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STABLE YANG-MILLS FIELDS FOR AN OVER-CRITICAL
POINT SOURCE

M. Magg

CERN -- Geneva

ABSTRACT

The classical Yang-Mills energy functional in the presence of an external particle is explored as a function of the coupling strength. In the over-critical coupling region besides the Coulomb field other new saddle points as well as new minima are found. Near the critical point approximate expressions for the new static and stable solutions of the Yang-Mills equations are derived.

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1. INTRODUCTION

During the last few years the study of classical, non-linear field theories has led to the discovery of many unexpected solutions both in Minkowskian and in Euclidian space-time. The treatment of the full nonlinearities of a classical system can give, via the semi-classical approximation, a better understanding on what kind of objects and interactions are described by its quantized version. For many field theories structures have been found which have not been foreseen on the basis of usual perturbation theory. Perhaps the most exciting challenge of this kind is presented by the SU_3 gauge theory, which is used to describe hadrons and their strong interactions in terms of quark fields coupled to gluon gauge fields¹⁾.

In this paper we examine the classical static limit of a non-Abelian gauge theory in the presence of a single charged particle. The no-particle case is known to be trivial²⁾. Any student of classical electrodynamics knows that in the Abelian case the Coulomb field is the unique and absolutely stable solution for an arbitrary number of charged particles of given positions. This establishes the dominant role played by the classical Coulomb interaction in the low energy regime of both classical and quantum-electrodynamics. The Abelian solution can be trivially translated to the Yang-Mills (Y-M) case. However, as Mandula has shown first, the Coulomb solution to the field equations becomes then unstable if the Y-M coupling to the particle exceeds a certain critical value^{3),4)}. The Coulomb field is then no longer acceptable for a physical solution. Existence and uniqueness of the stable Y-M field of a single particle are therefore open problems. The classical interaction of two external particles at a finite distance away from each other is a much harder unsolved problem⁵⁾. This should be carried in mind if one interprets the calculations of quantum corrections to the static potential starting with the Coulomb interaction in zeroth approximation.

The existence of a new static Y-M solution for a single particle source in the over-critical coupling region was pointed out in a previous letter⁶⁾. There the Y-M potentials satisfy a very special ansatz and stability with respect to general small perturbations was not studied. The present paper presents a systematic approach to find all static solutions of the Y-M field equations with a single particle source, which are contained within a certain neighbourhood (in function space) of the Coulomb field. Together with the existence of such new solutions their stability properties are displayed.

Our approach is based on an exploration of the energy functional for the Y-M fields. This functional acts on the potentials in a fixed gauge under Gauss' law as a constraint. This language and our notation will be explained in the next section.

The infinite self-energy of a point source requires either a short distance cut-off or the subtraction of the singular short distance contribution. Due to some peculiar difficulties in non-Abelian gauge theories the cut-off has to be introduced very carefully. This is done in Section 3.

Section 4 repeats the proof for the instability of the Coulomb field in the language of variational calculus. Analytic expressions for the relevant instability modes are derived.

In Section 5 the energy functional restricted to the relevant subspace of small fluctuations is calculated up to fourth order in the deviation from the Coulomb field. New static and stable Y-M solutions are found as local minima of the energy.

The last section summarizes our results and offers a comment on their relevance for the quantized Y-M system as well as a speculation on global instability.

2. YANG-MILLS FIELD WITH A STATIC SOURCE

In order to keep the group theory as simple as possible we take SU_2 for the gauge group. The classical Y-M field of a particle at rest has to solve the Y-M equations. The first set of equations describes the covariant analog of Gauss' law in electrodynamics:

$$D^i F^{0i}(\vec{r}, t) = \rho(\vec{r}, t) \quad (2.1)$$

The second set are the equations of motion

$$D^0 F^{0i}(\vec{r}, t) - D^j F^{ji}(\vec{r}, t) = 0 \quad (2.2)$$

The particle is represented by its non-Abelian source ρ . The symbols $F^{\mu\nu}$ and D^μ denote the field strength tensor and the covariant derivative respectively. The gauge potentials are rescaled such that the non-Abelian coupling constant g disappears in the left-hand sides of the Y-M equations. The source term gets then multiplied by a factor g^2 . The fields are linear

combinations of the three traceless anti-Hermitean matrices T_a related to the Pauli matrices σ_a by

$$T_a = -\frac{i}{2} \sigma_a$$

We are primarily interested in a point source ρ , since it is meant to describe an elementary particle in a relativistic theory.

$$g(\vec{r}, t) = n_a(t) T_a \cdot 4\pi Q \delta(\vec{r}) \quad (2.3)$$

The triple n_a is normalized to one.

For a point source the well-known linear divergence occurs in the classical field energy at short distances. Therefore one is forced to cut off the energy integral at the short distance end. In electrodynamics the electric field outside of a spherically symmetric, concentrated source depends only on the total charge but not on the internal charge distribution. Because of this fact the energy cut-off can be interpreted as a symmetric smearing of the point charge. This is no longer possible in the Y-M theory as we will explain in the following section. Let us continue for the moment with a general spherically symmetric source. The symmetry is defined by the rotation invariance of the gauge invariant distribution

$$\sqrt{\text{Tr} \{ g(\vec{r}, t) g(\vec{r}', t) \}}$$

The covariant analog to local charge conservation

$$D^0 g(\vec{r}, t) = 0 \quad (2.4)$$

implies the time independence of ρ in the temporal gauge

$$A^0 = 0 \quad (2.5)$$

By means of the remaining time independent gauge transformations the source can be brought into the following standard form, if the Kronecker index of the symmetric isospin source vanishes.

$$g = T_3 q(r) \quad (2.6)$$

Obviously conditions (2.5) and (2.6) do not fix the gauge completely.

For a source (2.6) the Abelian Coulomb field solves the Y-M equations

$$\begin{aligned} A^0 &= 0 \\ \vec{A}(\vec{r}, t) &= -t T_3 \vec{\nabla} \phi(r) \\ -\Delta \phi(r) &= q(r) \end{aligned} \quad (2.7)$$

If one applies a complicated gauge transformation to the solution (2.7) [even within the class of gauges characterized by (2.5), (2.6)] one may completely disguise the Abelian nature of this configuration. Still we would call it the (Abelian) Coulomb field, because we do not differentiate between gauge equivalent potentials. Note there is only one Coulomb solution if we demand a connected source.

In the temporal gauge the Y-M equations take the form of an initial value problem

$$\ddot{A}^i(\vec{r}, t) = D^j F^{ji}(\vec{r}, t) \quad (2.8)$$

The initial data at time $t = 0$, let us say, are restricted by Gauss' law

$$D^i \dot{A}^i(\vec{r}, 0) = T_3 q(r) \quad (2.9)$$

For any solution of the equations of motion the field energy W is conserved.

$$W = \int_{\mathbb{R}^3} dV T_r \left\{ \dot{A}^i \dot{A}^i + \frac{1}{2} F^{ij} F^{ij} \right\} \quad (2.10)$$

In gauge field theories a static configuration means that the time-dependence can be absorbed by a gauge transformation. In this sense we have the familiar correspondence between extremal energy configurations and static solutions of the equations of motion also for gauge theories: the energy integral (2.10) treated as a functional of those initial data ($A^0 = 0$) which satisfy the constraint (2.9) has a stationary point if and only if the corresponding initial data belong to a static solution of the Y-M equations.

To verify this statement one uses a Lagrangian multiplier field $\lambda(\vec{r})$ which takes care of the constraint.

$$W = \int dV \text{Tr} \left\{ \dot{A}^i \dot{A}^i + \frac{1}{2} F^{ij} F^{ij} - 2\lambda (D^i \dot{A}^i - g) \right\} \quad (2.11)$$

The Euler-Lagrange equations are of the following form:

$$\dot{A}^i = -\partial^i \lambda + [\lambda, A^i] \quad (2.12)$$

$$\partial^i \dot{A}^i + [A^i, \dot{A}^i] = g \quad (2.13)$$

$$D^j F^{ji} - [\lambda, \dot{A}^i] = 0 \quad (2.14)$$

With the identification

$$\begin{aligned} A'^0(\vec{r}, t) &= \lambda(\vec{r}) \\ A'^i(\vec{r}, t) &= A^i(\vec{r}) \end{aligned} \quad (2.15)$$

the equations (2.13) and (2.14) are simply the Y-M equations for the time-independent potential A'^{μ} . Equation (2.12) defines the electric field. The potentials in the temporal gauge are obtained from the time-independent ones by the gauge transformation g

$$\begin{aligned} \dot{g}(\vec{r}, t) &= g(\vec{r}, t) \lambda(\vec{r}) \\ g(\vec{r}, 0) &= 1 \end{aligned} \quad (2.16)$$

The characterization of the static solutions of the Y-M equations in terms of stationary points of the energy functional is quite appropriate for the discussion of their stability properties. We call a Y-M field locally stable if it is a local minimum of the energy. The stable Y-M fields are

a subset of the static solutions of (2.1), (2.2). For example, the screening solutions found by Sikivie and Weiss are unstable because they are genuinely time-dependent⁷⁾.

We end this section with a further remark. If the energy functional (2.9), (2.10) is restricted to initial data of the special form

$$\begin{aligned}\vec{A}(\vec{r}, 0) &= \vec{A}_1(\vec{r}) T_1 \\ \dot{\vec{A}}(\vec{r}, 0) &= \dot{\vec{A}}_2(\vec{r}) T_2 + \dot{\vec{A}}_3(\vec{r}) T_3\end{aligned}\quad (2.17)$$

then any minimum of this restricted energy functional is a stationary point of the full energy functional [with constraint (2.9)]. In general, however, this stationary point will be a saddle point in the large space.

3. BOUNDARY CONDITIONS

Next we will discuss a peculiar problem associated with extended sources in non-Abelian gauge theories. It does not seem to have an analog in electrodynamics.

The Coulomb solution (2.7) in the Y-M theory is always unstable if the source is not point-like. This statement does not depend of the value of the coupling constant. Therefore one has to be very careful if one tries to regularize the energy at short distances by some smearing of the source. Just this smearing procedure is responsible for mock instabilities which disappear in the point source limit. The origin of this type of instability is related to the existence of the screening solutions in Ref. 7) which can carry arbitrarily small amounts of energy.

Take any smooth, time-independent gauge transformation $g(\vec{r})$ normalized at the origin

$$g(\vec{0}) = 1$$

We use it to construct the initial data for a vector potential in a gauge defined by (2.5), (2.6)

$$\vec{A}(\vec{r}, 0) = \vec{g}^{-1}(\vec{r}) \vec{\nabla} g(\vec{r}) \quad (3.1)$$

Its time derivative at $t = 0$ will be specified in a moment. Now we

apply the gauge transformation g on this potential. The transformation will change the gauge condition (2.6) into a new one

$$\begin{aligned} g'(\vec{r}) &= m_a(\vec{r}) T_a q(r) \\ m_a(\vec{r}) T_a &= q(\vec{r}) T_3 g^{-1}(\vec{r}) \end{aligned} \quad (3.2)$$

The transformed initial data are given by

$$\begin{aligned} \vec{A}'(\vec{r}, 0) &= 0 \\ \dot{\vec{A}}'(\vec{r}, 0) &= q(\vec{r}) \dot{\vec{A}}(\vec{r}, 0) g^{-1}(\vec{r}) \end{aligned} \quad (3.3)$$

In the primed gauge Gauss' law is a linear condition on $\dot{\vec{A}}$

$$\vec{\nabla} \cdot \dot{\vec{A}}'(\vec{r}, 0) = g'(\vec{r}) \quad (3.4)$$

The purely longitudinal solution of (3.4) carries the following amount of field energy

$$W = \frac{1}{2} \sum_{a=1}^3 \int dV q(r) m_a(\vec{r}) \int dV' q(r') m_a(\vec{r}') \frac{1}{4\pi |\vec{r} - \vec{r}'|} \quad (3.5)$$

For $m_a = \delta_{a3}$ this formula describes also the energy stored by the Coulomb field (2.7). The latter is larger than the energy of the initial data just constructed because $q(r) \cdot q(r')$ is non-negative for a connected source. One sees that there are always small perturbations of the Coulomb solution for an extended source which lower the energy and still satisfy Gauss' law. For a point source this mechanism we have just explained reduces to a pure gauge transformation of the Coulomb field which is, of course, irrelevant for stability.

One avoids these mock instabilities caused by a smearing of the point source if one works with a cut-off radius r_0 in the energy integral and replaces the point source at the origin by appropriate boundary conditions on the cut-off sphere.

$$W = \int_{|\vec{r}| \geq r_0} dV \operatorname{Tr} \left\{ \dot{\vec{A}} \dot{\vec{A}} + \frac{1}{2} F^{ij} F^{ij} \right\} \quad (3.6)$$

$$\vec{\nabla} \dot{\vec{A}} + [\vec{A}, \dot{\vec{A}}] = 0 \quad (|\vec{r}| \geq r_0) \quad (3.7)$$

There are two sets of boundary conditions.

$$\vec{r} \times \vec{A}(\vec{r}) = 0 \quad |\vec{r}| = r_0 \quad (3.8)$$

$$\int_{|\vec{r}|=r_0} d\vec{f} \cdot \dot{\vec{A}}(\vec{r}) = 4\pi Q T_3 + o(r_0) \quad (3.9)$$

The electric boundary condition (3.9) contains the information on the point source at the origin. The magnetic side-condition (3.8) has to be introduced in order to get a well-defined variational problem. The length scale r_0 will show up in the solutions even if the cut-off for the energy integral (3.6) is removed. The functional (3.6)-(3.9) is invariant under those (time-independent) gauge transformations which become constant on the cut-off sphere. The Euler-Lagrange equations to this reformulated variational problem are equivalent to the static Y-M equations outside the cut-off sphere. Surface terms at $|\vec{r}| = r_0$ vanish because of the boundary conditions. The Coulomb solution (restricted to $|\vec{r}| \geq r_0$) is clearly a stationary configuration for this functional.

4. SECOND VARIATION OF THE ENERGY

The energy integral is not quadratic with respect to the field variables. Furthermore, Gauss' law is a non-linear constraint. These two difficulties will confine our exploration of the energy functional to a certain neighbourhood of the Coulomb field which is the maximally symmetric configuration compatible with Gauss' law (3.7), (3.9). This suggests the following change of variables

$$(\vec{A}, \dot{\vec{A}}) \longrightarrow (\vec{A}, \vec{\chi}) \quad (4.1)$$

$$\vec{\chi} := \dot{\vec{A}} + \vec{\nabla} \phi T_3 - [\phi(r) T_3, \vec{A}]$$

In terms of the new variables the energy takes the form

$$W - W_0 = \int_{|\vec{r}| \geq r_0} dV \operatorname{Tr} \left\{ \vec{\chi}^2 + \frac{1}{2} F^{ij} F^{ij} - [\phi T_3, \vec{A}]^2 \right\} \quad (4.2)$$

The energy W_0 stored by the Coulomb field in the region $|\vec{r}| \geq r_0$ is deducted. In contrast with the extended source case discussed in the previous section, it is now clear that $W - W_0$ can become negative only for configurations where $\vec{A}$ at $t = 0$ is not gauge equivalent to zero. The constraints (3.7)-(3.9) are reformulated as follows

$$\vec{\nabla} \vec{\chi} = - \vec{\nabla} [\phi T_3, \vec{A}] - [\vec{\nabla} \phi T_3, \vec{A}] - [\vec{A}, \vec{\chi} + [\phi T_3, \vec{A}]] \quad (4.3)$$

$$\vec{r} \times \vec{A}(\vec{r}) = 0 \quad |\vec{r}| = r_0 \quad (4.4)$$

$$\int_{|\vec{r}|=r} d\vec{f} \cdot (\vec{\chi}(\vec{r}) + [\phi(r) T_3, \vec{A}(\vec{r})]) = o(r) \quad (4.5)$$

Let us insert a remark on the role played by the length scale r_0 . If we start directly with the energy difference $W - W_0$ we need no cut-off in the integral (4.2). The length scale is introduced by the magnetic side-condition (4.4) alone. Without it $W - W_0$ would not be bounded from below in the over-critical coupling region^{3),4)} which we are interested in. This is a consequence of the scaling behaviour of the energy. In the

following we will retain the cut-off for the difference $W - W_0$, however, it should be kept in mind that the stationary configurations for the energy functional (4.2)-(4.5) with cut-off can be simply continued (up to the origin) to stationary configurations for $W - W_0$ without short distance cut-off. The fields we are going to present in this paper solve the static Y-M equations not only outside the sphere with radius r_0 but also inside this ball.

We just mentioned the critical coupling Q_{crit} where the Coulomb field turns from a local minimum of the energy into a saddle point^{3),4)}. We will derive analytic expressions for the complete sub-space of those small deformations $(\vec{\chi}, \vec{A})$ of the Coulomb field which point in the direction of steepest descent for the energy. For this purpose the fields $\vec{\chi}$ and $\vec{A}$ are taken to be small and we keep only the quadratic part W_2 , the so-called second variation of $W - W_0$:

$$W_2[\vec{A}, \vec{\chi}] = \int dV \text{Tr} \{ \vec{\chi}^2 + (\vec{\nabla} \times \vec{A})^2 - [\phi T_3, \vec{A}]^2 \} \quad (4.6)$$

$$\vec{\nabla} \vec{\chi} = -\vec{\nabla} [\phi T_3, \vec{A}] - [\vec{\nabla} \phi T_3, \vec{A}] \quad (4.7)$$

The small fluctuation $(\vec{\chi}, \vec{A})$ transforms under an infinitesimal gauge transformation according to

$$\begin{aligned} \vec{A}(\vec{r}) &\rightarrow \vec{A}(\vec{r}) + \vec{\nabla} \lambda(\vec{r}) \\ \vec{\chi}(\vec{r}) &\rightarrow \vec{\chi}(\vec{r}) + \vec{\nabla} [\lambda(\vec{r}), \phi(r) T_3] \end{aligned} \quad (4.8)$$

The quadratic form (4.6), (4.7) is invariant under these transformations. We may safely restrict the functional to purely longitudinal fields $\vec{\chi}$ since any transversal part can only increase the energy.

$$\vec{\chi}(\vec{r}) = -\vec{\nabla} \chi(\vec{r}) \quad (4.9)$$

The constraint (4.7) and the boundary condition (4.5) may be used to eliminate $\vec{\chi}$ as an independent variable. The second variation is then a quadratic non-local functional of $\vec{A}$ alone. The number of relevant variables is further reduced by the observation that the component $\vec{A}_3$ of $\vec{A}$ which points in the T_3 direction contributes only positively to the second variation. The minimal values will be obtained for a purely longitudinal field $\vec{A}_3$ which can be gauged to zero. The remaining components $\vec{A}_1$ and $\vec{A}_2$ do not interfere with each other in the quadratic form (4.6). It is therefore sufficient to take $\vec{A}$ parallel to T_1 say. The second variation is invariant under rotation and reflection in ordinary space; the angular dependence of the fluctuation $\vec{A}$ can be chosen as a definite vector spherical harmonics. It is either an axial vector mode

$$\vec{A}(\vec{r}) = T_1 h_j(r) \frac{\vec{r}}{r} \times \vec{\nabla} Y_{j,m}(\Omega) \quad \left(\begin{array}{l} j \geq 1 \\ |m| \leq j \end{array} \right) \quad (4.10)$$

or a proper vector mode.

$$\vec{A}(\vec{r}) = T_1 g_j(r) \vec{\nabla} Y_{j,m}(\Omega) + T_1 \vec{\nabla} (f_j(r) Y_{j,m}(\Omega)) \quad (4.11)$$

($j \geq 1, |m| \leq j$)

The function f_j describes precisely the gauge freedom.

For the axial vector mode the second variation reduces to the form

$$W_2 = \frac{j(j+1)}{2} \int_0^\infty dr \left\{ (h'_j(r))^2 - \frac{Q^2 - j(j+1)}{r^2} h_j^2(r) \right\} \quad (4.12)$$

For the proper vector mode we can choose a gauge. The following is a very convenient one:

$$f_j(r) = - \frac{j(j+1)}{Q^2} g_j(r) \quad (4.13)$$

In this gauge we obtain for the second variation

$$W_2 = \frac{j(j+1)}{2} \frac{Q^2 - j(j+1)}{Q^2} \int_{r_0}^{\infty} dr \left\{ (g'_j(r))^2 - \frac{Q^2 - j(j+1)}{r^2} g_j^2(r) \right\} \quad (4.14)$$

$$+ \frac{1}{2} \int_{|\vec{r}| > r_0} dV (\vec{\nabla} \chi)^2$$

The field $\chi(\vec{r})$ is a multipole of order j whose radial dependence can be calculated from Eq. (4.7) for given g_j . If instead of (4.13) we take the gauge condition

$$f_j(r) = -g_j(r) \quad (4.15)$$

then we would get the following expression for the second variation

$$W_2 = \frac{j(j+1) - Q^2}{2} \int_{r_0}^{\infty} dr (g'_j(r))^2 \quad (4.16)$$

$$+ \frac{1}{2} \int dV (\vec{\nabla} \chi')^2$$

The function $\vec{\nabla} \chi'$ is the gauge transformed of $\vec{\nabla} \chi$ [see Eq. (4.8)].

Let us first discuss definiteness of the second variation for the proper vector modes as a function of Q . Equation (4.16) shows immediately that instability for $Q^2 \leq 3$ can only happen for the $j = 1$ modes at $Q^2 > 2$. In this Q^2 interval we may use the form (4.14) to note that the second variation for the $j = 1$ proper vector mode can become indefinite only if the corresponding form (4.12) for the $j = 1$ axial vector mode is already indefinite. The second variation for the proper vector modes contains an additional positive definite term $(\vec{\nabla} \chi)^2$.

As it is well known⁸⁾, the form (4.12) for the axial vector modes becomes indefinite precisely at

$$Q^2 = \left(j + \frac{1}{2}\right)^2$$

The Coulomb field therefore changes from a local minimum of the field energy into a saddle-point if Q^2 exceeds the critical value Q_{crit}^2 (3), 4).

$$Q_{crit}^2 = \frac{g}{4} \quad (4.17)$$

In some neighbourhood of this critical point the proper vector fluctuations still will increase the field energy, because $\vec{\nabla}\chi$ does not vanish at $Q^2 = Q_{crit}^2$. The instability (near Q_{crit}^2) is solely due to the axial vector modes with $j = 1$. The second variation for these modes [Eq. (4.12)] can be diagonalized in $L^2[(r_0, \infty)]$. The eigenfunctions are expressed by modified Bessel functions of imaginary order

$$h_{1,n}(r) = \sqrt{r} K_{i\nu}(\sqrt{-E_n} r) \quad (n = 1, 2, \dots) \quad (4.18)$$

$$\nu = \sqrt{Q^2 - Q_{crit}^2}$$

The eigenvalue E_n is related to the n^{th} largest zero y_n of the modified Bessel function $K_{i\nu}$

$$E_n = - \left(\frac{y_n}{r_0} \right)^2 \quad (4.19)$$

As long as $0 < Q^2 - Q_{crit}^2 \ll 1$ these zeros are approximated by the following formula

$$y_n = 2 \cdot e^{-C} \cdot e^{-n \frac{\pi}{\nu}} \cdot (1 + O(\nu)) \quad (4.20)$$

The symbol C is Euler's constant.

The eigenvalues E_n accumulate at 0. This is an effect of the long range $1/r^2$ potential in the form (4.12). For $\nu \ll 1$ the "excited" levels E_n [$n > 1$] are practically zero compared with the "ground-state" energy E . A long-distance cut-off of order $r_0 [e^{(2\pi/\nu)}]$ pushes all the almost zero modes with $n > 1$ to positive energy values, whereas the ground-state $h(r)$ remains nearly unchanged and its eigenvalue stays negative.

This section can be summarized by the analytic expression for those small fluctuations $(\vec{\chi}, \vec{A})$ which point from the Coulomb field in the direction of steepest descent for the field energy. These fluctuations span a six-dimensional linear space, characterized by two spatial vectors $\vec{a}$ and $\vec{b}$.

$$\vec{A}(\vec{r}) = \sqrt{\frac{3}{4\pi}} \frac{\vec{r}}{r^2} h(r) \times (\vec{a} T_1 + \vec{b} T_2) \quad (4.21)$$

$$\vec{\chi} = 0$$

Here and in the following h is the short-hand for the function $h_{1,1}$ in Eq. (4.18). The second variation for these minimal fluctuations can be easily calculated in the limit $v \ll 1$.

$$W_2^{\min} = -\frac{1}{2} (\vec{a}^2 + \vec{b}^2) \quad (4.22)$$

Note that there is no naturally given norm in the space of small fluctuations which should be used to diagonalize the second variation. The diagonalization was simply a tool to find the modes (4.21) which cause the instability at Q_{crit}^2 .

5. THE ENERGY FUNCTIONAL NEAR THE COULOMB FIELD

Starting with a small fluctuation of steepest descent we allow now for higher order corrections. This enables us to solve Gauss' law up to quadratic terms and to determine the energy functional for the most favourable fluctuations around the Coulomb field up to fourth order. This will be sufficient to control the critical behaviour at $Q^2 = Q_{\text{crit}}^2$.

The corrected ansatz for a small fluctuation takes the following form [see Eq. (4.1)]:

$$\begin{aligned} \vec{A} &= \vec{A}^{(1)}(\vec{r}) + \vec{A}^{(2)}(\vec{r}) \\ \vec{\chi} &= 0 + \vec{\chi}^{(2)}(\vec{r}) \end{aligned} \quad (5.1)$$

The piece $\vec{A}^{(1)}$ is some small potential given by (4.21) whereas $\vec{A}^{(2)}$ and $\vec{\chi}^{(2)}$ are second order correction terms compared with the leading $\vec{A}^{(1)}$.

For the ansatz (5.1) Gauss' law (4.3) is equivalent to the equations

$$\vec{\nabla} \vec{\chi}_3^{(2)} T_3 = -2 \phi(r) T_3 \cdot \text{Tr} \{ \vec{A}^{(1)}(\vec{r}) \vec{A}^{(1)}(\vec{r}) \} \quad (5.2)$$

$$\vec{\nabla} \vec{\chi}_\perp^{(2)}(\vec{r}) = - [\vec{\nabla} \phi(r) T_3, \vec{A}^{(1)}(\vec{r})] - \vec{\nabla} [\phi(r) T_3, \vec{A}^{(1)}(\vec{r})] \quad (5.3)$$

The suffix $\perp$ denotes the projection of a field orthogonal to the direction T_3 . If one inserts the ansatz (5.1) into the energy integral (4.2) and keeps all terms up to fourth order one finds after a straightforward calculation:

$$\begin{aligned} W - W_0 &= W_2 [\vec{A}^{(1)} + \vec{A}_\perp^{(2)}, \vec{\chi}_\perp^{(2)}] = W_4 \\ W_4 &= \frac{1}{2} \int dV \text{Tr} \{ (\partial^i A_3^{(2)j} T_3 - \partial^j A_3^{(2)i} T_3 + [A^{(1)i}, A^{(1)j}])^2 \} \\ &\quad + 2 \int dV \text{Tr} \{ A_3^{(2)j} T_3 [A^{(1)i}, \partial^j A^{(1)i} - \partial^i A^{(1)j}] \} \\ &\quad + \int dV \text{Tr} \{ (\vec{\chi}_3^{(2)} T_3)^2 \} + \dots \end{aligned} \quad (5.4)$$

The contribution W_2 is the quadratic form (4.6), (4.7). The omitted terms are of fifth and higher order in the small fluctuation.

By choosing the spatial Cartesian coordinate system $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ appropriately the vectors $\vec{a}$ and $\vec{b}$ which fix $A^{(1)}$ [Eq. (4.21)] can be written as follows:

$$\begin{aligned} \vec{a} &= c \cdot \cos \alpha \vec{e}_1 \\ \vec{b} &= c \cdot \sin \alpha (\cos \beta \vec{e}_1 + \sin \beta \vec{e}_2) \\ &\quad \left(-\frac{\pi}{2} < \alpha, \beta \leq \frac{\pi}{2} \right) \end{aligned} \quad (5.5)$$

For given angles α and β we have now to determine the minimum of W_4 as a functional of $(\vec{A}_3^{(2)}, \vec{\chi}_3^{(2)})$ with the constraint (5.2). In the minimum of W_4 , $\vec{\chi}_3^{(2)}$ is the gradient of the function $-\chi_3^{(2)}$ which is determined by

$$\Delta \chi_3^{(2)}(\vec{r}) = 2 \phi(r) \text{Tr} \{ \vec{A}^{(1)}(\vec{r}), \vec{A}^{(1)}(\vec{r}) \} \quad (5.6)$$

Using the explicit form for $\vec{A}^{(1)}$ we can calculate the last part of W_4 .

$$\frac{1}{2} \int dV \vec{\chi}_3^{(2)}(\vec{r}) \vec{\chi}_3^{(2)}(\vec{r}) = c^4 \frac{Q^2}{\pi} \frac{1}{\sqrt{-E}} \int_{\frac{1}{2}}^{\infty} \frac{dx}{x} K_{iv}^2(x) \int_{\frac{1}{2}}^x dx' K_{iv}^2(x') \quad (5.7)$$

The small quadrupole contribution has been neglected in this formula. In the limit $Q^2 \rightarrow Q_{\text{crit}}^2$ we get for the double integral the numerically estimated value 45.

The remaining part of the functional W_4 is abbreviated by the symbol X :

$$\begin{aligned} X[\vec{A}_3^{(2)}] = & \\ & \frac{1}{2} \int dV \text{Tr} \left\{ \left(\partial^i A_3^{(2)i} - \partial^i A_3^{(2)i} \right) T_3 + [A^{(2)i}, A^{(2)j}] \right\}^2 \\ & + 2 \int dV \text{Tr} \left\{ A_3^{(2)j} T_3 [A^{(2)i}, \partial^j A^{(2)i} - \partial^i A^{(2)j}] \right\} \end{aligned} \quad (5.8)$$

The functional X remains invariant under infinitesimal gauge transformations

$$\vec{A}_3^{(2)} \rightarrow \vec{A}_3^{(2)} + \vec{\nabla} \varphi \quad (5.9)$$

Somewhat more explicitly it looks like:

$$X[\vec{A}_3^{(2)}] = \frac{1}{2} \int dV \left(\vec{\nabla} \times \vec{A}_3^{(2)} + \gamma \frac{\vec{r}}{r} \left(\frac{h(r)}{r} \right)^2 Y_1^-(\Omega) \right)^2 - 2 \gamma \int dV \left(\frac{h(r)}{r} \right)^2 \left(\frac{\vec{r}}{r} \times \vec{\nabla} Y_1^-(\Omega) \right) \vec{A}_3^{(2)} \quad (5.10)$$

The constant γ is a short-hand for

$$\gamma = \sqrt{\frac{3}{4\pi}} \frac{c^2}{2} (\sin 2\alpha) (\sin \beta) \quad (5.11)$$

and Y_1^- is a Cartesian component of the first spherical harmonic.

$$Y_1^-(\Omega) = \sqrt{\frac{3}{4\pi}} \cdot \vec{e}_3 \cdot \frac{\vec{r}}{r} \quad (5.12)$$

The invariance (5.9) allows us to choose $\vec{A}_3^{(2)}$ to be of the form

$$\vec{A}_3^{(2)}(\vec{r}) = \frac{\vec{r}}{r} \tilde{g}(r) + \sum_{j,m} \tilde{f}_{j,m}(r) \frac{\vec{r}}{r} \times \vec{\nabla} Y_j^m(\Omega) \quad (5.13)$$

The index m is thought to label the Cartesian components of the spherical harmonics of order j . The radial piece and all the axial vector terms in the infinite sum, except for Y_1^- contribute only to a sum of squares in the part $(\vec{\nabla} \times \vec{A}_3^{(2)})^2$ of X . The minimal configuration for X is therefore of the form

$$\vec{A}_3^{(2)}(\vec{r}) = \tilde{f}(r) \frac{\vec{r}}{r} \times \vec{\nabla} Y_1^-(\Omega) \quad (5.14)$$

After integrating over the angle Ω , X boils down to the integral:

$$X = \int_{r_0}^{\infty} dr \left\{ (\tilde{f}'(r))^2 - 6\gamma \left(\frac{h(r)}{r} \right)^2 \tilde{f}(r) + \frac{2}{r^2} \tilde{f}^2(r) + \frac{\gamma^2}{2} r^2 \left(\frac{h(r)}{r} \right)^4 \right\} \quad (5.15)$$

To find the minimum for this form we could, in principle, solve the Euler-Lagrange equations for $\tilde{f}$. In practice this is not the most convenient way to get a rough estimate for this minimum. We prefer a direct method. Since $h(r)$ vanishes exponentially for large r , we may assume that the minimal configuration for X can be approximated with functions $\tilde{f}$ which satisfy the inequality

$$\int_{r_0}^{\infty} dr (\tilde{f}'(r))^2 \leq \frac{1}{2} \int_{r_0}^{\infty} dr \frac{1}{r^2} \tilde{f}^2(r) \quad (5.16)$$

On the other side the following inequality is true for all functions $\tilde{f}$ which vanish at the boundary

$$\int_{r_0}^{\infty} dr (\tilde{f}'(r))^2 \geq \frac{1}{4} \int_{r_0}^{\infty} dr \frac{1}{r^2} \tilde{f}^2(r) \quad (5.17)$$

We use both inequalities to eliminate $\tilde{f}'$ from the functional X and we arrive at the following estimate for the minimum $X^{\min}$ of X

$$X^{\min} = -\kappa \cdot \gamma^2 \int_{r_0}^{\infty} dr \cdot r^2 \left(\frac{h(r)}{r} \right)^4 \quad (5.18)$$

$$3.1 \leq \kappa \leq 3.5$$

The integral

$$\int_{r_0}^{\infty} dr \cdot r^2 \left(\frac{h(r)}{r} \right)^4 = \int_{r_0}^{\infty} dr K_{iv}^4(\sqrt{-E} r) \quad (5.19)$$

can be evaluated numerically in the limit $Q^2 \rightarrow Q_{\text{crit}}^2$, the result is

$$\int_{r_0}^{\infty} dr \cdot r^2 \left(\frac{h(r)}{r} \right)^4 \approx \frac{27}{\sqrt{-E'}} \quad (5.20)$$

The quadratic form $W_2[\vec{A}^{(1)} + \vec{A}^{(2)}, \vec{\chi}^{(2)}]$ in the formula (5.4) for the field energy gets its lowest value according to the discussion in the previous section if $\vec{\chi}^{(2)}$ and $\vec{A}^{(2)}$ as well as $\chi^{(1)} = 0$ and $\vec{A}^{(1)}$ are of the form (4.21). Therefore $\vec{A}^{(2)}$ can be absorbed into a redefinition of the parameters α, β and c [Eq. (5.5)].

Putting together the results (5.7), (5.18) and (5.20) we find the field energy (5.4) for the lowest lying field configurations in a certain neighbourhood of the Coulomb field as a function of the parameters α, β and c .

$$W = W_0 - \frac{1}{2} c^2 + \frac{c^4}{4\pi\sqrt{-E'}} (180 Q_{crit}^2 - 67 \sin^2 2\alpha \sin^2 \beta) \quad (5.21)$$

The expression is valid for $0 < Q^2 - Q_{crit}^2 \ll 1$. The Q^2 dependence is contained in the equations (4.19), (4.20) for $E = E_1$. The energy functional just above the critical coupling, restricted to the sub-space of most favourable fluctuations is shown for fixed $\sin\beta \neq 0$ in our figure. The fourth order contribution is positive, however, its magnitude depends on α and β .

For a slightly under-critical coupling $0 < Q_{crit}^2 - Q^2 \ll 1$ the quadratic part of the field energy as well as the quartic part both are positive. The fourth order correction W_4 for an axial vector fluctuation around the Coulomb field would become negative in general if $Q^2 \lesssim 1$, because then the negative "magnetic" contribution X wins over the positive "electric" term $\frac{1}{2} \int dV (\vec{\chi}_3^{(2)} \vec{\chi}_3^{(2)})$.

Besides the Coulomb field ($c = 0$) the energy functional (5.21) has other stationary points $c \neq 0$.

$$c^2 = \pi \cdot \sqrt{-E'} \cdot (180 Q_{crit}^2 - 67 \sin^2 2\alpha \sin^2 \beta)^{-1} \quad (5.22)$$

The energy has saddle points if

$$\sin^2 2\alpha \quad \sin^2 \beta = 0 \quad (5.23)$$

The field configurations for these parameter values are all gauge equivalent to

$$\begin{aligned} \vec{A}(\vec{r}) &= c \cdot \sqrt{\frac{3}{4\pi}} T_1 h(r) \frac{\vec{r}}{r^2} \times \vec{n} + O(c^3) \\ \dot{\vec{A}}(\vec{r}) &= -T_3 \vec{\nabla} (\phi(r) - 2\Delta^{-1}(\phi \text{Tr}\{\vec{A}\vec{A}\})) + [\phi(r)T_3, \vec{A}(\vec{r})] \\ &\quad + O(c^4) \end{aligned} \quad (5.24)$$

The unit vector $\vec{n}$ is arbitrary. These solutions were pointed out previously⁶⁾. These solutions are stable only with respect to variations of $\vec{A}$ along the T_1 axis (the "static" fluctuations in the terminology of Ref. 6)). In the full space (5.1) of lowest possible fluctuations they are saddle points. From this observation one can conclude that the ansatz for the field of two point sources which is studied in Ref. 5) can lead only to an unstable solution of this problem, if the point sources are far apart from each other.

The parameter values

$$\sin 2\alpha = 1, \quad \sin \beta = 1 \quad (5.25)$$

describe new (locally) stable field configurations for an over-critical Q

$$\begin{aligned} \vec{A}(\vec{r}) &= c \sqrt{\frac{3}{8\pi}} h(r) \frac{\vec{r}}{r^2} \times (T_1 \vec{e}_1 + T_2 \vec{e}_2) \\ &\quad + \frac{3}{4\pi} c^2 h^2(r) \frac{\vec{r}}{r^2} \times \vec{e}_3 T_3 + O(c^3) \\ \dot{\vec{A}}(\vec{r}) &= -T_3 \vec{\nabla} (\phi(r) + 2\Delta^{-1}(\phi \text{Tr}\{\vec{A}_1 \vec{A}_1\})) + [\phi(r)T_3, \vec{A}(\vec{r})] + O(c^4) \end{aligned} \quad (5.26)$$

Their energy $W^{\min}$ may be compared with the energy W_0 stored by the Coulomb field in the region $|\vec{r}| \geq r_0$.

$$\frac{W^{\min} - W_0}{W_0} \approx \frac{-1}{12000} \cdot e^{-\frac{\pi}{v}} \cdot (1 + O(v^4)) \quad (5.27)$$

The transition $Q^2 \rightarrow Q_{\text{crit}}^2$ is very soft with respect to field energy. The solution (5.26) exhibits screening. The induced source

$$- 2 \phi(r) \text{Tr} \{ \vec{A}_1 \vec{A}_1 \} \quad (5.28)$$

has the opposite sign relative to the external point source. However, it is important to note that the Y-M electric field is not perfectly screened in the sense that it does not decrease faster than any power of r^{-1} . With the technique presented here it is not possible to calculate whether the screening is "total" or only "partial" in the terminology of Ref. 9). There the screening mechanism is called total if the electric field vanishes faster than r^{-2} ; if only the coefficient of the r^{-2} dependence is weakened it is called partial screening. The difference between the last mentioned forms of screening is related to the large distance behaviour of the induced screening source (5.28). In the presence of a large distance cut-off, as we used it in the previous section, this distinction becomes void.

The almost zero modes $h_{1,n}$ ($n > 1$) [Eq. (4.18)] which are eliminated by a large distance cut-off decrease much slower than $h(r)$. Without the long-distance cut-off they would not contribute much to the energy, however, they would add to the screening source (5.28) at large distances, perhaps until they achieve total screening like it was shown to happen in scalar electrodynamics⁹⁾.

6. CONCLUSION AND COMMENT

For $Q^2 > Q_{\text{crit}}^2 = 9/4$ the Coulomb field of a point source changes from a local minimum of the energy functional to a saddle point. New local minima (and further saddle points) show up near the Coulomb configuration. For $0 < Q^2 - Q_{\text{crit}}^2 \ll 1$ it is possible to write down the new static and stable solutions of the Y-M equations in an approximate form. These

solutions are characterized by a symmetry axis and a length which enters into the multipole moments of the electric field. The length originates from a boundary condition on the vector potential and it determines the scale for the energy difference between the Coulomb field and the new locally stable solution. The electric field does not decrease exponentially however, it is weaker than the corresponding Coulomb field.

For two equally strong point sources separated from each other by a large distance d the locally stable solution would correspond to the Coulomb field of two oppositely charged point sources (up to gauge transformations). The other Abelian solution which corresponds to equally charged point sources is highly unstable. If the coupling strength exceeds the critical value the previously stable Coulomb field will break down. The field energy as a function of d is bounded from above by the energy stored in the Coulomb field and approaches the asymptotic value $2W^{\min}$ [Eq. (5.27)]. Let us add a comment on how we think that the classical instability affects the quantized version of the Y-M theory. In a Hamiltonian formulation of both Abelian and non-Abelian gauge theories Gauss' law acts as a constraint on the canonical momenta of the fields. Together with a gauge choice it eliminates a pair of canonically conjugate field variables. One is left with the independent, dynamical fields which get quantized and with a non-local, instantaneous interaction for the charges which remains classically¹⁰⁾. In electrodynamics the Coulomb solution of Gauss' law carries the minimal energy. It is therefore very natural to choose the Coulomb gauge and to eliminate the longitudinal degrees of freedom from the independent variables. For vanishing transversal fields one reaches the absolute minimum of the classical field energy. The quantization of the oscillations in the transversal fields can be carried out canonically. If one follows this pattern in the Y-M theory coupled to heavy particles the longitudinal electric field $-\vec{\nabla}\varphi$ has to be eliminated by Gauss' law which now takes the form

$$-\Delta\varphi + [\vec{A}, -\vec{\nabla}\varphi] = -[\vec{A}, \vec{E}_{tr}] + \rho$$

One may use this equation formally to define φ as a functional of the particle source ρ , $\vec{A}$ and the transversal electric field $\vec{E}_{tr}$. For sufficiently large coupling the classical static energy functional is now minimized by $\vec{A} \neq 0$, $\vec{E}_{tr} \neq 0$ [see Eq. (5.26)]. Both the iterative

procedure to solve Gauss' law for the longitudinal electric field and the perturbative handling of the resulting Hamiltonian in the transverse field variables have to start from a non-trivial background. This may add to the long-lasting doubts (hopes) on whether a Y-M system beyond the weak coupling limit describes interacting, massless quanta and a long-range classical Coulomb force.

Finally let us speculate on global stability of the classical Coulomb field for a small coupling constant. As we had mentioned in the previous section the fourth order correction W_4 to the energy functional near the Coulomb field will be negative for many fluctuations if Q^2 is small. If one uses blindly the approximation of the energy functional up to quartic terms, in a region where it is no longer justified, one would find that the Coulomb solution for a single, point-like particle is separated only by a finite energy barrier from configurations which have lower energy. In this case the electrodynamics oriented perturbation theory for quantized Y-M systems could be misleading even in the small coupling region.

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Figure caption

The energy functional at the critical coupling strength, restricted to the sub-space of lowest lying fluctuations with fixed $\sin\beta \neq 0$ [Eqs. (5.5), (5.21)]. The energy is plotted vertically in arbitrary units. The horizontal dimensions parametrize the deviation of the configuration from the Coulomb field which is represented by the origin 0. The front bisegment, which is cut away describes configurations which are gauge equivalent to those shown.