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U(1) WITHOUT INSTANTONS

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A B S T R A C T

Witten's recent proposal that the $U(1)$ problem might be solved in $1/N$ expanded QCD, is shown to be automatically consistent with expected θ dependences and anomalous Ward identities, if a (modified) Kogut-Susskind mechanism is used. Ward identities are algebraically saturated for large N . A sort of "partial conservation of the $U(1)$ current" is found to hold for the " η " field.

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1. INTRODUCTION

The history of the $U(1)$ problem¹⁾ is a particularly frustrating one: although a number of solutions were proposed at various moments, each time they were claimed not to be viable at some deeper level.

The most recent example of this pattern is 't Hooft's solution²⁾, i.e., his fundamental observation that topologically non trivial field configurations can provide a non-zero value for the (integrated, zero momentum) Adler-Bell-Jackiw axial anomaly:

$$\partial_\mu J_{\mu,5}^{N_f} = (2N_f) Q(x) + (\text{normal soft divergence}) \quad (1)$$

$$Q(x) \equiv \frac{g^2}{32\pi^2} F_{\mu\nu}^{(x)} F_{\mu\nu}^{*(x)} ; \quad \int d^4x Q(x) \equiv \nu$$

where N_f is the number of quark flavours and ν is the topological charge³⁾ taking integer values for finite action field configurations (e.g. instantons).

Although this is most probably the way out of the $U(1)$ problem, the explicit arguments of 't-Hooft, based on quantized ν configurations have been criticized by Crewther^{4),1)} who pointed out some apparent difficulties in satisfying anomalous Ward Identities (WI) and in obtaining the right θ periodicity of vacuum expectation values such as $\langle \bar{q}q \rangle$ in the θ vacuum⁵⁾.

Recently Witten⁶⁾, drawing on some analogies between $1/N$ expanded QCD^{7),8)} and a class of $1/N$ expanded non-linear (CP(N) invariant) σ models in two dimensions^{6),9)}, has proposed that the $U(1)$ problem should be solved, in QCD, at the lowest non-planar level, i.e., at the next-to-leading order of its $1/N$ expansion, due to quantum fluctuations in the topological charge ν .

Indeed, this attitude fits very well with the one based on the dual topological expansion (TE) of the S matrix¹⁰⁾. A straightforward argument, based on the high energy, Regge analysis of pseudoscalar exchange in the simplest, non-planar (cylinder) topology, shows¹¹⁾ that the π - η planar degeneracy has to be removed at that level. Barring extra accidental zeroes of Reggeon propagators and a smart ensuing cancellation, the proof is rather tight. Even the sign of the shift ($\alpha_\eta(0) < \alpha_\pi(0)$) can be correctly predicted¹²⁾.

In this paper we argue that a modified Kogut-Susskind mechanism¹³⁾, assumed to be active at leading order in $1/N_c$ (N_c here is the number of colours) in pure Yang-Mills (YM) theory, allows one to remove the difficulties found by Crewther^{1),4)} in the case of quantized ν solutions.

We shall proceed as follows: We first formulate precisely (but in a set of "ideal worlds") the $U(1)$ problem in $1/N$ language and discuss which is the relevant large N limit to be taken; Second, we show how the normal and anomalous Ward Identities are algebraically saturated for any number of light quarks, and we shall recover a recent " η mass formula" of Witten¹⁴⁾; Third, we discuss possible problems^{1),4)} connected with θ dependences and find that they are not there; finally, we discuss how these "ideal world" results can be applied, with considerable success, to the physical situation.

2. $1/N$ FORMULATION OF THE $U(1)$ PROBLEM

Consider a set of "ideal worlds" described by the usual QCD Lagrangian with an $SU(N_c)$ colour gauge group, and N_f quark flavours. Out of the N_f quarks, L are taken to be light quarks with common mass m and the remaining ones have masses $M_i \gg \Lambda \gg m$, where $\Lambda \sim 500$ Mev is the renormalization group invariant scale of QCD.

Consider the TE of QCD discussed in Ref. 8) and obtained, formally, by taking $N_c, N_f \rightarrow \infty$ with a fixed ratio and with fixed $\alpha_s N_c$. Assume both confinement and the Nambu - Goldstone (NG) mechanism¹⁵⁾ to appear at the leading (planar) level. Then, whereas all other hadrons made of light quarks have masses of order Λ (times an N independent factor), there is an L^2 pllet of light NG pseudoscalars of mass:

$$m_{NS}^2 = F_{NS}^{-2} \langle -2m \bar{q}_a q_a \rangle_0 \equiv F_{NS}^{-2} \langle \Delta_2 \rangle_0, \quad (2)$$

$a = 1, 2 \dots L$; no sum over a

where NS stands for non-singlet of $SU(L)$ flavour, F_{NS} is the analog of F_π , and $F_{NS}, \langle \bar{q}q \rangle_0$ are^{7),8)} of order N_c (so that $m_{NS}^2 \propto m\Lambda$). Although m_{NS} stands for the non-singlet mass (e.g. $m_{NS} = m_\pi$), at this planar level it is also the mass of the $SU(L)$ singlet: $m_S = m_{NS}$.

This degeneracy, known from old duality arguments, can be understood group-theoretically as coming from a (larger) invariance, possessed by the planar theory under $SU(L)_q \otimes SU(L)_{\bar{q}}$, which rotates separately quarks and antiquarks^{*)}.

*) This was recognized after a very illuminating discussion with M. Gell-Mann. I then realized that the relevance of this group had been noticed earlier by H.J. Lipkin, Nucl. Phys. B1, 597 (1967).

The larger symmetry holds because there are no OZI violating $q\bar{q}$ annihilation diagrams at leading order in $1/N$. As a consequence, particles appear in N_f^2 -plets which, for broken $SU(N_f)$, are ideally mixed (i.e. physical particles are diagonal in the $q\bar{q}$ basis). Another consequence of planarity is that decay couplings (F_π , F_K , etc.) are all related. With the definitions of Ref. 1 we find:

$$\begin{aligned} \langle NS\text{-Pseudoscalar}, a | \mathcal{J}_{\mu,5}^b(0) | 0 \rangle &= -i \delta_{ab} g_\mu F_{NS} \\ \langle S\text{-Pseudoscalar}, L | J_{\mu,5}^L(0) | 0 \rangle &= -i g_\mu \sqrt{2L} F_{NS} \end{aligned} \quad (3)$$

These symmetries for the couplings appear to be consistent with the data¹⁶⁾.

The U(1) problem can now be formulated in $1/N$ language as follows:
Is the L^2 (planar) degeneracy broken down by next-to-leading (non-planar) diagrams to an (L^2-1) degeneracy of NG bosons, accompanied by a higher lying $SU(L)$ singlet of finite mass, even in the limit $m(m_{NS}^2) \rightarrow 0$?

The answer to this question can be affirmative only if the anomalous contribution to $\partial_\mu J_{\mu,5}^L$ ($J_{\mu,5}^L \equiv \sum_{i=1}^L \bar{q}_i \gamma_\mu \gamma_5 q_i$)

$$\begin{aligned} \partial_\mu J_{\mu,5}^L &= 2L Q(x) + D_L(x), \\ D_L &= 2im \sum_{i=1}^L \bar{q}_i \gamma_5 q_i \end{aligned} \quad (4)$$

becomes non-trivial at zero momentum due to quantum fluctuations of topological charge. Unlike the contribution of integer v configurations these fluctuations do not have to vanish exponentially for large N ⁶⁾ and the S-matrix arguments mentioned above support the conjecture that such effects are already present at $O(1/N_c)$. Since one can define a gauge-non-invariant current K_μ :

$$K_\mu = \frac{g^2}{16\pi^2} \epsilon_{\mu\alpha\beta\gamma} A_a^\alpha (\partial^\beta A_a^\gamma + \frac{1}{3} g f_{abc} A_b^\beta A_c^\gamma) \quad (5)$$

such that:

$$\partial_\mu K_\mu = Q(x) = \frac{1}{2L} \left[\partial_\mu J_{\mu,5}^L - D_L \right] \quad (6)$$

the non-vanishing of $\int d^4x Q(x) = v$ implies unphysical poles at $q^2=0$ in the matrix elements of K_μ (the analog of Kogut Susskind ghosts¹³⁾). We shall then assume the existence of such poles in the large N_c limit of YM theory and we

shall work out the consequences of this assumption vis-à-vis anomalous Ward Identities and θ dependence problems.

Before turning to this we have to briefly discuss the exact nature of the large N limit that we shall take. Because of the presence of another (besides $1/N$) small parameter in the problem i.e. of the light quark mass in units of Λ m/Λ (which we shall let approach zero) and because of the need to consider Green's functions at zero momentum (hence near light particle poles), it will turn out that the $N_c \rightarrow \infty$ limit of 't Hooft⁷⁾, selecting diagrams with no quark loops, is not a smooth one^{*)}. We shall have to consider instead the TE of QCD⁸⁾, at L/N_c fixed with inclusion of an arbitrary number of quark loops at each order, and then we shall let (for simplicity) $L/N_c \rightarrow 0$. To be even more precise, we shall take the double limit:

$$m_{NS}^2 \sim m\Lambda \ll \frac{L}{N_c} \Lambda^2 \ll \Lambda^2 \quad (7)$$

since, in this limit, we shall be able to achieve a completely algebraic saturation of the Ward Identities

3. SATURATION OF WARD IDENTITIES.

In the notations of Ref. 1 and eq. (1), the relevant WI take the form:

$$\int d^4x T \langle \partial^\mu \mathcal{F}_{\mu,5}^a(x), \partial^\nu \mathcal{F}_{\mu,5}^b(0) \rangle_0 = -i \delta_{ab} \langle \Delta_2 \rangle_0 \quad (8)$$

$$2L \int d^4x T \langle Q(x), D_L(0) \rangle_0 = -2iL \langle \Delta_2 \rangle_0 - \int d^4x T \langle D_L(x) D_L(0) \rangle_0 \quad (9)$$

$$4L^2 \int d^4x T \langle Q(x), Q(0) \rangle_0 = 2iL \langle \Delta_2 \rangle_0 + \int d^4x T \langle D_L(x) D_L(0) \rangle_0 \quad (10)$$

*) This crucial point is due, in this context, to Witten¹⁴⁾. It is not however the first example of a non-smooth $N_c \rightarrow \infty$ limit [see, e.g., D. Amati, R. Petronzio and G. Veneziano, Nucl. C Phys. B140, 54 (1978), Sect. 4; D. Amati and G. Veneziano CERN TH. 2620 (1978)]. Even ordinary perturbation theory is occasionally non-smooth (e.g., in the presence of NG bosons). By contrast the TE of QCD is expected to be smooth with the only exception of forward, high-energy scattering, where it leads⁸⁾ to Reggeon Field Theory.

Eq. (8) is the usual, anomaly free, WI for SU(L) non-singlet currents. It implies the existence of NG bosons of mass m_{NS} given by eq. (2). Of the remaining two anomalous WI the difficult one to satisfy is eq. (10). For instance, in the instanton solution a WKB approximation shows⁴⁾ that the left-hand side is of order m^2 and not of order m as the right-hand side. Furthermore, the right-hand side depends on θ as

$$\langle \Delta_2 \rangle_\theta = \cos \theta / L \langle \Delta_2 \rangle_{\theta=0} \quad (11)$$

because of its chirality, and it is hard to see⁴⁾ how integer topological charge can give anything but periodicity 2π .

Assume instead, with Witten⁶⁾, that the left-hand side of eq. (10) is non-zero at leading order in $1/N_c$ of pure YM theory. We may get that by introducing an axial 4-vector ghost^{*)} of propagator $-ig_{\mu\nu}/q^2$ coupled to K_μ of eq. (5) with strength $i\lambda_{YM}g_{\mu\nu}$, where $\lambda_{YM} = O(1)$. Consequently, in pure YM and large N_c , the left-hand side of eq. (10) is:

$$4L^2 (i\lambda_{YM})^2 (ig_\mu)(-ig_\nu) \frac{-ig_{\mu\nu}}{q^2} = 4iL^2 \lambda_{YM}^2 = O(L^2) \quad (12)$$

in contrast with the right-hand side which, in the absence of SU(L) singlets of mass² $O(m)$, is of $O(mLN_c)$. The paradox is solved by noticing that quark loops, although formally down by $(L/N_c)^n$, contain intermediate states of mass of $O(m)$ making a resummation necessary. This can easily be done if the ghost dominates again at $q = 0$ by coupling to the (planar) SU(L) singlet (which we shall denote by η_L) with strength $i\sqrt{L/N_c} \lambda_\eta q_\mu$ (with λ_η again independent of L, N_c). All quark loops, inserted on the leading non-planar (cylinder) topology, can then be summed with the following replacement^{**) in eq. (12) :}

$$\begin{aligned} 4iL^2 \lambda_{YM}^2 &\rightarrow 4iL^2 \lambda_{YM}^2 \left(1 + \frac{L}{N_c} \lambda_\eta^2 (q^2 - m_{NS}^2)^{-1} + \dots \right) = \\ &= \frac{4iL^2 \lambda_{YM}^2 (q^2 - m_{NS}^2)}{q^2 - m_{NS}^2 - \frac{L}{N_c} \lambda_\eta^2} \end{aligned} \quad (13)$$

*) The original KS dipole¹³⁾ seems to be not as useful in this formulation.

**) This can be done even if q^2 dependences of $\lambda_{YM}, \lambda_\eta$ are taken into account since the range of q^2 to be considered is $0 \leq q^2 \leq L/N_c \Lambda_c^2 \ll \Lambda_c^2$ and variations in this range are non-leading in L/N_c if we assume no massless glueballs.

where we have computed everything at a q^2 value to be sent to zero at the end. We see clearly that each loop is "down" by a power of $(L/N_c \lambda_\eta^2) / (q^2 - m_{NS}^2)$ which is a "small" number of $O(L/N_c)$ at finite q^2 but becomes $O[(L/N_c) / (m_{NS}^2) \gg 1/\Lambda^2]$ as $q^2 \rightarrow 0$. After resummation of the geometric series the singularity at $q^2 = m_{NS}^2$ produces a zero at that same point, or rather something of order m_{NS}^2 at $q = 0$, as required by the WI (10). Furthermore, this is so independently of L . Eq. (13) shows that the η_L mass m_S is given by

$$m_{\eta_L}^2 \equiv m_S^2 = m_{NS}^2 + \frac{L}{N_c} \lambda_\eta^2 \quad (14)$$

Furthermore, by going on the m_L pole and using eqs. (3), one obtains :

$$\langle 0 | Q(0) | \eta \rangle = \frac{F_{NS}}{\sqrt{2L}} m_S^2 = \lambda_{YM} \lambda_\eta \sqrt{\frac{L}{N_c}} \quad (15)$$

which, combined with eq. (14), gives:

$$\lambda_{YM}^2 = \lambda_\eta^2 \frac{F_{NS}^2}{2N_c} = m_S^2 \frac{F_{NS}^2}{2L} \quad (16)$$

Going then to $q^2 = 0$ in eq. (13) we get:

$$\begin{aligned} \text{l.h.s. (10)} &= \frac{4iL^2 \lambda_{YM}^2 m_{NS}^2}{m_{NS}^2 + \frac{L}{N_c} \lambda_\eta^2} \simeq 4iL N_c m_{NS}^2 (\lambda_{YM}/\lambda_\eta)^2 = \\ &= 2iL m_{NS}^2 F_{NS}^2 = \text{r.h.s. (10)} \end{aligned} \quad (17)$$

It is easy to check, at this point, that also eq. (8) is completely saturated by the η_L pole for $L/N_c \rightarrow 0$. This fact is not surprising because the η_L mass is small $O(\sqrt{L/N_c})$ compared to that of all other hadrons with its quantum numbers, which therefore contribute negligibly in the large N_c limit^{*}). We wish to note, finally that eq. (16) can be rewritten as :

^{*}) The reader may wonder about the rôle of the ghost, since this never appears as a pole in physical matrix elements. One can easily see that the ghost ensures a factorization constraint among three $q = 0$ amplitudes, which is crucial for consistency. It is quite possible, however, that one can find an analogous argument completely in terms of gauge invariant quantities, hence without introducing the ghost.

$$m_s^2 = 2L F_{NS}^{-2} \lambda_{YM}^2 = 2L F_{NS}^{-2} \frac{\partial}{\partial \theta} \langle Q(x) \rangle_{\theta} \Big|_{\theta=0}^{YM, N_c \rightarrow \infty} \quad (18)$$

a formula relating the η_L mass to the θ dependence of large N_c YM theory. This relation was obtained first in a different way by Witten¹⁴⁾, who noticed that quark loops were crucial (even for $N_c \rightarrow \infty$) in order to avoid a paradox with θ dependence.

4. CHECKS OF θ DEPENDENCES

We start with a simple check of consistency with eq. (11) i.e.,

$$-\frac{\partial^2}{\partial \theta^2} \langle \Delta_L \rangle \Big|_{\theta=0} = \int d^4x d^4y T \langle Q(x) Q(y) \Delta_L(0) \rangle = \frac{1}{L^2} \langle \Delta_L \rangle_{\theta=0} \quad (19)$$

In leading order in $1/N_c$ the left-hand side is given by QCD diagrams on a sphere with two $Q(x)$ sources and a $\bar{q}q$ insertion. Naively, the left-hand side is $O(L/N_c)$ whereas the right-hand side is of $O(N_c/L)$. The difference is made up by two low lying poles in the $Q(x), Q(y)$ channels. Again using dominance of the ghosts, one gets, after insertion of an arbitrary number of quark loops,

$$\begin{aligned} \text{l.h.s. (19)} &= \lambda_{YM}^2 \frac{L}{N_c} \lambda_{\eta}^2 \left(\frac{i}{q^2 - m_s^2} \right)^2 \langle S | \Delta_L | S \rangle \xrightarrow{q \rightarrow 0} \\ &\rightarrow - \frac{F_{NS}^2}{2L} \langle S | \Delta_L | S \rangle = - \frac{F_{NS}^2}{2L} \langle \pi_0 | \Delta_2 | \pi_0 \rangle = \frac{m_{NS}^2 F_{NS}^2}{2L} = \text{r.h.s. (19)} \end{aligned} \quad (20)$$

where we have related the planar couplings of singlet and non-singlet pseudoscalars by trivial Clebsch Gordan and we have used some standard current algebra result¹⁷⁾. It is clear that $\langle \bar{q}q \rangle_{\theta}$, in spite of being a one point function, has a θ dependence given by n point functions with Q insertions. The θ dependence is sensitive therefore to poles near $q^2 = 0$, whose exact location depends however on L . This seems to answer Crewther's puzzles on these questions^{1), 4)}.

In a similar way one checks that :

$$\frac{\partial}{\partial \theta} \langle \bar{q}_i \gamma_5 q_i \rangle \Big|_{\theta=0} = -\frac{i}{L} \langle \bar{q}_i q_i \rangle_{\theta=0} \quad (21)$$

We now look, finally, at the θ dependence of $\langle 0|0 \rangle_\theta$ or rather at that of $\langle 0|Q(x)|0 \rangle_\theta$ which, from eqs. (10), (11) we expect to be :

$$\langle 0|Q(x)|0 \rangle_\theta = \frac{1}{2} m_{NS}^2 F_{NS}^2 \sin \frac{\theta}{L} \quad (22)$$

To this aim we differentiate the left-hand side of eq. (22) $2n-1$ times with respect to θ at $\theta=0$ and obtain:

$$\frac{\partial^{2n-1}}{\partial \theta^{2n-1}} \langle 0|Q(x)|0 \rangle_\theta \Big|_{\theta=0} = (-i)^{2n-1} \int d^4 x_i T \langle Q(x_1) Q(x_2) \dots Q(x_{2n-1}) Q(0) \rangle_{\theta=0} \quad (23)$$

We have to show that this is equal to $-L/2 m_{NS}^2 F_{NS}^2 L^{-2n} (-1)^n$. The right-hand side of eq. (23) can be evaluated first at $q_i \neq 0$ by inserting ghost propagators in each external leg. The result, after some algebra and topological reasoning, is

$$\text{r.h.s. (23)} = i \sum_{\{b\} = \text{boundary structure}} \lambda_{YM}^{2n} \prod_i \frac{\lambda_2 \sqrt{\frac{L}{N_c}}}{(q_i^2 - m_S^2)} \prod_j \frac{\sqrt{\frac{N_c}{L}} \lambda_2^{-1} (q_j^2 - m_{NS}^2)}{(q_j^2 - m_S^2)} A_{2n-\eta_L}^{\{b\}} \quad (24)$$

Here the concept of boundary structure is as defined in Ref. 8; the index j selects channels with a single η on one boundary, with the index i running over all other channels. On the $2n$ -ple η_L pole, $q_j^2 = m_S^2 = L/N_c \lambda_\eta^2$, so that each boundary contribution is equally weighted and one obtains the desired factorization with the right, known coefficient.

If, instead, we go to $q_i^2 = 0$, we see that all terms in which the set of indices j is not empty are at most $O(m_{NS}^4)$. In all other terms the $2n - \eta_L$ amplitude can be replaced by a known Clebsch-Gordan times the corresponding contribution to some $2n - \pi$ amplitude. For large N_c one finds that terms with $b > 1$ provide just a shift of the pion poles appearing in the leading term with $b = 1$ so that:

$$A_{2n-\eta_L}^{(b=1)} = \left(\frac{2}{L}\right)^{n-1} A_{2n-\pi_0}'^{(b=1)} \quad (25)$$

where the prime indicates that all pion poles have been removed. Using standard current algebra techniques¹⁷⁾ it is then easy to show that:

$$A_{2n-\pi_0}'^{(b=1)} = i m_{NS}^2 F_{NS}^{2-2n} (-1)^n \quad (26)$$

and, after use of eq. (16), we obtain:

$$\text{r.h.s. (23)} = i \left(\frac{\lambda_{YM}^2}{m_s^2}\right)^n \left(\frac{2}{L}\right)^{n-1} i m_{NS}^2 F_{NS}^{2-2n} (-1)^n = -\frac{L}{2} m_{NS}^2 F_{NS}^2 L^{-2n} (-1)^n \quad (27)$$

which is the desired result.

It is interesting to see what happens in pure YM theory at $N_c \rightarrow \infty$. This is obtained from eq. (24) by setting $(q_j^2 - m_{NS}^2)/(q_j^2 - m_s^2) = 1$. The result is:

$$\frac{\partial^{2n-1}}{\partial \theta^{2n-1}} \langle Q(x) \rangle_\theta \bigg|_{\theta=0}^{YM, N_c \rightarrow \infty} = i \left(\frac{F_{NS}^2}{2L}\right)^n A_{2n-\eta_L} \bigg|_{q_i \rightarrow 0} \quad (28)$$

since all other terms can be related to soft pion amplitudes, which vanish in the soft limit. Eq. (28) is to be contrasted with our previous result:

$$\frac{\partial^{2n-1}}{\partial \theta^{2n-1}} \langle Q(x) \rangle_{\theta} \bigg|_{\substack{\text{QCD} \\ \frac{L}{N_c} \rightarrow 0 \\ \theta=0}} = i \frac{L}{2} \left(\frac{F_{\pi}^2}{L^2} \right)^n A'_{2n-\pi_0} \bigg|_{q_i \rightarrow 0} \quad (29)$$

Eq. (28) generalizes^{*)} Witten's formula eq. (18) ($A_{2n} \equiv -im_S^2$) and shows that the θ dependence in large N_c YM theory is related to QCD soft η scattering amplitudes! It is easy to show, by trivial N power counting, that the right-hand side of eq. (28) is of $O(N_c^{2-2n})$ and independent of L , as should be the case for a YM quantity. This implies that (non-trivial) θ dependence in pure YM is through θ/N_c and could possibly indicate a θ periodicity proportional to N_c . A check of this possibility in the $CP(N)$ models is under way¹⁸⁾.

Incidentally, it has been explicitly checked by Di Vecchia¹⁸⁾ that Ward Identities are saturated in $1/N$ expanded $CP(N)$ models with fermions in exactly the same way as we have guessed here to be the case for QCD.

5. RELEVANCE TO THE PHYSICAL WORLD

Unfortunately, the physical world is not as simple as one of our "ideal worlds". First $N_c = 3$ and not ∞ ; second, m_u and m_d are really small compared to $\Lambda \approx 500$ Mev, but m_s is not so different. Finally, we believe m_c , m_b etc to be considerably larger than Λ .

In this situation already discussed in the literature from several view points¹⁹⁾ we cannot decide a priori which is the quark content of our physical pseudoscalars,

*) This generalization was also obtained by Witten¹⁴⁾ in an effective Lagrangian approach. A further generalization of eq. (28) can be found by noticing that both η_L amplitudes and $Q(x)$ insertions are dominated, at $q = 0$, by the ghost. One thus obtains a sort of Partial Conservation of $U(1)$ Current (PCU_1):

$$(\partial_\mu K_\mu)_{YM} \simeq F_\pi m_{\eta_L}^2 (2L)^{-1/2} \phi_{\eta_L}(x).$$

because the non-planar ($\bar{q}q$ annihilation) diagrams, besides shifting some levels, also mix them. Since off diagonal mass²-matrix elements are of $O(1/N_c \Lambda^2)$ and differences of diagonal matrix elements can be of $O[(m_s - m_u) \Lambda]$, large mixing can occur due to the (accidental?) fact, often emphasized by Gell Mann, that m_s and Λ are comparable^{*)}. We shall argue that the phenomenological trend, as well as the semiquantitative nature of the pseudoscalar spectrum, can be understood if nature indeed chooses quark masses such that:

$$m_{u,d} \Lambda \ll m_s \Lambda \lesssim \text{Non Planar Mixing } O(\Lambda^2/N_c) \ll \Lambda m_{c,t,b,\dots} \quad (30)$$

which seems quite reasonable.

On the contrary, for other multiplets (e.g., vector and tensor mesons) unshifted (planar) masses are of $O(\Lambda^2)$ and the large N expansion gives the correct pattern of ideally mixed multiplets both at the $SU(3)$ and at the $SU(4)$ level. This phenomenon is probably related to the property of "Asymptotic Planarity", found in the S-matrix TE approach¹⁰⁾, which predicts quenching of non-planar effects at large masses.

We finally give here some numerical computations for the pseudoscalar nonet (π, K, η, η'). We consider two bases, in which π and K do not mix with anything else (isospin is taken to be exact), and the remaining states are taken to be one of the following:

$$I \equiv \frac{1}{\sqrt{2}} (u\bar{u} + d\bar{d}) \quad ; \quad S \equiv s\bar{s} \quad (31)$$

$$\underline{8} \equiv \frac{1}{\sqrt{3}} (I - \sqrt{2} S) \quad ; \quad \underline{1} = \frac{1}{\sqrt{3}} (\sqrt{2} I + S) \quad (32)$$

*) Because of this there will be also corrections of $O(m_c^2)$ to the mass² of $s\bar{s}$ and $(u,d)\bar{s}$ pseudoscalars. Indeed, for $m \gg \Lambda$ we know from charmonium that $m_{c\bar{c}} \sim (2m_c)$ (non-relativistic limit).

In terms of m_π^2 , m_K^2 and the ghost coupling λ_η introduced earlier, the mass² matrix is easily computed to be^{*)} :

$$\begin{pmatrix} m_\pi^2 + \frac{2\lambda_\eta^2}{N_c} & \frac{\sqrt{2}\lambda_\eta^2}{N_c} \\ \frac{\sqrt{2}\lambda_\eta^2}{N_c} & 2m_K^2 - m_\pi^2 + \frac{\lambda_\eta^2}{N_c} \end{pmatrix} ; \begin{pmatrix} \frac{4}{3}m_K^2 - \frac{1}{3}m_\pi^2, -\frac{2\sqrt{2}}{3}(m_K^2 - m_\pi^2) \\ -\frac{2\sqrt{2}}{3}(m_K^2 - m_\pi^2), \frac{2}{3}m_K^2 + \frac{1}{3}m_\pi^2 + \frac{3\lambda_\eta^2}{N_c} \end{pmatrix} \quad (33)$$

in the basis (31) and (32), respectively.

Diagonalization then yields:

$$m_{\eta_\pm}^2 = \left(m_K^2 + \frac{3}{2} \frac{\lambda_\eta^2}{N_c} \right) \pm \frac{1}{2} \sqrt{\left(2m_K^2 - 2m_\pi^2 - \frac{\lambda_\eta^2}{N_c} \right)^2 + 8 \frac{\lambda_\eta^4}{N_c^2}} \quad (34)$$

$$\eta_- = \eta \quad , \quad \eta_+ = \eta'$$

with mixing angles ($|\eta\rangle = \cos\theta|8\rangle + \sin\theta|1\rangle$ etc...) :

$$\tan \theta = \sqrt{2} - \frac{3}{2\sqrt{2}} \frac{m_{\eta'}^2 - m_\pi^2}{m_K^2 - m_\pi^2} \quad (35)$$

where $\tan \theta = \sqrt{2}$ ($\theta = 0$) corresponds to ideal mixing (pure SU(3) representations). For the extreme case $m_K^2 - m_\pi^2 \gg \lambda_\eta^2 / N_c$ (e.g. $N_c \rightarrow \infty$) one gets $\tan \theta \sim \sqrt{2}$ and $m_{\eta'}^2 \simeq m_\pi^2 + \frac{2}{N_c} \lambda_\eta^2 + O(\lambda_\eta^2 / N_c m_K^2)$ corresponding to our "ideal world" with $L = 2$; for the other extreme case, $(m_K^2 - m_\pi^2) \ll \lambda_\eta^2 / N_c$, (e.g. very good SU(3)) $\theta \rightarrow 0$ and $m_{\eta'}^2 \simeq m_{\pi,K}^2$ whereas $m_{\eta}^2 = m_\pi^2 + 3/N_c \lambda_\eta^2$, corresponding again to an "ideal world" this time with $L = 3$ (see eq. (14)).

It is easy to see where the actual world approximately lies. Since one has from eq. (34):

$$\frac{3}{N_c} \lambda_\eta^2 \stackrel{N_c=3}{=} \lambda_\eta^2 = m_{\eta'}^2 + m_{\eta}^2 - 2m_K^2 \stackrel{\text{exp.}}{=} 0.726 \text{ GeV}^2 \quad (36)$$

*) Such a mass matrix leads automatically to Schwinger's formula²⁰⁾ for the masses of a nonet of mesons.

one sees that, in the square root of eq. (34), the first term is very small compared with the second one, giving approximately:

$$m_{\eta}^2 \simeq m_K^2 + \frac{(3-2\sqrt{2})}{6} \lambda_{\eta}^2 \simeq 0.267 \text{ GeV}^2 \text{ (Exp. 0.30)} \quad (37)$$

$$m_{\eta'}^2 \simeq m_K^2 + \frac{(3+2\sqrt{2})}{6} \lambda_{\eta}^2 \simeq 0.951 \text{ GeV}^2 \text{ (Exp. 0.917)} \quad (38)$$

and a mixing angle of $+14^\circ$. We regard these as very encouraging results (compare e.g. with the value $\theta = 11^\circ$ found in Ref. 16) due to the approximations involved, in particular to the use of PCAC for strangeness-carrying currents. Hence the whole scheme is consistent with the inequalities (30).

Finally, with the aid of these numbers and of eq. 18, one finds:

$$\frac{\partial}{\partial \theta} \langle Q(x) \rangle_{\theta} \Big|_{\substack{YM \\ N_c \rightarrow \infty \\ \theta=0}} = \lambda_{YM}^2 \simeq (180 \text{ MeV})^4 \quad (39)$$

which is perhaps a reasonable (and in any case interesting) number.

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