1)_j = \text{Cart}(SU(n))$$

We shall hereafter exploit this property in order to select the irreducible $SU(n)$ representations in which can be found a vector invariant under an $SU(n)$ subgroup of the type

$$\prod_{\substack{i=1 \\ p_i > 1}}^l SU(p_i) \times \prod_{j=1}^{n+l-1-\sum_{i=1}^l p_i} U(1)_j \quad \text{or} \quad \prod_{j=1}^{n-1} U(1)_j$$

i.e., an $SU(n)$ subgroup containing the Cartan subgroup $\text{Cart}(SU(n))$.

Among this set of subgroups, the groups $U(n-1)$ and $S[U(n-p) \times U(p)]$, $p > 1$, are maximal in $SU(n)$. (Let us recall that G' is a maximal Lie subgroup of G if there exists no Lie subgroup $\tilde{G}$ such that $G' \subsetneq \tilde{G} \subsetneq G$.)

Our results can be summarized in the following propositions.

Proposition 1

The irreducible representation $R(m_{1n}, m_{2n}, \dots, m_{n-1,n})$ of $SU(n)$ contains a vector the stabilizer of which is $S[U(n-p) \times U(p)]$ $n-p \geq p > 1$ if and only if the two following conditions are satisfied :

- i) $m_{in} + m_{n-i+1,n} = 2\lambda$ for $i=1, 2, \dots, n-1$ and λ a positive integer,
- ii) $m_{n-p,n} \geq \lambda$.

Note that condition i) implies $R(m_{1n}, \dots, m_{n-1,n})$ to be self-conjugate and such that $\sum_{i=1}^{n-1} m_{in} = n\lambda$.

Proposition 2

The irreducible representation $R(m_{1n}, \dots, m_{n-1,n})$ of $SU(n)$ contains a vector the stabilizer of which is of the form

$$\prod_{\substack{i=1 \\ p_i > 1 \\ p_1 > p_2 > \dots > p_l > 1}}^l SU(p_i) \times \prod_{j=1}^{n+l-1-\sum_{i=1}^l p_i} U(1)_j$$

(except $S[\overline{U}(n-p) \times U(p)]$, $p > 1$) if and only if :

- i) $\sum_{i=1}^{n-1} m_{in} = n\lambda$ with λ a positive
- ii) $m_{p_1, n} \geq \lambda$ if $n+1 > 2p_1 \geq 2$ and $m_{k, n} = \lambda$ for $k = n-p_1+1, \dots, p_1$ if $n+1 \leq 2p_1$. This last condition has also to be satisfied by the ordered set of integers $(m'_{1n}, \dots, m'_{n-1, n})$ corresponding to the $SU(n)$ representation conjugate to $R(m_{1n}, \dots, m_{n-1, n})$.

Proposition 3

The irreducible representation $R(m_{1n}, \dots, m_{n-1, n})$ of $SU(n)$ contains a vector the stabilizer of which is $\prod_{j=1}^{n-1} U(1)_j = \text{Cart}(SU(n))$ if and only if $\sum_{i=1}^{n-1} m_{in} = n\lambda$, λ positive integer.

The proofs of these propositions are given in the Appendix. The case where the little group is the maximal one $SU(n-1) \times U(1)$ appears as a particular case of Proposition 2 :

Corollary 1

The irreducible $R(m_{1n}, \dots, m_{n-1, n})$ of $SU(n)$ contains a vector invariant under $U(n-1)$ if and only if $m_{2n} = \dots = m_{in} = \dots = m_{n-1, n} = \frac{1}{2} m_{1n}$. The representation $R(m_{1n}, \dots, m_{n-1, n})$ will contain a vector stabilized by $SU(n-1)$ if and only if $m_{2n} = \dots = m_{n-1, n} \neq \frac{1}{2} m_{1n}$.

From Propositions 1, 2 and 3, one can also deduce :

Corollary 2

The set of little groups of the self-conjugate $SU(n)$ representations of the form $R(2\lambda, \lambda, \dots, \lambda)$ $\lambda \geq 1$ contains as a subset all the $SU(n)$ subgroups of rank $(n-1)$.

Let us note that a simple method to find the $SU(n)$ representations with a vector invariant at least under $SU(n-p) \times SU(p)$ $p > 1$ has been proposed in Ref. 14) using the decomposition of tensor product of representations into irreducible ones. The relevant representations are simply those satisfying $\alpha(n-p) + \beta p = \sum_{i=1}^{n-1} m_{in}$ with α and β integers ≥ 0 , and such that their corresponding Young tableaux appear in the decomposition of the outer product

Let us remark that the above-mentioned procedure does not bring any information on the $U(1)$ part. Of course, one can conclude that the $SU(n)$ representations obtained in this way and not satisfying the conditions of Proposition 1, contain vectors stabilized by $SU(n-p) \times SU(p)$, $p > 1$. It is, however, not possible to conclude that an $SU(n)$ representation satisfying the conditions of Proposition 1 does not contain a vector with little group $SU(n-p) \times SU(p)$, $p > 1$. As an example, let us consider the representation $(2,2,0)$ of $SU(4)$: it is a simple exercise to check that the little group of the vector (in Gel'fand-Zetlin basis)

$$\begin{pmatrix} 2 & 2 & 0 & 0 \\ & 2 & 2 & 0 \\ & & 2 & 2 \\ & & & 2 \end{pmatrix}$$

is $SU(2) \times SU(2)$ while the little group of the vector

$$\begin{pmatrix} 2 & 2 & 0 & 0 \\ & 2 & 1 & 0 \\ & & 1 & 1 \\ & & & 1 \end{pmatrix}$$

is $SU(2) \times SU(2) \times U(1)$.

Let us conclude this section by considering the simplest representations of $SU(n)$, i.e., the fundamental $R(1,0,\dots,0)$ representation and the adjoint one $R(2,1,\dots,1)$. From the above propositions, one deduces that the fundamental representation admits $SU(n-1)$ as a little group while any $SU(n)$ subgroup of rank $(n-1)$ is a little group for the adjoint one. Actually these subgroups exhaust the sets of the stabilizers for these representations. Indeed, one can easily check that all the vectors of a same length are on the same orbit in the fundamental $SU(n)$ representation and consequently that $SU(n-1)$ is the only little group associated to $R(1,0,\dots,0)$. For the representation $R(2,1,\dots,1)$ a direct proof is the following : the adjoint representation of $SU(n)$ which can be seen also as its Lie algebra can be represented by the set of (n^2-1) Hermitian traceless matrices H . By action of the group $SU(n)$ on its Lie algebra : $H \rightarrow UHU^{-1}$, $U \in SU(n)$, any H can be diagonalized. It is then a simple exercise to verify that the set of little groups is exactly the set of all the $SU(n)$ subgroups which contain the Cartan subgroup $\text{Cart}(SU(n))$.

4. - THE MAXIMAL SUBGROUPS $SO(n)$, $Sp(n)$ AND $SU(r) \times SU(s)$ WITH $r.s = n$ AS LITTLE GROUPS IN $SU(n)$ REPRESENTATIONS

In addition to the subgroups $S[U(n-p) \times U(p)]$, $p > 1$, and $U(n-1)$ (note that this number is $n/2$ for n even and $n-1/2$ for n odd), the following subgroups are also maximal in $SU(n)$:

- i) $SO(n)^d$, the Dynkin index ¹⁵⁾ of which is $d=4$ for $n=3$ and $d=2$ for $n > 3$;
- ii) $Sp(n)$ if n is even ;
- iii) the "tensorial" subgroups $SU(r)^s \times SU(s)^r$ with $r.s=n$.

To this list must be added two "exceptions" ¹⁵⁾ : the maximal subgroups $SU(n-1)^{n-1}$ in $SU(\frac{1}{2}(n-1)(n+2)-1)$ for $n > 3$ and $SU(n-1)^{n+3}$ in $SU(\frac{1}{2}n(n+3)-1)$ with $n > 1$. Note that the smallest groups containing such maximal subgroups are $SU(6)$, $SU(10)$, $SU(15)$. These exceptions will not be considered hereafter ¹⁶⁾.

Let us recall that all these above maximal subgroups are also called S subgroups while those of the family $SU(n-1) \times SU(p) \times U(1)$, $p > 1$ and $SU(n-1) \times U(1)$ are called "regular" in Dynkin's theory ¹⁵⁾. General properties for $SU(n)$ representations containing vectors with stabilizer of the type i), ii) or iii) can be found in the literature.

Stabilizer of the type : $SU(r)^s \times SU(s)^r$ with $r.s=n$

The construction of $SU(n)$ representations in which $SU(r)^s \times SU(s)^r$, $r.s=n$, is the little group of a vector can be deduced from the study of Ref. 14). Such vectors can only be found in the $SU(n)$ representations $R(m_{1,n}, \dots, m_{n-1,n})$ satisfying the conditions :

$$\sum_{i=1}^{n-1} m_{in} = \lambda r = \lambda' s$$

with λ and λ' positive integers, and appearing in the decomposition into irreducible representations of the inner product of representations (considered as representations of the symmetric group S_{n-1}) :

$$r \left\{ \begin{array}{c} \square \quad \square \\ \square \quad \square \end{array} \right\} \times \lambda' \left\{ \begin{array}{c} \square \quad \square \\ \square \quad \square \end{array} \right\}$$

$\underbrace{\hspace{10em}}_{\lambda} \qquad \qquad \underbrace{\hspace{10em}}_{\lambda'}$

As an example, the $SU(6)$ irreducible representations $R(m_{16}, \dots, m_{56})$ with $\sum_{i=1}^5 m_{i6} = 6$ containing a vector invariant under $SU(3)^2 \times SU(2)^3$ are those appearing in the right-hand side of the relation :

i.e., after taking away the trivial representation $[1^6]$, the three representations $[2^2, 1^2]$, $[3^2]$ and $[4, 1^2]$ of dimensions 189, 490 and 840, respectively.

Stabilizer of the type $SO(n)^d$ ($d=4$ for $n=3$ and $d=2$ for $n>3$) and of the type $Sp(n)$ if n even

The reduction of the representation $R(m_{1n}, \dots, m_{n-1,n}) = [\lambda]$ of $SU(n)$ into representations (η) of the maximal subgroups $SO(n)^d$ and $Sp(n)$, n even, may be determined by a theorem due to Littlewood¹⁷⁾. This theorem states that if in the reduction into irreducible representations of the product $[\sigma] \times [\beta]$ of $SU(n)$ representations, the $SU(n)$ representation $[\lambda]$ appears $\Gamma_{\sigma\beta\lambda}$ times, then the representation $[\lambda]$ can be decomposed into representations (σ) of $SO(n)^d$ - or $Sp(n)$ - according to the formula :

$$[\lambda] = \sum_{\beta, \sigma} \Gamma_{\sigma\beta\lambda} (\sigma)$$

the summation being over all partitions $[\beta]$ of the form :

$$\begin{aligned} [\beta] &= [0], [2], [4], [2, 2], [4, 2] \dots \text{ for } SO(n) \\ &= [0], [11], [22], [1111], [2222] \dots \text{ for } Sp(n) \end{aligned}$$

Being concerned only with the case $[\sigma] = [0]$, we can deduce :

Proposition 4

The maximal subgroup $SO(n)^d$ ($d=4$ for $n=3$ and $d=2$ for $n>3$) of $SU(n)$ stabilizes a vector in $R(m_{1n}, \dots, m_{n-1,n})$ if and only if all the integers $m_{1n}, \dots, m_{n-1,n}$ are even.

The maximal subgroup $Sp(n)$ of $SU(n)$ (n even) stabilizes a vector in $R(m_{1n}, \dots, m_{n-1,n})$ if and only if $m_{1n} = m_{2n}, m_{3n} = m_{4n}, \dots, m_{n-1,n} = 0$.

Of course, these results could be re-obtained in the framework of Gel'fand-Zeltin basis. Indeed, one can express the generators of these subgroups in terms of the E_{ij} and proceed along the same lines as those of the Appendix.

5. - LITTLE GROUPS IN $SU(3)$ REPRESENTATIONS

Although it appears difficult to obtain simple general results for types of subgroups not considered in Sections 3 and 4, the Gel'fand-Zeltin basis can anyway be very useful as soon as we are faced with a particular problem. As an illustration, we will see that in the particular case of $SU(3)$, the use of the Gel'fand-Zeltin basis allows the determination of all the continuous stabilizers of $\mathcal{S}(R_{SU(3)})$ for any irreducible $R_{SU(3)}$.

Let $R(m_1, m_2) = R(p, q)$ be an irreducible $SU(3)$ representation. The correspondence between the usual labels associated with the decomposition $SU(2) \times U(1)$ and the Gel'fand-Zeltin labels is the following :

$$\begin{aligned} j &= \frac{1}{2} (m_{12} - m_{22}) \\ j_2 &= m_{11} - \frac{1}{2} (m_{12} + m_{22}) \\ Y &= \frac{1}{2} (m_{12} + m_{22}) - \frac{1}{3} (p+q) \end{aligned} \quad (5.1)$$

A classification up to a conjugation of the continuous subgroups of $SU(3)$ can be found in Ref. 18).

From Corollary 1 of Section 3, it follows that :

- i) all the self-conjugated $SU(3)$ representations (i.e., $p=2q$) and only these ones admit $SU(2) \times U(1)$ as a stabilizer ;
- ii) all the non-self-conjugate $SU(3)$ representations (i.e., $p \neq 2q$) and only these ones admit $SU(2)$ as a stabilizer ;

while from Proposition 3, one deduces :

- iii) the $SU(3)$ representations which admit the Cartan subgroup $U(1) \times U(1)$ as a stabilizer are the triality zero ones (i.e., $p+q=3\lambda$, λ positive integer).

Finally, from Proposition 4, Section 4, one gets :

- iv) the subgroup $SO(3)^4$ is the stabilizer of a vector in $R_{SU(3)}(p,q)$ if and only if p and q are both even.

We note that for the fundamental representations $(1,0)$ and $(0,1)$ the only stabilizer is $SU(2)$, while the self-adjoint $(2,1)$ representation admits two stabilizers, namely $SU(2) \times U(1)$ and $U(1) \times U(1)$.

To exhaust the set of continuous subgroups of $SU(3)$, one has to add the family of non-conjugate one-dimensional $U(1)$ subgroups which can be represented, up to a conjugation, as generated by : λ_3, λ_8 and $\lambda_3 + \alpha\lambda_8$ ($\alpha > 0$, $\alpha \neq \sqrt{3}$ and $\neq 1/\sqrt{3}$) respectively, making use of the Gell-Mann matrices λ_i , $i=1, \dots, 8$ with $[\lambda_i, \lambda_j] = 2i f_{ijk} \lambda_k$.

How to select the $SU(3)$ representations with a vector invariant under the $U(1)$ subgroup generated, up to a conjugation, by λ_3 ? One can answer this question by constructing such vectors. It appears that the following vectors satisfy the condition :

$$\begin{pmatrix} p & q & 0 \\ m & 0 & 0 \end{pmatrix} \quad p+q \neq 1,3 \quad \text{as well as for} \quad \begin{pmatrix} 3 & 0 & 0 \\ 3 & 0 & 0 \\ 1 & & \end{pmatrix} \quad (5.2)$$

$m \neq 0$ the decuplet

In Eq. (5.2), only the fundamental $(1,0)$ and $(0,1)$ as well as the self-adjoint $(2,1)$ representations are excluded. Except for the last above vector which belongs to the decuplet and which is invariant under $\lambda_3 + (1/\sqrt{3})\lambda_8$, these other vectors are invariant under λ_3 . One can check that the stabilizer of these vectors is not bigger than $U(1)_{\lambda_3}$. In order to solve this problem, one first notes that the $SU(3)$ subalgebras containing λ_3 are the $SU(2)$ and $SU(2) \times U(1)$ algebras generated by $\{\lambda_1, \lambda_2, \lambda_3\}$ and $\{\lambda_1, \lambda_2, \lambda_3, \lambda_8\}$ and the $SO(3)$ ones generated by $\{\lambda_3; \cos\theta(\lambda_4 + \lambda_2) - \sin\theta(\lambda_5 - \lambda_6); \sin\theta(\lambda_4 - \lambda_7) + \cos\theta(\lambda_5 + \lambda_6)\}$. In the case where p and q are not both even, one has just to check the non-invariance of the vectors under $SU(2)$ and $SU(2) \times U(1)$, while in the case when p or q are both even, one must also take into account the $SO(3)$ subgroups. This can be done as before by considering the action of the E_{ij} operators of Refs. (11)-(13) on the vectors. It follows :

- v) any representation (p,q) of $SU(3)$ contains a vector the little group of which is $U(1)_{\lambda_3}$ except the fundamental ones $(1,0)$ and $(0,1)$ and the self-adjoint octet $(2,1)$.

What about the representations containing a vector with little group $U(1)_{\lambda_8}$?

From the relations given in (5.1) we must have :

$$m_{12} \neq 0 \quad m_{12} \neq m_{22} \quad m_{11} \neq \frac{1}{2}(m_{12} + m_{22})$$

and

$$3(m_{12} + m_{22}) = 2(p+q).$$

It then follows that the $SU(3)$ representation must be of triality 0. Moreover, looking for the $SU(3)$ subalgebras containing the generator λ_8 , one finds only the $U(1) \times U(1)$ and $SU(2) \times U(1)$ algebras generated by $\{\lambda_3, \lambda_8\}$ and $\{\lambda_1, \lambda_2, \lambda_3, \lambda_8\}$, respectively. But the above relation also implies $j' = \frac{1}{2}(m_{12} - m_{22})$ to be an integer. One knows that in the three-dimensional representation $j=1$ of $SU(2)$ any vector is invariant under an $U(1)$ subgroup, but from Ref. 9), one also knows that in $SO(3)$ representations with $j > 1$ there exist vectors invariant under only a discrete subgroup (and even under only the identity in the case $j=4$ for example). Because only continuous symmetries are considered in this exercise, we can conclude, after some calculations :

- vi) a necessary and sufficient condition for the representation (p,q) of $SU(3)$ to contain a vector stabilized by $U(1)_{\lambda_8}$ is that $p+q=3n$ and $p \geq 4$.

In other words all the 0 triality representations contain a vector stabilized by $U(1)_{\lambda_8}$ except the octet and the decuplet (and antidecuplet)⁸⁾.

It is more difficult to give a general result for stabilizers of the form $U(1)_{\lambda_3 + \alpha \lambda_8}$ with $\alpha > 0$, $\alpha \neq \sqrt{3}$ and $\neq 1/\sqrt{3}$. However, in a $SU(3)$ Gel'fand basis, the canonical decomposition $SU(2) \times U(1)$ appears explicitly [see Eq. (5.1)] and also each basis vector is an eigenvector of λ_3 and λ_8 : from these simple properties, it is therefore possible to propose a

method in order to determine such stabilizers for each $SU(3)$ representation. These techniques can be summarized as follows :

- vii) any stabilizer of the form $U(1)_{\lambda_3 + \alpha \lambda_8}$ $\alpha > 0$, $\alpha \neq \sqrt{3}$ and $1/\sqrt{3}$ of a vector in the representation (p,q) of $SU(3)$ must stabilize a Gel'fand basis vector satisfying $j_z \pm \sqrt{3} \alpha y = 0$, with j_z and y defined in Eq. (5.1) and non-simultaneously zero.

As an example, let us look at the $(4,2)$ representation of $SU(3)$. Considering the 27 basis vectors, one can deduce that the stabilizers of the form $U(1)_{\lambda_3 + \alpha \lambda_8}$ in this representation are generated up to a conjugation by :

$$\{ \lambda_3 + 1/\sqrt{3} \lambda_8 \} ; \quad \{ \lambda_3 + 2/\sqrt{3} \lambda_8 \} ; \quad \{ \lambda_3 + 5/\sqrt{3} \lambda_8 \} .$$

6. - HIGGS REPRESENTATIONS IN GAUGE THEORIES

Now, let us briefly discuss our results in the context of $SU(5)$ ^{3),4)} and $SU(6)$ ¹⁹⁾ gauge models. In the $SU(5)$ gauge model, the pattern of spontaneous symmetry breaking $SU(5) \rightarrow SU(3) \times SU(2) \times U(1) \rightarrow SU(3) \times U(1)$ is physically favoured ²⁰⁾. From our results, the smallest representations admitting $SU(3) \times SU(2) \times U(1)$ as a stabilizer are the adjoint representations $(2,1^3)$ of dimension 24, and then the following ones : $(2^2,1)$, $(4,2^3)$ of respective dimension 75 and 200 ; the next ones are all of a dimension bigger than 1000. It is clear that the most economical and natural choice has been done in Refs. 3), 4) where the Higgs are supposed to belong to the sum $\underline{24} \oplus \underline{5}$, the fundamental representation $\underline{5}$ admitting $SU(4)$ as a stabilizer. In the case of the $SU(6)$ gauge model, there are several possible patterns of breaking ²¹⁾, as the first step can be :

- i) $SU(6) \rightarrow SU(4) \times SU(2) \times U(1)$; or
- ii) $\rightarrow SU(3) \times SU(3) \times U(1)$; or
- iii) $\rightarrow SU(3) \times SU(2) \times U(1)$.

The smallest $SU(6)$ representations relevant for the realization of such schemes are :

- i) $(2,1^4)$, dim 35 and $(2^2,1^2)$, dim 189 ;
- ii) $(2,1^4)$, dim 35, (2^3) , dim 175 and $(2^2,1^2)$, dim 189 ;
- iii) $(2,1^3)$, dim 84, and $(2^2,1)$, dim 210.

7. - CONCLUSIONS

We have solved the problem of the classification of all the irreducible $SU(n)$ representations $R_{SU(n)}$ such that the set of stabilizers $\mathcal{S}(R_{SU(n)})$ contains a subgroup which is maximal or of rank $(n-1)$. The relevance of such results in the choice of Higgs representations is shown in Section 6. We want to stress that the Gel'fand-Zetlin basis is a very useful tool which can also be applied to find stabilizers of $R_{SU(n)}$ which are not of the above-mentioned type. Of course, such a research is very complicated in the general case, the first difficulty being that a complete and explicit classification of all the $SU(n)$ subgroups is not known. In Section 5 we have found all the continuous little groups for any irreducible $SU(3)$ representation. Even in this simplest case the study of the $U(1)$ stabilizers is rather delicate.

It is clear that the same type of methods can be used in order to find out the stabilizers for the representations of the groups $SO(n)$, for which the Gel'fand-Zetlin basis is also known.

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APPENDIX

The main part of this Appendix is devoted to the proof of Proposition 1, Section 3. As we shall see, the framework of this study could be directly used to deduce the results of Propositions 2 and 3 of the same Section.

Actually the proof is the construction of a vector invariant under a maximal $SU(n)$ subgroup $S[U(n-p) \times U(p)]$ $n \geq 2p > 2$ in an irreducible representation $R_{SU(n)}$ of $SU(n)$. The main steps of this construction are the following : (i) selection of the Gel'fand-Zetlin basis vectors of $R_{SU(n)}$ invariant under the $(n-1)$ rank subgroup $SU(n-p) \times \prod_{j=1}^p U(1)_j \subset S[U(n-p) \times U(p)]$; (ii) determination of the conditions on $R_{SU(n)}$ allowing to form a linear combination of vectors satisfying (i) which is invariant under $SU(p)$.

Once the Gel'fand-Zetlin basis is given, it is always possible to choose the $SU(n-p)$ group in the canonical channel

$$SU(n) \supset SU(n-1) \times U(1) \supset \dots \supset SU(n-p) \times U(1) \times \dots \times U(1)$$

The (m) vectors invariant under this subgroup $SU(n-p)$ - but not under $SU(n-p+1)$ - must satisfy :

$$m_{ij} = \rho \quad \rho \leq m_{n-p, n} \quad (A.1)$$

with $i < j = 1, 2, \dots, n-p$ and $m_{1, n-p+1} \neq \rho$.

Moreover, the invariance of a vector (m) under the Cartan subgroup $\prod_{j=1}^{n-1} U(1)_j$ implies, using Eq. (2.6)

$$\sum_{k=1}^l m_{kl} = l\lambda \quad \text{for } l=1, 2, \dots, n. \quad (A.2)$$

Then the (m) vectors invariant under $SU(n-p) \times \prod_{j=1}^p U(1)_j$ will be such that :

$$m_{ij} = \lambda \quad 0 \leq \lambda \leq m_{n-p, n} \quad i < j = 1, 2, \dots, n-p$$

and:

$$\sum_{k=1}^l m_{kl} = l\lambda \quad \text{for } l = n-p+1, \dots, n- \quad (\text{A.3})$$

As the Casimir operators of $SU(p)$ are not diagonal in the Gel'fand-Zetlin basis, we have to find, in this subset I of states satisfying Eq. (A.3), a linear combination

$$\phi = \sum_{(m) \in I} \alpha(m) (m). \quad (\text{A.4})$$

such that ϕ is an eigenstate with eigenvalue zero of the Casimir operators of $SU(p)$. Because of the commutation relations given in Eq. (2.4), it is sufficient to verify that the action on ϕ of each raising (resp. lowering) operator of the form $E_{k, k+1}$ ($E_{k+1, k}$) for $k = n-p+1, \dots, n-1$ gives zero.

Applying the raising operator (RO) $E_{k, k+1}$ on ϕ , one gets :

$$E_{k, k+1}(\phi) = \sum_{(m) \in I} \sum_{\tau=1}^k \alpha(m) \zeta_{k, k+1}^{\tau}(m) (m')_{\tau, k} \quad (\text{A.5})$$

where $(m')_{\tau, k}$ denotes the Gel'fand-Zetlin vector specified by

$$\begin{aligned} m'_{ij} &= m_{ij} && \text{for all } i, j \text{ except} \\ m'_{\tau k} &= m_{\tau k} + 1 && i = \tau, \quad j = k \end{aligned} \quad (\text{A.6})$$

and $\zeta_{k, k+1}^{\tau}(m)$ is the matrix element :

$$\zeta_{k, k+1}^{\tau}(m) = \langle (m')_{\tau, k} | E_{k, k+1} | (m) \rangle.$$

In order for $E_{k,k+1}(\emptyset)$ to vanish, the coefficient of each state (m') in the right-hand side of Eq. (A.5) must be zero. Due to the action of the one-label RO $E_{k,k+1}$ the same state (m') in this expression may appear once, or, at most, twice. So let us look for the condition which guarantees that the coefficient of a state vector appearing once in the right-hand side of Eq. (A.5) is zero. For better insight, let us start with $k=n-p+1$. From Eq. (A.3) only the labels $m_{1,n-p+1}$ and $m_{n-p+1,n-p+1}$ are free and such that

$$m_{1,n-p+1} + m_{n-p+1,n-p+1} = 2\lambda \quad (A.7)$$

Therefore $\mathcal{E}_{n-p+1,n-p+2}^\tau$ is not vanishing only for $\tau=1,n-p+1$, and the state which can appear only once in the right-hand side of Eq. (A.5) is the raised state : $(m')_{n-p+1,n-p+1}$, with the maximum allowed value for the label $m_{n-p+1,n-p+1}$, that is : $2\lambda - m_{1,n-p+1}$ with $m_{1,n-p+1} = m_{2,n-p+2}$. Then in order for $\mathcal{E}_{n-p+1,n-p+2}^{(m)}$ to vanish in this case, $m_{n-p+1,n-p+1}$ has to be equal to $m_{n-p+1,n-p+2}$, so we shall have :

$$m_{2,n-p+2} + m_{n-p+1,n-p+2} = 2\lambda \quad (A.8)$$

From Eq. (A.3), it then follows

$$m_{1,n-p+2} + m_{n-p+2,n-p+2} = 2\lambda. \quad (A.9)$$

Considering $k=n-p+2$ in Eq. (A.5) one gets with the same argument :

$$m_{i,n-p+3} + m_{n-p+4-i,n-p+3} = 2\lambda \quad (A.10)$$

for $i=1,2,3$

Going on with other values of k , one gets a set of analogous relations, i.e. :

$$m_{ij} + m_{j-i+1,j} = 2\lambda$$

$$\text{for } j = n-p+1, \dots, n \text{ and } \begin{cases} i = 1, \dots, j/2 & \text{if } j \text{ even} \\ i = 1, \dots, j+1/2 & \text{if } j \text{ odd} \end{cases} \quad (A.11)$$

Let us note that Eq. (A.11) with $j=n$ implies that the representation $(m_{11}, \dots, m_{n-1,n})$ is self-conjugated.

Moreover, Eq. (A.11) ensures that the $(m')_{\tau k}$ states on the right-hand side of Eq. (A.5) all appear twice. Therefore the condition for Eq. (A.5) to vanish can be expressed as a set of relations of the type :

$$\alpha(m) \mathcal{G}_{k,k+1}^{\tau}(m) + \alpha(\bar{m}) \mathcal{G}_{k,k+1}^{\tau'}(\bar{m}) = 0 \quad (\text{A.12})$$

where (m) and $(\bar{m})$ are state vectors which differ only for :

$$\begin{aligned} m_{\tau k} &= \bar{m}_{\tau k} + 1 \\ m_{\tau'k+1} &= \bar{m}_{\tau'k} \end{aligned} \quad (\text{A.13})$$

If N is the number of different (m) states appearing in Eq. (A.4), there will be at most $(N-1)$ relations involving the $N \alpha(m)$ coefficients : so, for fixed k , it will always be possible to find a set of α 's such that $E_{k,k+1}(\emptyset) = 0$. Now, considering another value of k , another set of relations of the same type as those given by Eq. (A.12) will be obtained : we have to check that these new relations are compatible with the previous ones. For such a purpose, let us imagine a triplet of states (m) , (m') and (m'') such that

$$\begin{aligned} \alpha(m) &= X_r(m, m') \alpha(m') \\ \alpha(m') &= X_s(m', m'') \alpha(m'') \end{aligned} \quad (\text{A.14})$$

with X_r (resp. X_s) function of the matrix elements of $E_{r,r+1}$ (resp. $E_{s,s+1}$). Then, it is easy to prove that there is no RO $E_{t,t+1}$ yielding the condition

$$\alpha(m) = X_t(m, m'') \alpha(m''). \quad (\text{A.15})$$

Indeed Eq. (A.14) implies that (m) and (m') $\llbracket m'$ and (m'') differ only for two labels in the r row (s row), while Eq. (A.15) would imply that (m) and (m'') differ only for two labels in the t row.

Thus we have shown that it is possible to find a vector ϕ_0 such that every RO of $SU(p)$ gives zero on it if Eq. (A.11) holds. In order to complete the proof, i.e., to show that the stabilizer of ϕ_0 is $S[U(n-p) \times U(p)]$ we have to check that the action of the lowering operator of $SU(p)$ on ϕ_0 gives zero. This follows at once from the following symmetry property :

Lemma A

The matrix element $\mathcal{Y}_{k,k+1}^T(m)$ with (m) such that the labels of its rows $k-1, k, k+1$ satisfying Eq. (A.11) is equal to the matrix element $\mathcal{Y}_{k+1,k}^{k-\tau+1}(m)$.

The proof of this relation follows straightforwardly but tediously from the form of the matrix elements of the lowering and raising operators (see Section 3).

As a particular case, one can study the $SU(n)$ representations containing a vector the stabilizer of which is $U(n-1)$.

Choosing $SU(n-1)$ in the canonical chain, it follows that

$$m_{i,n-1} = \lambda \quad \text{for any } i=1, \dots, n-1 \quad (\text{A.16})$$

Therefore, using the branching law :

$$\begin{aligned} m_{i,j} &= \lambda & j \neq n, i \leq j. \\ m_{i,n} &= \lambda & i \neq 1, n. \end{aligned} \quad (\text{A.17})$$

The invariance for the $(n-1)^{\text{th}}$ $U(1)$ requires :

$$n \sum_{i=1}^{n-1} m_{i,n-1} - (n-1) \sum_{i=1}^n m_{i,n} = 0 \quad (\text{A.18})$$

Because $m_{n,n} = 0$, from the above relations, one finally obtains :

$$m_{1n} = 2\lambda ; \quad m_{2n} = \dots = m_{n-1,n} = \lambda \quad (\text{A.19})$$

which completely proves Corollary 1.

The proof of Proposition 2 (Section 3) goes along the same lines. First choosing $SU(p_1)$ in the canonical chain, the subspace invariant under $SU(p_1)$ is constituted by the set of vectors (m) satisfying

$$m_{ij} = \lambda \quad 0 < \lambda \leq \pi_{p_1, n} \quad \text{with } i < j = 1, 2, \dots, p_1. \quad (A.21)$$

Then making linear combinations ϕ of such vectors also satisfying Eq. (A.2) as well as Eq. (A.11) one can build, using the same techniques as above, a vector stabilized under

$$\prod_{i=1}^{\ell} SU(p_i) \times \prod_{j=1}^{n+\ell-1-\sum_{i=1}^{\ell} p_i} U(1)_j.$$

REFERENCES

- 1) For a recent review on the subject, see :
H. Fritzsch - CERN Preprint TH. 2579 (1978), talk given at the XIX International Conference on High Energy Physics, Tokyo (1978).
- 2) S. Weinberg - Phys.Rev.Letters 19, 1204 (1967) ;
A. Salam - Elementary Particle Theory (Ed. N. Svartholm, 1968) ;
S.L. Glashow - Nuclear Phys. 22, 579 (1961).
- 3) H. Georgi and S.L. Glashow - Phys.Rev.Letters 32, 439 (1975).
- 4) M.S. Chanowitz, J. Ellis and M.K. Gaillard - Nuclear Phys. B123, 506 (1977) ;
A.J. Buras, J. Ellis, M.K. Gaillard and D.V. Nanopoulos - Nuclear Phys. B135, 66 (1978).
- 5) Very recently, F. Wilczek and A. Zee (Princeton University Preprint, 1978) suggested to extend the standard model of $SU(2)_V \times U(1) \times SU(2)_H$. Their arguments for such an extension are the lack of information in the $SU(2) \times U(1)$ model for the values of the Cabibbo-like mixing angles and of the quark masses as well as the necessity, because of the proliferation of quarks, of adding more left-handed doublets and right-handed singlets.
- 6) G.D. Mostow - Ann.Math. 65, 432 (1957).
- 7) Let us recall that the orbit of $m \in R_G$ is the set $\{g(m) | g \in G\}$. If m and m' are on the same orbit, their little groups are conjugate. But if two points m and m' have conjugate little groups they need not be on the same orbit : by definition they are on the same stratum. Note that being on the same orbit is an equivalence relation between the elements of R_G . By the same way, to have conjugate little groups is an equivalence relation for the elements of R_G . Thus R_G is partitioned into orbits, as well as into strata. For more details about action of compact Lie groups on manifolds, see, e.g. : L. Michel - "Statistical Mechanics and Field Theory", Haifa Summer School (1971), p. 133, Eds. R.N. Sen and C. Weil, Halsted Press (1972).
- 8) J. Lieberman - Phys.Rev. D8, 2545 (1973).
- 9) B.A. Ovrut - J.Math.Phys. 19, 418 (1978).
- 10) L.F. Li - Phys.Rev. D9, 1723 (1974).
- 11) I.M. Gel'fand and M.L. Zetlin - Dokl.Akad.Nauk SSSR 71, 825 (1950).
- 12) G.E. Baird and L.O. Biedenharn - J.Math.Phys. 4, 1449 (1963) ; *ibid.* 5 1723, 1730 (1964).
- 13) J.D. Louck - Amer.J.Phys. 38, 3 (1970).
- 14) C. Itzykson and M. Nauenberg - Revs.Modern Phys. 38, 95 (1966).
- 15) E. Dynkin - Mat.Sb. 30, 349 (1952) ; Amer.J.Soc.Transl.Ser. 2,6,111 (1957).

- 16) In the case of the $SU(6)$ group, the representations of dimension less than 1000 with a vector stabilized under $SU(3)^5$ are the following : (3) , (2^3) , $(4,2^4)$, (6) and $(3,2^3)$ of dimension 56, 175, 405, 462 and 560, respectively. This result can be learned from J. Patera and D. Sankoff - "Tables of Branching Rules for Representations of Simple Lie Algebras", Presses de l'Université de Montréal (1973).
- 17) D. Littlewood - "The Theory of Group Characters", Oxford Clarendon Press (1940).
- 18) L. Michel and L.A. Radicati - Ann.Inst.Henri Poincaré XVIII, 185 (1973).
- 19) M. Abud, F. Buccella, H. Ruegg and C.A. Savoy - Phys.Letters 67B, 313 (1977).
- 20) D.V. Nanopoulos - CERN Preprint TH. 2534 (1978).
- 21) M. Abud and C.A. Savoy - Nuovo Cimento 42A, 67 (1977)