HIGH-ENERGY BEHAVIOUR IN A NON-ABELIAN GAUGE THEORY - I $T_{n \rightarrow m}$ IN THE LEADING - $\ln s$ APPROXIMATION

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A B S T R A C T

This is the first part of an attempt to find a unitary high-energy description of a spontaneously broken non-Abelian gauge theory. In this paper we calculate the high-energy behaviour of $n \rightarrow m$ amplitudes in the leading $\ln s$ approximation. We start from tree approximations and then only use dispersion relations and unitarity equations for elastic and inelastic amplitudes in the multi-Regge limit. The resulting amplitudes have multi-Regge behaviour with simple pole exchange and satisfy s channel unitarity in all channels. There is, however, no vacuum quantum number exchange (Pomeron) in this approximation. We briefly outline a scheme which may lead to a unitary description of the Pomeron.

1. INTRODUCTION

Non-Abelian gauge field theories (NAGFT) have become an extremely promising candidate for not only weak and electromagnetic but also strong interaction physics. Main attention has been given to asymptotic freedom and the possibility of quark confinement. Much less is known, however, about the behaviour of these theories in other regions of interest, in particular the high-energy and small-angle scattering regime (Regge limit). Several groups of authors¹⁻⁶⁾ have calculated the high-energy behaviour of spontaneously broken non-Abelian gauge field theories in the leading- $\ln s$ approximation (or the weak coupling limit). They all agree in that the (massive) gauge vector meson reggeizes, and the $2 \rightarrow 2$ scattering amplitude with the quantum number of the gauge particle in the exchange channel has simple Regge pole behaviour (Fig. 1a). The vacuum exchange channel (Pomeron) can most easily be illustrated in terms of the reggeon diagrams of Fig. 1b^{2,7)}. The resulting j -plane singularity is a fixed cut to the right of $j = 1$. This shows that the leading- $\ln s$ approximation is insufficient to give a satisfactory high-energy description.

The basic reason for this is the fact that the leading- $\ln s$ approximation for the vacuum exchange channel does not satisfy s -channel unitarity⁸⁾. This can easily be seen from Fig. 1b: if we take the energy discontinuity of these diagrams, say of the first one (in the leading- $\ln s$ approximation only the diffractive cut contributes), on both sides of the cut we find only non-zero quantum number exchange amplitudes but no vacuum exchange. The reason why the diagrams of Fig. 1b are not sufficient for a satisfactory high-energy description can also be understood from the point of view of t -channel unitarity. In the high-energy limit, t -channel unitarity takes the form of reggeon unitarity, and a solution to this set of unitarity equations is obtained by the use of reggeon field theory. All results of reggeon field theory calculations emphasize the vital role of the triple Pomeron coupling, i.e. contributions where one Pomeron breaks up into two. Figure 1b does not contain such contributions, and this explains why these diagrams can violate the Froissart bound. The appearance of a triple Pomeron coupling (Fig. 1c), on the other hand, is closely connected with a particular part of s -channel unitarity, namely subchannel unitarity of inelastic amplitudes. If we take an s -discontinuity of Fig. 1c, we find a production amplitude which, in one subchannel, contains the same set of reggeon diagrams as the $2 \rightarrow 2$ amplitude in Fig. 1b, whereas all other subchannels have simple Regge pole exchange. In the same way as Fig. 1b is obtained by squaring a $T_{2 \rightarrow n}$ amplitude, the production amplitude in Fig. 1c represents a particular contribution to the unitarity in this subchannel.

All this emphasizes the necessity of finding a high-energy description of the gauge field theory which satisfies, at least asymptotically, s-channel unitarity. The connection between the triple Pomeron vertex and subenergy unitarity in the s-channel makes it clear that it is essential to satisfy s-channel unitarity not only in the total energy but simultaneously also in all subenergy variables. In order to satisfy also t-channel unitarity, this high-energy description should have the form of a reggeon calculus. The most satisfactory solution to this task would, no doubt, be obtained if we would be able to calculate, for any given order perturbation theory, not only the leading but *all* powers of $\ln s$; this, however, at present seems to be beyond any hope. More modestly, we therefore might try to set up a scheme which, while going beyond the leading- $\ln s$ approximation, at least satisfies the unitarity requirements and, in the weak coupling limit, agrees with the leading- $\ln s$ approximation, even if it does not include *all* non-leading powers of $\ln s$. The use of conventional techniques for studying high-energy behaviour in field theories (α -parameter techniques, infinite-momentum variables) is not the best way to have control over s-channel unitarity; it appears to be easier if we make extensive use of dispersion relations and unitarity equations.

The present paper contains the first part of such a program. The requirement that s-channel unitarity should be satisfied in all subchannels makes it necessary to deal simultaneously with all $2 \rightarrow 2$, $2 \rightarrow 3$, $2 \rightarrow 4$, $3 \rightarrow 3$, ..., amplitudes. As a starting point, we therefore construct these amplitudes in the multiregge limit, still staying in the leading- $\ln s$ approximation. Our method will be the following: starting from the tree approximations for $T_{2 \rightarrow 2}$, $T_{2 \rightarrow 3}$, etc., we use (elastic and inelastic) dispersion relations and unitarity equations for constructing, order by order perturbation theory, the high-energy limit of these amplitudes in the leading- $\ln s$ approximation. In connection with dispersion relations for inelastic amplitudes we find it also useful to make use of the analytic structure of multi-regge theory. We shall carry out these calculations for $T_{2 \rightarrow 2}$, $T_{2 \rightarrow 3}$, $T_{2 \rightarrow 4}$, and $T_{3 \rightarrow 3}$ up to eighth-order perturbation theory and then extrapolate to infinite order. We then demonstrate for a few examples that this set of amplitudes $\{T_{n \rightarrow m}$ in the multiregge limit} satisfies unitarity in all subenergy variables; this justifies the extrapolation to infinite-order perturbation theory from low-order calculations.

The results and their implications are simply summarized. In the leading- $\ln s$ approximation, no vacuum exchange (Pomeron) appears. All exchange channels carry the quantum number of the reggeizing vector particle, and the amplitudes $T_{n \rightarrow m}$ have multiregge behaviour with simple Regge pole exchange. These amplitudes satisfy s-channel unitarity, for example:

$$\text{disc}_s T_{2 \rightarrow 2} = \sum_n \int d\Omega_n T_{2 \rightarrow n}(s+i\epsilon) T_{2 \rightarrow n}(s-i\epsilon) \quad (1.1)$$

or

$$\text{Diagram 1} - \text{Diagram 2} = \text{Diagram 3}$$

Diagram 1: A circle with a '+' sign inside. It has two external lines on the left labeled 'a' and 'b', and one external line on the right labeled 'c'. There is a small cross on the line 'b'.

Diagram 2: A circle with a '+' sign inside. It has two external lines on the left labeled 'a' and 'b', and one external line on the right labeled 'c'. There is a small cross on the line 'b'.

Diagram 3: Two circles connected by a double horizontal line. The left circle has a '+' sign and two external lines on the left labeled 'a' and 'b'. The right circle has a '-' sign and one external line on the right labeled 'c'.

$$\text{disc}_{s_{b_c}} T_{2 \rightarrow 3} = \sum_n \int d\Omega_n T_{2 \rightarrow n+1}(s_{b_c} + i\epsilon) T_{2 \rightarrow n}(s_{b_c} - i\epsilon) \quad (1.2)$$

$$\text{Diagram 1} - \text{Diagram 2} = \text{Diagram 3}$$

Diagram 1: A circle with a '+' sign inside and two external lines on the left.

Diagram 2: A circle with a '-' sign inside and two external lines on the left.

Diagram 3: Two circles connected by three horizontal lines. The left circle has a '-' sign and two external lines on the left. The right circle has a '+' sign and two external lines on the right.

(1.3)

$$\text{disc}_s T_{2 \rightarrow 3} = \sum_n \int d\Omega_n T_{2 \rightarrow n}(s - i\epsilon) T_{3 \rightarrow n}(s + i\epsilon).$$

Reggeon unitarity in the t-channel is trivially satisfied: in each t-channel of $T_{n \rightarrow m}$ only a single Regge pole appears. We conclude from this that, in a world which only knows scattering with exchange of the quantum number of the gauge particle, the leading- $\ln s$ approximation yields a completely acceptable high-energy theory. However, as soon as a scattering amplitude with vacuum quantum number exchange is formed, unitarity in the s-channel is lost. In order to restore it, a perturbative scheme can be defined which results in the formulation of a reggeon calculus with infinitely many couplings. This part of our program is in progress and will be presented in a forthcoming paper.

The use of dispersion relations for calculating the high-energy behaviour in the leading- $\ln s$ approximation has first been introduced by Lipatov²⁾ for constructing the $T_{2 \rightarrow 2}$ amplitude up to sixth-order perturbation theory. For these calculations, only the dispersion relation and unitarity equation for the $2 \rightarrow 2$ amplitude have been used. The extension of this method to inelastic amplitudes is the main issue of our paper. It makes use of some recent results of the theory of dispersion relations⁹⁾ and of the analytic structure of inelastic amplitudes in the multiregge limit^{10,11)}. Many elements of the leading- $\ln s$ calculation contained in this paper will be needed in the following part of our program. Some of the results of this paper have been derived before by other authors¹⁻⁶⁾, using different methods.

The organization of this paper is the following. In Section 2 we describe the model and sketch our method of calculation. Section 3 summarizes the tree approximations for $T_{n \rightarrow m}$, most of which are taken from Ref. 2. Section 4 contains the main part of our calculations. Successively, we calculate the amplitudes $T_{2 \rightarrow 2}$, $T_{2 \rightarrow 3}$ and $T_{2 \rightarrow 4}$ up to eighth order, and each order has its own subsection. In the final part (4.6) of Section 4, we give a brief summary of the results.

Section 5 illustrates the unitarity properties of the infinite-order extrapolations. In the final Section 6, we summarize and discuss the implications of our results.

2. THE MODEL AND THE METHOD OF CALCULATION

We choose our model to be as simple as possible. We take SU(2) as the gauge group and restrict ourselves to the bosonic sector. In order to avoid problems with infrared divergences, we invoke the Higgs mechanism which gives the gauge mesons the mass M. This requires a complex isodoublet of Higgs fields. Our basic Lagrangian is:

$$\begin{aligned}\mathcal{L}_0 &= -\frac{1}{2} F_{\mu\nu}^i F^{\mu\nu i} + \frac{1}{2} (\partial_\mu \phi)^\dagger \partial_\mu \phi - \frac{\mu^2}{2} \phi^\dagger \phi - \lambda (\phi^\dagger \phi)^2 \\ F_{\mu\nu}^i &= \partial_\mu A_\nu^i - \partial_\nu A_\mu^i + g \varepsilon^{ikl} A_\mu^k A_\nu^l \\ \partial_\mu \phi &= (\partial_\mu - i \frac{g}{2} \vec{\tau} \cdot \vec{A}_\mu) \phi.\end{aligned}\tag{2.1}$$

As usual we take $\mu^2 < 0$ such that ϕ develops a non-zero vacuum expectation value. The full Lagrangian then consists of three pieces:

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{gauge fixing}} + \mathcal{L}_{\text{Faddeev-Popov}}.\tag{2.2}$$

In the unitary gauge this theory describes the interaction of three massive vector particles (isotriplet) and one massive scalar Higgs boson. The Feynman rules in the unitary gauge are given in Fig. 2. For our calculations we will use this unitary gauge, and Faddeev-Popov ghosts will not appear. We shall comment on this point at the end of this section.

In order to illustrate which method of calculating the high-energy behaviour in perturbation theory is the most convenient one for the case of NAGFT, we consider vector-vector scattering in lowest-order perturbation theory, in order g^2 . The Feynman diagrams that contribute to this order are shown in Fig. 3. The explicit calculation shows that some of them have very bad behaviour (for example, Fig. 3a grows like s^2 , Fig. 3b does not go to zero for large momentum transfer t), which should not be present in the full second-order scattering amplitude. Indeed when the sum is taken over all diagrams of Fig. 3 (except Fig. 3e), none of these

undesirable features persists: the result grows like s for large energies, and the leading coefficient falls off like t^{-1} for large t . This demonstrates the existence of extensive cancellations among diagrams of the same-order perturbation theory. In higher order the number of interfering diagrams grows rapidly, and it becomes increasingly difficult to keep track of all cancellations. For this reason the conventional procedure of computing the high-energy behaviour of individual diagrams and then summing them all up is not very suitable for the case of NAGFT.

The alternative method, which avoids these difficulties, is the use of dispersion relations. This method has first been used by Lipatov²⁾ in order to construct the leading- $\ln s$ behaviour of the $2 \rightarrow 2$ scattering amplitude up to sixth-order perturbation theory. For our program of constructing a unitary high-energy behaviour it is necessary to consider not only the $2 \rightarrow 2$ scattering amplitude but also the $T_{n \rightarrow m}$ inelastic amplitudes in the multiregge limit. We therefore have to extend the use of dispersion relations to inelastic amplitudes. We now briefly illustrate how the use of such techniques allows us to find the leading- $\ln s$ behaviour of $T_{n \rightarrow m}$ amplitudes.

We start with the lowest-order amplitude, vector-vector scattering in the tree approximation (order g^2), which has to be calculated from Feynman diagrams. In the next order, g^3 , only the $2 \rightarrow 3$ production amplitude in the tree approximation exists, which again has to be calculated from Feynman diagrams. To order g^4 , there is a contribution to the $2 \rightarrow 2$ scattering amplitude, and for its calculation we make use of the dispersion relation^{*)}:

$$T_{2 \rightarrow 2}(s, t) = T_{2 \rightarrow 2}(s_0, t) + s T'_{2 \rightarrow 2}(s_0, t) + \frac{s^2}{\pi} \int_{\frac{s}{2}}^{\infty} ds' \frac{\text{disc}_s T_{2 \rightarrow 2}(s', t)}{s'^2 (s - s')} + \text{left hand cut contr.}, \quad (2.3)$$

where

$$\text{disc}_s T_{2 \rightarrow 2}(s, t) = \frac{1}{2i} \left[T_{2 \rightarrow 2}(s + i\varepsilon, t) - T_{2 \rightarrow 2}(s - i\varepsilon, t) \right]. \quad (2.4)$$

*) The way we have written the dispersion relation (2.3), it does not exhibit the particle pole contribution. In lowest-order perturbation theory, the pole diagram Fig. 3a yields a contribution to the subtraction term. In higher order, we only retain the powers of $\ln s$ coming from the dispersion integral, and the pole term never contributes to powers of $\ln s$. In that sense, relation (2.3) is an "asymptotic" dispersion relation.

In a theory with vector exchange the high-energy behaviour in a given order perturbation theory is of the form $s(\ln s)^P$. This requires two subtractions in the dispersion relation (2.3). The energy discontinuity is given by the unitarity relation:

$$\text{Diagram with a circle containing a '+' sign} - \text{Diagram with a circle containing a '-' sign} = \sum \text{Diagram with two circles containing '+' and '-' signs connected by three horizontal lines}. \quad (2.5)$$

To order g^4 only the two-particle intermediate state contributes to the right-hand side of Eq. (2.5), and for the $2 \rightarrow 2$ amplitudes, which must be of the order g^2 , we use the results obtained before. For large s , $\text{disc}_s T$ grows like s and the dispersion integral in (2.3) behaves as $s \ln s$. Therefore, for the leading behaviour the subtraction constants in (2.3) are not needed, and the results from previous steps are sufficient to determine the high-energy behaviour of $T_{2 \rightarrow 2}$ in fourth order. In order to complete the discussion of this order perturbation theory, we have to calculate the amplitudes $T_{2 \rightarrow 4}$, $T_{3 \rightarrow 3}$. This again is the tree approximation and has to be extracted from Feynman diagrams.

In the next order perturbation theory, g^5 , we have -- apart from the tree approximations of $T_{2 \rightarrow 5}$, $T_{3 \rightarrow 4}$ -- the $T_{2 \rightarrow 3}$ production amplitude in the one-loop approximation. We now need the dispersion relation for such an amplitude. It is known that in the double Regge limit¹⁰⁻¹²⁾ (for notation see Fig. 5a):

$$s, s_{ab}, s_{bc} \rightarrow \infty ; \quad s_{ab}/s, s_{bc}/s \rightarrow 0 ; \quad (2.6)$$

$$t_1, t_2, \eta = s_{ab}s_{bc}/s \quad \text{fixed}$$

the $T_{2 \rightarrow 3}$ amplitude splits into two pieces, each of them having the maximal number of simultaneous energy discontinuities allowed by the Steinmann relations:

$$T_{2 \rightarrow 3}(s, s_{ab}, s_{bc}, t_1, t_2, \eta) = T_{ab}(s, s_{ab}, t_1, t_2, \eta) + T_{bc}(s, s_{bc}, t_1, t_2, \eta). \quad (2.7)$$

The first term T_{ab} has discontinuities in both s and s_{ab} , the second one T_{bc} in s and s_{bc} . For each of them a double dispersion relation holds, for example for T_{bc} :

$$T_{bc}(s, s_{bc}) = T_{bc}(s_0, s_{bc0}) + \frac{s_{bc}}{\pi} \int_{s_{bc}}^{\infty} ds'_{bc} \frac{\text{disc}_{s_{bc}} T_{bc}(s_0, s'_{bc})}{s'_{bc} (s'_{bc} - s_{bc})} + \text{cont.}$$

$$\begin{aligned}
 & + s \frac{\partial T_{bc}}{\partial s} \Big|_{s_0, s_{bc0}} + \frac{s s_{bc}}{\pi} \int_{\bar{s}_{bc}}^{\infty} ds'_{bc} \frac{\text{disc}_{s_{bc}} \frac{\partial T_{bc}(s_0, s'_{bc})}{\partial s}}{s'_{bc} (s'_{bc} - s_{bc})} \\
 & + \frac{s^2}{\pi} \int_{\bar{s}}^{\infty} ds' \frac{\text{disc}_s T_{bc}(s', s_{bc0})}{s'^2 (s' - s)} \\
 & + \frac{s^2 s_{bc}}{\pi^2} \int_{\bar{s}}^{\infty} ds' \int_{\bar{s}_{bc}}^{\infty} ds'_{bc} \frac{\text{disc}_s \text{disc}_{s_{bc}} T_{bc}(s', s'_{bc})}{s'^2 (s' - s) s'_{bc} (s'_{bc} - s_{bc})} + \text{e.h. cut contr.}
 \end{aligned} \tag{2.8}$$

Such a dispersion relation arises if we first disperse in s , keeping s_{bc} fixed, then in s_{bc} with fixed s' . In s we need two subtractions, in s_{bc} only one. This, again, results from inspecting Feynman diagrams in low-order perturbation theory. For the energy discontinuities in (2.8) we use unitarity equations such as:

$$\begin{array}{c} \diagup \diagdown \end{array} \begin{array}{c} + \\ + \end{array} - \begin{array}{c} \diagup \diagdown \end{array} \begin{array}{c} - \\ + \end{array} = \sum \begin{array}{c} \diagup \diagdown \end{array} \begin{array}{c} - \\ - \end{array} \begin{array}{c} \diagup \diagdown \end{array} \begin{array}{c} + \\ + \end{array} \tag{2.9}$$

and

$$\begin{array}{c} \diagup \diagdown \end{array} \begin{array}{c} + \\ + \end{array} - \begin{array}{c} \diagup \diagdown \end{array} \begin{array}{c} + \\ - \end{array} = \sum \begin{array}{c} \diagup \diagdown \end{array} \begin{array}{c} + \\ + \end{array} \begin{array}{c} \diagup \diagdown \end{array} \begin{array}{c} - \\ - \end{array} . \tag{2.10}$$

With these equations we find $T_{2 \rightarrow 3}$ in order g^5 .

In the next order g^6 we first compute the $2 \rightarrow 2$ amplitude: on the right-hand side of the unitarity equation (2.5) we have a contribution from the three-particle intermediate state ($T_{2 \rightarrow 3}$ appears to order g^3 , the tree approximation, and is already known), and another one from the two-particle intermediate state. For the latter we need the $2 \rightarrow 2$ amplitude to orders g^2 and g^4 . Both have been calculated in previous steps of our scheme. Finally, we have to consider one-loop corrections to $T_{2 \rightarrow 4}$ ($T_{3 \rightarrow 3}$). Dispersion relations for $T_{2 \rightarrow 4}$ in the multiregge limit are quite analogous to (2.7) and (2.8): first one decomposes $T_{2 \rightarrow 4}$ into five pieces, each of which represents an allowed set of simultaneous energy discontinuities, and for each of these terms a multiple dispersion relation (with an appropriate number of subtractions) holds. For a more detailed discussion of $T_{2 \rightarrow n}$ amplitudes in the multiregge limit we refer to Ref. 11. With these dispersion relations, together with unitarity equations similar to (2.9) and (2.10), one computes the one-loop corrections to $T_{2 \rightarrow 4}$; $T_{3 \rightarrow 3}$ follows from $T_{2 \rightarrow 4}$ via crossing.

It is now clear how this method allows us to calculate, order by order perturbation theory, the leading- $\ln s$ approximation of all amplitudes $T_{n \rightarrow m}$. The ingredients are:

- i) $T_{2 \rightarrow 2}$, $T_{2 \rightarrow 3}$, ..., $T_{n \rightarrow m}$ in the tree approximation, computed in the kinematical limit where all subenergies are as large as possible (multiregge region). These amplitudes have to be calculated directly from Feynman diagrams.
- ii) Dispersion relations which, in the multiregge region, correctly reproduce the analytic structure in all subenergies.
- iii) Unitarity equations for the discontinuities in all subenergy variables.

The second ingredient requires a few comments. Recently⁹⁾, considerable progress has been made, within analytic S-matrix theory, in understanding the structure of multiparticle scattering amplitudes. Most important for our goals is the result that in the asymptotic domain, in particular the multiregge region, the analytic structure greatly simplifies. At finite energies, the presence of Landau singularities leads to a rather complicated singularity structure, but asymptotically these contributions are either down by powers of s or do not contribute to the kinematic limit, and what remains is the normal threshold cut structure. What this implies for $T_{2 \rightarrow 3}$ in the double Regge limit is contained in (2.7) and (2.8), and analogous results hold for general $T_{n \rightarrow m}$: each amplitude can be decomposed into several pieces, each of which has a simple singularity structure and satisfies a multiple dispersion relation [the analogue of (2.7) for $T_{2 \rightarrow 4}$ will be discussed in Section 4.3]. For a more detailed discussion we refer to the lectures of H.P. Stapp and A.R. White, given at the Les Houches Conference 1975⁹⁾. Models which illustrate the analytic structure of multiparticle amplitudes in the multiregge region are dual amplitudes and ϕ^3 ladder models¹⁰⁻¹¹⁾.

To conclude the discussion of our calculational scheme we note the following. When computing tree approximations to on-shell scattering amplitudes, one is allowed to work directly in the unitary gauge, and in the unitarity equations one also deals only with on-shell S-matrix elements. As a result of this, Faddeev-Popov ghosts never appear. This indicates that, as far as the leading- $\ln s$ approximation is concerned, one could work entirely in the unitary gauge (which, in general, of course is not allowed; speaking in terms of the R_ξ gauge, the limit $\xi \rightarrow 0$ has to be taken only at the end of all loop integrations). It is only beyond the leading- $\ln s$ approximation that Faddeev-Popov ghosts come in, for example if one wants to calculate subtraction constants in the dispersion relation (2.3). The fact that ghosts do not enter leading- $\ln s$ calculations has been verified in Refs. 4 and 5.

In the following we shall use the calculational scheme that we have described and compute the amplitudes $T_{n \rightarrow m}$ in the leading- $\ln s$ approximation. To be more precise, we shall carry out the calculations up to order g^6 and then find the general form. A proof that this form is correct to all orders follows from the discussion of its unitarity content which will be given in Section 5.

3. HIGH-ENERGY BEHAVIOUR OF TREE DIAGRAMS

To begin our calculations we need the high-energy behaviour of $T_{n \rightarrow m}$ in the tree approximation. We therefore start by giving a brief description of these amplitudes. Most of them have been calculated by Lipatov²⁾, and for details we refer to his paper.

For vector-vector scattering to order g^2 the Feynman diagrams are shown in Fig. 3. In the limit $s \rightarrow \infty$, t fixed, the sum of these diagrams is:

$$T_{2 \rightarrow 2}^{(2)} = 2s [g H_{VV} \epsilon_{a_1 c b}] \frac{1}{t - M^2} [g H_{VV} \epsilon_{a_2 b_2 c}], \quad (3.1)$$

where H_{VV} denotes the helicity matrix

$$H_{VV} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix} \quad (3.2)$$

(the first two components are the transverse polarizations, the third one is the longitudinal one). The labels $a_1, \dots$, refer to the SU(2) orientation of the particles. For the process vector + vector $\rightarrow$ Higgs scalar + vector the analogue to (3.1) is

$$T_{2 \rightarrow 2}^{(2)} = 2s [-ig H_{VS} \delta_{a_1 c}] \frac{1}{t - M^2} [g H_{VV} \epsilon_{a_2 b_2 c}] \quad (3.3)$$

with

$$H_{VS} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}. \quad (3.4)$$

In order to illustrate these results it is convenient to use two kinds of diagrams: one represents the spin and momentum structure (Fig. 4a for elastic vector-vector scattering), the other one (Fig. 4b) the group structure. In Fig. 4a, the elements are: a propagator $1/(t-M^2)$ for the exchange, and helicity matrices $gH_{\nu\nu}$ [cf. Eq. (3.2)] for the two vertices. In Fig. 4b, each vertex has a tensor ϵ_{abc} . For the process vector + vector $\rightarrow$ scalar + vector, the two corresponding diagrams are shown in Figs. 4c and 4d. The new elements in these diagrams are $-igH_{\nu s}$ [cf. Eq. (3.4)] and the tensor δ_{ab} . In the following we shall refer to these two kinds of diagrams as "transverse momentum diagram" and "group structure diagram", respectively.

The Feynman diagrams for the process two vector particles $\rightarrow$ three vector particles (Fig. 5a) are given in Fig. 5b. The sum of these diagrams has been calculated by Lipatov:

$$\begin{aligned} T_{2 \rightarrow 3}^{(3)} = & 2s \left[g H_{\nu\nu} \epsilon_{a_1 c_1 b_1} \right] \frac{1}{t_1 - M^2} \left[g \epsilon_{c_1 c_2 b_2} \vec{\Gamma}(q_1, -q_2) \right] \\ & \cdot \frac{1}{t_2 - M^2} \left[g H_{\nu\nu} \epsilon_{a_2 b_3 c_2} \right], \end{aligned} \quad (3.5)$$

The three-component vector $\vec{\Gamma}$ stands for the production vertex, and its components refer to the helicity states of the produced vector particle. The components are:

$$\begin{aligned} \Gamma_\sigma e_\sigma^1 &= \frac{-i}{2\sqrt{s}|\vec{q}|} \left\{ |\vec{q}_\perp| \left[S_{bc} \left(1 + 2 \frac{t_1 - M^2}{-q_\perp^2 - M^2} \right) + S_{ab} \left(1 + 2 \frac{t_2 - M^2}{-q_\perp^2 - M^2} \right) \right] + \frac{S_{bc} - S_{ab}}{|\vec{q}_\perp|} (t_2 - t_1) \right\} \\ \Gamma_\sigma e_\sigma^2 &= \frac{i}{|\vec{q}_\perp|} \sqrt{-\lambda(q_{1\perp}^2, q_{2\perp}^2, q_\perp^2)} \\ \Gamma_\sigma e_\sigma^3 &= \frac{-i}{2\sqrt{s}|\vec{q}|} \left\{ S_{ab} \left(1 + 2 \frac{t_2 - M^2}{-q_\perp^2 - M^2} \right) - S_{bc} \left(1 + 2 \frac{t_1 - M^2}{-q_\perp^2 - M^2} \right) \right\}, \end{aligned} \quad (3.6)$$

where $e^i(q_1 - q_2)$ are the polarization vectors (see the appendix for details), and

$$\Gamma_\sigma = -i \left[-(q_1 + q_2)_\sigma + 2p_{1\sigma} \left(\frac{S_{bc}}{s} + \frac{t_1 - M^2}{S_{ab}} \right) - 2p_{2\sigma} \left(\frac{S_{ab}}{s} + \frac{t_2 - M^2}{S_{bc}} \right) \right] \quad (3.7)$$

The two diagrams that illustrate (3.5) are given in Figs. 6a and 6b. The circle in Fig. 6a denotes the vector $\vec{\Gamma}$. If the outgoing particle at the l.h.s. external vertex is a scalar rather than a vector, we replace

$$[g H_{VV} \epsilon_{a_1 c_1 b_1}] \rightarrow [-ig H_{Vs} \delta_{a_1 c_1}]. \quad (3.8)$$

At the production vertex the corresponding substitution is (Figs. 6c, 6d):

$$g \vec{\Gamma} \epsilon_{c_1 c_2 b_2} \rightarrow g M \vec{e}_3 \delta_{c_1 c_2}. \quad (3.9)$$

For the production amplitude $T_{2 \rightarrow 4}$ a consideration similar to that of Ref. 2 leads to the simple result (see Fig. 7):

$$\begin{aligned} T_{2 \rightarrow 4}^{(4)} = & 2s [g H_{VV} \epsilon_{a_1 c_1 b_1}] \frac{1}{t_1 - M^2} [g \epsilon_{c_1 c_2 b_2} \Gamma_c(q_1, q_2) e_\sigma^i] \\ & \cdot \frac{1}{t_2 - M^2} [g \epsilon_{c_2 c_3 b_3} \Gamma_s(q_2, q_3) e_s^j] \frac{1}{t_3 - M^2} [g H_{VV} \epsilon_{c_3 a_2 b_4}], \end{aligned} \quad (3.10)$$

where

$$\Gamma_c(q_1, q_2) = -i \left[-(q_1 + q_2)_c + 2 p_{1c} \left(\frac{s_{bcd}}{s} + \frac{t_1 - M^2}{s_{ab}} \right) - 2 p_{2c} \left(\frac{s_{abc}}{s} + \frac{t_2 - M^2}{s_{bd}} \right) \right] \quad (3.11)$$

and

$$\Gamma_s(q_2, q_3) = -i \left[-(q_2 + q_3)_s + 2 p_{1s} \left(\frac{s_{cd}}{s} + \frac{t_2 - M^2}{s_{abc}} \right) - 2 p_{2s} \left(\frac{s_{abc}}{s} + \frac{t_3 - M^2}{s_{cd}} \right) \right] \quad (3.12)$$

We shall assume that this factorization property holds also for higher-order amplitudes $T_{2 \rightarrow n}$ in the limit where all subenergies are large.

Finally, we need the amplitude $T_{3 \rightarrow 3}$ in the limit where all subenergies are large but small compared with the total energy s . Furthermore, we require (see Fig. 8 for notation):

$$s_{23} \ll s_{56}, \quad s_{45} \ll s_{12}, \quad \text{all } t_i \text{ fixed} \quad (3.13)$$

(note that the limit $s_{23} \sim s_{56}$, $s_{45} \sim s_{12}$ is not needed for our leading- $\ln s$ calculation). In this limit $T_{3 \rightarrow 3}$ is simply obtained from $T_{2 \rightarrow 4}$ in (3.9) by crossing. This completes our review of the tree approximation.

4. THE $T_{n \rightarrow m}$ AMPLITUDES IN THE LEADING- $\ln s$ APPROXIMATION

4.1 Order g^4

We are now ready to calculate, order by order perturbation theory, the $T_{n \rightarrow m}$ amplitudes in the leading- $\ln s$ approximation. The lowest-order amplitude which goes beyond the tree approximation is the $2 \rightarrow 2$ scattering amplitude in fourth order. We first compute the discontinuity:

$$\text{disc}_s T_{2 \rightarrow 2}^{(4)} = \sum \int \frac{d^2 k_\perp}{16\pi^2 s} T_{2 \rightarrow 2}^{(2)}(s+i\epsilon) T_{2 \rightarrow 2}^{(2)}(s-i\epsilon). \quad (4.1)$$

On the r.h.s. of this equation the sum goes over vector and scalar intermediate states. For the amplitudes we use the results (3.1) and (3.3):

$$\begin{aligned} \text{disc}_s T_{2 \rightarrow 2}^{(4)} = & 2\pi s \left[g^2 (H_{VV} \cdot H_{VV} \epsilon_{a_1 c_1 d_1} \epsilon_{d_1 c_2 b_1} - H_{VS} H_{SV} \delta_{a_1 c_1} \delta_{c_2 b_1}) \right] \\ & \cdot \left[g^2 (H_{VV} \cdot H_{VV} \epsilon_{a_2 d_2 c_2} \epsilon_{d_2 b_2 c_2} - H_{VS} H_{SV} \delta_{a_2 c_2} \delta_{b_2 c_2}) \right] \\ & \cdot \int \frac{d^2 k_\perp}{(2\pi)^3} \frac{1}{-k_\perp^2 - M^2} \frac{1}{-(q-k)_\perp^2 - M^2}. \end{aligned} \quad (4.2)$$

Following Lipatov²⁾ we introduce SU(2) projection operators:

$$\begin{aligned} \mathcal{P}_0(a_1 b_1 | a_2 b_2) &= \frac{1}{3} \delta_{a_1 b_1} \delta_{a_2 b_2} \\ \mathcal{P}_1(a_1 b_1 | a_2 b_2) &= \frac{1}{2} \epsilon_{a_1 b_1 c} \epsilon_{a_2 b_2 c} \\ \mathcal{P}_2(a_1 b_1 | a_2 b_2) &= \frac{1}{2} \left(\delta_{a_1 a_2} \delta_{b_1 b_2} + \delta_{a_1 b_2} \delta_{a_2 b_1} - \frac{2}{3} \delta_{a_1 b_1} \delta_{a_2 b_2} \right) \end{aligned} \quad (4.3)$$

with

$$\mathcal{P}_i(a_1 b_1 | a_2 b_2) \mathcal{P}_k(a_2 b_2 | a_3 b_3) = \delta_{ik} \mathcal{P}_k(a_1 b_1 | a_3 b_3) \quad (4.4)$$

and

$$\sum_{i=0,1,2} \mathcal{P}_i(a_1 b_1 | a_2 b_2) = \delta_{a_1 a_2} \delta_{b_1 b_2}. \quad (4.5)$$

With the identities

$$\epsilon_{a,c,d}, \epsilon_{d,c_2,b_1} = -2 P_0(a,b_1|c_1,c_2) - P_1(a,b_1|c_1,c_2) + P_2(a,b_1|c_1,c_2) \quad (4.6)$$

$$\delta_{a,c_1} \delta_{b,c_2} = P_0(a,b_1|c_1,c_2) + P_1(a,b_1|c_1,c_2) + P_2(a,b_1|c_1,c_2) \quad (4.7)$$

we then decompose (4.2) into terms with definite SU(2) quantum number in the t-channel:

$$\begin{aligned} disc_s T_{2 \rightarrow 2}^{(4)} &= 2\pi s \beta(q^2) \cdot \\ &\left\{ P_1(a,b_1|a_2,b_2) [g(-H_{vv} \cdot H_{vv} - H_{vs} \cdot H_{sv})] \cdot [g(-H_{vv} \cdot H_{vv} - H_{vs} \cdot H_{sv})] \right. \\ &+ P_0(a,b_1|a_2,b_2) [g(-2H_{vv} \cdot H_{vv} - H_{vs} \cdot H_{sv})] \cdot [g(-2H_{vv} \cdot H_{vv} - H_{vs} \cdot H_{sv})] \\ &\left. + P_2(a,b_1|a_2,b_2) [g(H_{vv} \cdot H_{vv} - H_{vs} \cdot H_{sv})] \cdot [g(H_{vv} \cdot H_{vv} - H_{vs} \cdot H_{sv})] \right\} \end{aligned} \quad (4.8)$$

with

$$\beta(q^2) = g^2 \int \frac{d^2 k_\perp}{(2\pi)^2} \frac{1}{-k_\perp^2 - M^2} \frac{1}{-(q-k)_\perp^2 - M^2} \quad (4.9)$$

For the leading- $\ln s$ approximation only the term which is odd under $s \leftrightarrow u$ crossing contributes; from (4.3)-(4.5) one sees that P_0 and P_2 are symmetric under $a_1 \leftrightarrow b_1$, whereas P_1 is antisymmetric. We therefore keep in (4.8) only the term proportional to P_1 (isospin-one exchange):

$$disc_s T_{2 \rightarrow 2}^{(4)} \sim 2\pi s P_1(a,b_1|a_2,b_2) [g H_{vv}] \beta(q^2) [g H_{vv}], \quad (4.10)$$

where we have used:

$$-H_{vv} \cdot H_{vv} - H_{vs} \cdot H_{sv} = H_{vv}. \quad (4.11)$$

With this result we evaluate the dispersion relation (2.3). For large energies we use

$$\frac{s^2}{n!} \int_0^\infty ds' \frac{(\ln s')^n}{s'(s'+s)} \sim s \frac{(\ln s)^{n+1}}{(n+1)!} . \quad (4.12)$$

This leads us to the result

$$T_{2 \rightarrow 2}^{(4)} \sim (s \ln(-s) - u \ln(-u)) [g \epsilon_{a_1 c b_1} H_{\nu\nu}] \beta(q^2) [g \epsilon_{a_2 b_2 c} H_{\nu\nu}]. \quad (4.13)$$

If the outgoing particle at the l.h.s. vertex is a scalar, a similar calculation leads to

$$T_{2 \rightarrow 2}^{(4)} \sim (s \ln(-s) - u \ln(-u)) [-ig \delta_{a_1 c} H_{\nu s}] \beta(q^2) [g \epsilon_{a_2 b_2 c} H_{\nu\nu}]. \quad (4.14)$$

Within our diagrammatic notation, we then show the transverse momentum diagram belonging to (4.13) in Fig. 9. For the two external vertices we again have $[gH_{\nu\nu}]$; the loop integration is $g^2 \int d^2 k_\perp / (2\pi)^3$, and the group structure is the same as in Fig. 4b. All this is multiplied by $(s \ln s - u \ln u)$.

As has been pointed out before²⁻⁶⁾, the results (4.13) and (4.14) indicate the reggeization of the vector particle. The expressions (3.1) and (4.13) for the process vector + vector $\rightarrow$ vector + vector are the lowest-order terms of

$$T_{2 \rightarrow 2} = - \frac{e^{-i\pi\alpha(t)} - 1}{t - M^2} s^{\alpha(t)} [g \epsilon_{a_1 c b_1} H_{\nu\nu}] [g \epsilon_{a_2 b_2 c} H_{\nu\nu}] \quad (4.15)$$

with

$$\alpha(t) = 1 + (t - M^2) \beta(t). \quad (4.16)$$

Similarly, for the process vector + vector $\rightarrow$ scalar + vector, we find that (3.3) and (4.14) are correctly reproduced from

$$T_{2 \rightarrow 2} = - \frac{e^{-i\pi\alpha(t)} - 1}{t - M^2} s^{\alpha(t)} [-ig \delta_{a_1 c} H_{\nu s}] [g \epsilon_{a_2 b_2 c} H_{\nu\nu}]. \quad (4.17)$$

Within our following calculations we shall show the correctness of (4.15) and (4.17) up to eighth-order perturbation theory.

4.2 Order g^5

The quantity we are now going to calculate is the $T_{2 \rightarrow 3}$ amplitude in the one-loop approximation. According to our discussion in Section 2, we expect this

amplitude to have two terms, the one having discontinuities in s_{ab} and s but not s_{bc} , the other one in s_{bc} and s but not s_{ab} . In order to construct, say, the T_{bc} term in (2.7) we therefore calculate the s_{bc} discontinuity. T_{ab} then follows from symmetry arguments.

For the s_{bc} discontinuity, we use the unitarity equation (2.10), putting together the $T_{2 \rightarrow 3}$ amplitude (3.5) and the $T_{2 \rightarrow 2}$ amplitude (3.1) (see Fig. 10 for notation). First, we transform the amplitude $T_{2 \rightarrow 2}$ for the process $b(q) + c(p_2') \rightarrow b(q') + c(p_2'')$ from the centre-of-mass system of the (bc) -channel into the overall centre-of-mass system. We show in the appendix that the result of this transformation is (for the moment we ignore the group structure):

$$T_{2 \rightarrow 2}^{(2)} = 2 s_{bc} [g \mathcal{D}_{bc}(q) H_{\nu\nu} \mathcal{D}_{bc}(q')^T] \frac{1}{-k_1^2 - M^2} [g H_{\nu\nu}] \quad (4.18)$$

(note that there is no transformation for the r.h.s. vertex). The matrix $\mathcal{D}_{bc}(q)$ consists of a boost part and a rotation part:

$$\mathcal{D}_{bc}(q) = L_{bc}(q) R(q_\perp, k_\perp) \quad (4.19)$$

with

$$L_{bc}(q) = \begin{pmatrix} A & 0 & -B \\ 0 & 1 & 0 \\ B & 0 & A \end{pmatrix} \quad (4.20)$$

$$A = \frac{\sqrt{s'}}{|\vec{q}| s_{bc}} \left(-M^2 + \frac{s_{bc}}{2s} (s_{ab} + s_{bc}) \right) \quad (4.21)$$

$$B = \frac{|q_\perp| M \sqrt{s'}}{|\vec{q}| s_{bc}} \quad (4.22)$$

$$R(q_\perp, k_\perp) = \begin{pmatrix} \sin \vartheta & -\cos \vartheta & 0 \\ \cos \vartheta & \sin \vartheta & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (4.23)$$

$$\cos \vartheta = \frac{k_{\perp} \cdot q_{\perp}}{|k_{\perp}| |q_{\perp}|} \quad (4.24)$$

$$\sin \vartheta = \frac{\sqrt{-\lambda(q_{\perp}^2, (q-k)_{\perp}^2, k_{\perp}^2)}}{2 |k_{\perp}| |q_{\perp}|} \quad (4.25)$$

θ is the angle between $k_{\perp}$ and $q_{\perp}$, and because of $A^2 + B^2 = 1$ both matrices L and R are unitary. With (4.19) we rewrite (4.18):

$$T_{2 \rightarrow 2}^{(2)} = 2 s_{b_c} [g L_{b_c}(q) R(q_{\perp}, q'_{\perp}) H_{\nu\nu} L_{b_c}^T(q')] \frac{1}{-k_{\perp}^2 - M^2} [g H_{\nu\nu}], \quad (4.26)$$

where

$$R(q_{\perp}, q'_{\perp}) = \begin{pmatrix} \cos(q_{\perp}, q'_{\perp}) & \sin(q_{\perp}, q'_{\perp}) & 0 \\ -\sin(q_{\perp}, q'_{\perp}) & \cos(q_{\perp}, q'_{\perp}) & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (4.27)$$

is a rotation from $\vec{q}_{\perp}$ to $\vec{q}'_{\perp}$.

Now we consider the "product" of $T_{2 \rightarrow 3}$ and $T_{2 \rightarrow 2}$ in Fig. 10. We first concentrate on the spin structure at the production vertex. From the production amplitude we have the three-component vector $\vec{\Gamma}$ [Eq. (3.6)], and from $T_{2 \rightarrow 2}$ the first bracket term of Eq. (4.26). We therefore have to evaluate:

$$\vec{\Gamma}^T L_{b_c}(q) R(q_{\perp}, q'_{\perp}) H_{\nu\nu} L_{b_c}^T(q'). \quad (4.28)$$

For the first part of this expression we obtain

$$L_{b_c}^T(q) \vec{\Gamma} = (-i) \left[\vec{V}(q_{\perp}, q_{\perp}) - M \vec{e}_3 - \frac{t_1 - M^2}{-q_{\perp}^2 - M^2} (\vec{V}(q_{\perp}, q_{\perp}) - 2 M \vec{e}_3) \right] \quad (4.29)$$

with

$$\vec{V}(q_{\perp}, q_{\perp}) = \begin{pmatrix} 2 |q_{\perp}| \cos(q_{\perp}, q_{\perp}) \\ 2 |q_{\perp}| \sin(q_{\perp}, q_{\perp}) \\ 0 \end{pmatrix} \quad (4.30)$$

$$\vec{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \quad (4.31)$$

The expression (4.28) now allows a simple interpretation: $L_{bc}^T \vec{\Gamma}$ is the production vertex in the (bc) centre-of-mass system, the matrix $R(q_\perp, q'_\perp) H_{vv}$ stands for vector-vector scattering in this system, and $L_{bc}(q')$ takes us back to the over-all centre-of-mass system. An important feature of this result is that in the (bc) system $L_{bc}^T \vec{\Gamma}$ has no kinematical singularities.

Before we go on with our calculations we translate this into our diagrammatic language. In the over-all centre-of-mass system we denote the production vertex $\vec{\Gamma}$ by a circle (Figs. 6a and 11a). When transformed into the centre-of-mass system of the (bc)-subchannel, $\vec{\Gamma}$ is changed to (4.29), and its transverse momentum structure is illustrated in Fig. 11b: in the first diagram, the production vertex is $\vec{V}(q_{1\perp}, q_\perp) - M\vec{e}_3$, in the second one $[\vec{V}(q_\perp, q_\perp) - 2M\vec{e}_3]$. In the second diagram of Fig. 11b, there is also a propagator for the produced particle.

Returning to the evaluation of the phase-space integral of Fig. 10, we complete the calculation of (4.28). We use

$$R^T(q_\perp, q'_\perp) \vec{V}(q_{1\perp}, q_\perp) = \vec{V}(q_{1\perp}, q'_\perp) \quad (4.32)$$

$$R^T(q_\perp, q'_\perp) \vec{V}(q_\perp, q_\perp) = \vec{V}(q_\perp, q'_\perp) \quad (4.33)$$

$$R^T(q_\perp, q'_\perp) \vec{e}_3 = \vec{e}_3 \quad (4.34)$$

and find for (4.28):

$$(4.28) = i \left[\vec{V}(q_{1\perp}, q'_\perp) - \frac{M}{2} \vec{e}_3 - \frac{t_1 - M^2}{-q_\perp^2 - M^2} (\vec{V}(q_\perp, q'_\perp) - M \vec{e}_3) \right]^T L_{bc}^T(q'). \quad (4.35)$$

Together with the other parts of the amplitudes in Fig. 10 (except for the group structure) we obtain for the s_{bc} discontinuity

$$\text{disc}_{s_{bc}} T_{bc}^{(5)} = 2\pi s [g H_{vv}] \frac{1}{t_1 - M^2} [ig (\vec{V}(q_{1\perp}, q'_\perp) - \frac{M}{2} \vec{e}_3)^T L_{bc}^T(q')].$$

cont.

$$\begin{aligned}
 & \cdot \beta(t_2) [g H_{VV} \cdot H_{VV}] \\
 & - 2\pi s [g H_{VV}] g^2 \int \frac{d^2 k_\perp}{(2\pi)^3} \frac{1}{-(q_2 - k_\perp)^2 - M^2} \frac{1}{-q_\perp^2 - M^2} \frac{1}{-k_\perp^2 - M^2} \quad (4.36) \\
 & \cdot [ig(\vec{V}(q_\perp, q'_\perp) - M\vec{e}_3)^T L_{bc}^T(q')] \cdot [g H_{VV} \cdot H_{VV}].
 \end{aligned}$$

In order to find the group structure we have to multiply the two graphs of Figs. 4b and 6b. Again it is convenient to decompose into isospin 0, 1, and 2 exchange in the t_2 -channel. For the leading- $\ln s$ approximation the antisymmetric combination ($I = 1$) contributes, and we obtain the group structure diagram of Fig. 6b with a weight factor $-\frac{1}{2}$. Finally, we add the contribution of scalar intermediate states. Altogether, the result is:

$$\begin{aligned}
 disc_{s_{bc}} T_{bc}^{(5)} = & -\pi s \left\{ [g H_{VV}] \frac{1}{t_1 - M^2} [-ig(\vec{V}(q_{1\perp}, q'_\perp) - M\vec{e}_3)^T L_{bc}^T(q')] \right. \\
 & \cdot \beta(t_2) [g H_{VV}] \\
 & - [g H_{VV}] g^2 \int \frac{d^2 k_\perp}{(2\pi)^3} \frac{1}{-(q_2 - k_\perp)^2 - M^2} \frac{1}{-q_\perp^2 - M^2} \frac{1}{-k_\perp^2 - M^2} \cdot [-ig \cdot \\
 & \left. (\vec{V}(q_\perp, q'_\perp) - M\vec{e}_3)^T L_{bc}^T(q')] [g H_{VV}] \right\} \cdot \text{group structure of Fig. 6b.} \quad (4.37)
 \end{aligned}$$

The transverse momentum structure of these two terms is shown in Fig. 12a. Because of the L_{bc} transformation in (4.37), it represents the structure in the (bc) centre-of-mass system. In order to find the structure in the over-all centre-of-mass system, we use the transformation law which is illustrated in Figs. 11a and 11b. From this it follows that (4.37) is identical to (Fig. 12b):

$$\begin{aligned}
 disc_{s_{bc}} T_{bc}^{(5)} = & -\pi s \left\{ [g H_{VV}] \frac{1}{t_1 - M^2} [g \vec{F}(q_1, -q_2)] \beta(t_2) [g H_{VV}] \right. \\
 & - [g H_{VV}] g^2 \int \frac{d^2 k_\perp}{(2\pi)^3} \frac{1}{-k_\perp^2 - M^2} \frac{1}{-(q_1 - k_\perp)^2 - M^2} \frac{1}{-(q_2 - k_\perp)^2 - M^2} \\
 & \left. \cdot [g \vec{F}(q_1 - k_\perp, -q_2 + k_\perp)] [g H_{VV}] \right\} \cdot \text{group structure of Fig. 6b.} \quad (4.38)
 \end{aligned}$$

This is our final result for the s_{bc} discontinuity. The s_{ab} discontinuity has, by symmetry arguments, the structure shown in Fig. 12c.

We now use these results to construct the $T_{2 \rightarrow 3}$ amplitude. The analogue to the way we found the high-energy behaviour of $T_{2 \rightarrow 2}$ in Section 4.1 would be the following. With our result for the s_{bc} discontinuity we calculate in (2.8) the right-hand cut contribution of the s_{bc} dispersion integral. Then we compute the s discontinuity and insert this into the s dispersion integral of (2.8) (note that in this order of perturbation theory there is no double discontinuity). Finally, we add the corresponding pieces for the left-hand energy cuts, and this completes the calculation of $T_{2 \rightarrow 3}$. It is, however, much simpler if we anticipate the result and start with a multiregge ansatz for $T_{2 \rightarrow 3}$ ^{10,14)}:

$$T_{2 \rightarrow 3} = \frac{(-s)^{\alpha_1}}{T_1} \frac{(-s_{bc})^{\alpha_2 - \alpha_1}}{B_{21}} \overline{F}_L + \frac{(-s)^{\alpha_2}}{T_2} \frac{(-s_{ab})^{\alpha_1 - \alpha_2}}{B_{12}} \overline{F}_R \quad (4.39)$$

with

$$\begin{aligned} \alpha_i &= \alpha(t_i) \quad (\text{eq. (4.16)}) \\ T_i &= t_i - M^2 \\ B_{ij} &= T_i \beta(t_i) - T_j \beta(t_j) \end{aligned} \quad (4.40)$$

Equation (4.39) is an unsigned 2+3 amplitude with double Regge pole exchange. It has only right-hand cuts in the energy variables and satisfies the dispersion relation (2.8). F_L and F_R are expected to factorize into residue functions for the three vertices [the subscripts "L" and "R" stand for "left" and "right", respectively: the term in (4.39) with F_L has a particle pole only in the left exchange channel (see Fig. 5) but not in the right-hand-side t -channel, whereas for the term with F_R the opposite is true]. In order to obtain the full signed amplitude we have to add the four amplitudes shown in Fig. 13a-d: each of them has an expression analogous to (4.39), with appropriate changes of the energy variables. After analytic continuation into the double Regge limit of Fig. 13a, the four terms add up to

$$\begin{aligned} T_{2 \rightarrow 3} &= s^{\alpha_1} s_{bc}^{\alpha_2 - \alpha_1} \frac{\xi_{\alpha_1}}{T_1} \frac{\xi_{\alpha_2 - \alpha_1}}{B_{21}} \overline{F}_L \\ &+ s^{\alpha_2} s_{ab}^{\alpha_1 - \alpha_2} \frac{\xi_{\alpha_2}}{T_2} \frac{\xi_{\alpha_1 - \alpha_2}}{B_{12}} \overline{F}_R \end{aligned} \quad (4.41)$$

with

$$\begin{aligned} \sum_{\alpha_1} e^{-i\pi\alpha_1} &= T_1 \\ \sum_{\alpha_1, \alpha_2} e^{-i\pi(\alpha_1 - \alpha_2)} &= T_1 T_2. \end{aligned} \quad (4.42)$$

From (4.42) it becomes clear that in the weak coupling limit where $\alpha_i \rightarrow 1$ only $\tau_1 = \tau_2 = -1$ contributes. This is why we have to keep only the antisymmetric combinations of group factors in both the t_1 - and t_2 -channels ($I = 1$ exchange).

In order to determine F_L and F_R we calculate the s_{bc} discontinuity of (4.41) to lowest order (disregarding the group structure):

$$\text{disc}_{s_{bc}} T_{2 \rightarrow 3}^{(5)} = 2\pi \frac{s}{T_1} F_L \quad (4.43)$$

and compare this with (4.38). This leads to:

$$\begin{aligned} F_L = - [g H_{\nu\nu}] \frac{1}{2} \left\{ g \vec{\Gamma}(q_1, -q_2) \beta_2 - T_1 K(q_1, -q_2) \right\} [g H_{\nu\nu}] \\ \cdot \text{group structure of Fig. 6b,} \end{aligned} \quad (4.44)$$

where

$$\begin{aligned} K(q_1, -q_2) = g^2 \int \frac{d^2 k_1}{(2\pi)^3} \frac{1}{-k_1^2 - M^2} \frac{1}{-(q_1 - k_1)^2 - M^2} \frac{1}{-(q_2 - k_1)^2 - M^2} \\ \cdot [g \vec{\Gamma}(q_1 - k_1, -q_2 + k_1)]. \end{aligned} \quad (4.45)$$

By symmetry it follows that F_R has the form:

$$\begin{aligned} F_R = - [g H_{\nu\nu}] \frac{1}{2} \left\{ \beta_1 g \vec{\Gamma}(q_1, -q_2) - T_2 K(q_1, -q_2) \right\} [g H_{\nu\nu}] \\ \cdot \text{group structure of Fig. 6b.} \end{aligned} \quad (4.46)$$

F_R and F_L have the expected factorizing form, and the expression in the curly brackets represents the production vertex.

It follows from (4.41) that the s discontinuity is not independent of the s_{ab} and s_{bc} discontinuity. It is therefore a check of the correctness of the ansatz (4.41) to compare the s discontinuity of (4.41) with the result we obtain from the unitarity equation (2.9). Direct calculation of (2.9) leads to:

$$\begin{aligned} \text{disc}_s T_{2 \rightarrow 3}^{(5)} = -\pi s [g H_{\nu\nu}] K(q_1, -q_2) [g H_{\nu\nu}] \\ \cdot \text{group structure of Fig. 6b.} \end{aligned} \quad (4.47)$$

The same result follows from (4.39) if we use (4.42) and (4.44) and combine the two terms. By power expansion in g we can also verify that the terms in the dispersion relation (2.8) are correctly reproduced. We conclude from this that (4.41) represents our amplitude $T_{2 \rightarrow 3}$ up to order g^5 . In the following we shall show that (4.41) also holds in higher-order perturbation theory.

We conclude our discussion of the order g^5 with a comment on the leading- $\ln s$ approximation. Equation (4.41) is that representation of $T_{2 \rightarrow 3}$ which exhibits the full analytic structure. Our method has enabled us to calculate F_L and F_R in lowest-order perturbation theory. It is, however, well known that (4.41) can be cast into a factorizing form which will be more convenient for future calculations. By combining the two terms in (4.41) one finds, to leading order g^2 :

$$T_{2 \rightarrow 3} = 2s [g H_{VV}] \frac{s_{ab}^{\alpha_1-1}}{T_1} \left\{ g \vec{\Gamma}(q_1, -q_2) - \ln \eta T_1 T_2 K(q_1, -q_2) \right\} \cdot \frac{s_{bc}^{\alpha_2-1}}{T_2} [g H_{VV}] \cdot \text{group structure of Fig. 6b} \quad (4.48)$$

with $\eta = (s_{ab} s_{bc})/s$. Since η is finite and $K(q_1, -q_2)$ is of the order g^3 , in the weak coupling limit only the first part of the production vertex contributes, and we have

$$T_{2 \rightarrow 3} = 2s [g H_{VV}] \frac{s_{ab}^{\alpha_1-1}}{T_1} [g \vec{\Gamma}(q_1, -q_2)] \frac{s_{bc}^{\alpha_2-1}}{T_2} [g H_{VV}] \cdot \text{group str.} \quad (4.49)$$

4.3 Order g^6

In this order we have to calculate two amplitudes: the $2 \rightarrow 2$ amplitude in the two-loop approximation, and the $2 \rightarrow 4$ amplitude on the one-loop level. For the s discontinuity of the $2 \rightarrow 2$ amplitude we illustrate in Fig. 14 which terms contribute. With the results (3.1), (3.3) and (4.13), (4.14) for the $2 \rightarrow 2$ amplitude to order g^2 and g^4 , respectively, we find for the contribution from the two-particle intermediate state (again summing over vector and scalar particles):

$$-2\pi s \ln s [g H_{VV}] g^4 \int \frac{d^2 k_{1\perp} d^2 k_{2\perp}}{(2\pi)^6} \frac{1}{-k_1^2 - M^2} \frac{1}{-k_2^2 - M^2} \frac{1}{-(q - k_1 - k_2)_\perp^2 - M^2} \quad (4.50)$$

$\cdot [g H_{VV}] \cdot \text{group structure of Fig. 4b.}$

The transverse momentum structure of this contribution is shown in Fig. 15a.

In order to find the contribution of the three-particle intermediate state we use (Fig. 16)

$$\vec{\Gamma}(q-k_1-q+k_2)^T \vec{\Gamma}(k_1, k_2) = -2 \left[t - \frac{3}{2} M^2 - \frac{(k_1^2 - M^2)((q-k_2)^2 - M^2) + (k_2^2 - M^2)((q-k_1)^2 - M^2)}{-(k_1-k_2)_+^2 - M^2} \right]. \quad (4.51)$$

The phase-space integral is taken from:

$$n \text{ particle phase space integral} \sim \frac{\pi}{s} \frac{(\ell u s)^{n-2}}{(n-2)!} \int \prod_{i=1}^{n-1} \frac{d^2 k_{i+}}{2(2\pi)^3} \quad (4.52)$$

Summing over vector and scalar production at each vertex, and taking the isospin combination which is antisymmetric under $s \leftrightarrow u$ crossing ($I = 1$ exchange), one finds for the energy discontinuity (Fig. 15b):

$$\begin{aligned} & -\pi s \ell u s \left\{ [g H_{VV}] \beta(t) (t - M^2) \beta(t) [g H_{VV}] \right. \\ & - 2 [g H_{VV}] g^4 \int \frac{d^2 k_1 d^2 k_2}{(2\pi)^6} \frac{1}{-k_1^2 - M^2} \frac{1}{-k_2^2 - M^2} \frac{1}{-(q-k_1-k_2)_+^2 - M^2} \\ & \left. \cdot [g H_{VV}] \right\} \cdot \text{group structure of Fig. 4b} \end{aligned} \quad (4.53)$$

In the sum of (4.50) and (4.53) the t-channel three-particle intermediate state cancels and the result is

$$\text{disc}_s T_{2 \rightarrow 2}^{(6)} = -\pi s \ell u s [g H_{VV}] \frac{1}{t - M^2} [(t - M^2) \beta(t)]^2 [g H_{VV}]. \quad (4.54)$$

Inserting this result into the dispersion relation (2.3) and using the approximation (4.12) we obtain:

$$T_{2 \rightarrow 2}^{(6)} = \frac{s \ell u (-s)^2 - u \ell u (-u)^2}{2! (t - M^2)} [g H_{VV}] [(t - M^2) \beta(t)]^2 [g H_{VV}]. \quad (4.55)$$

This is the sixth-order term in the power series expansion of Eq. (4.15).

For the one-loop correction to the $2 \rightarrow 4$ amplitude we proceed in the same way as we did for $T_{2 \rightarrow 3}$: rather than writing down the (rather lengthy) dispersion

relation and computing, term by term, the amplitude, we make a multiregge ansatz and show that it correctly reproduces all possible energy discontinuities. Let us start by computing unitarity integrals for the discontinuities in s_{ab} , s_{abc} , s_{bc} , s_{bcd} , and s_{cd} . They are illustrated in Fig. 17. In sixth-order perturbation theory, only the tree approximations for $T_{2 \rightarrow 2}$ and $T_{2 \rightarrow 4}$ are needed. All ingredients for the calculation of these unitarity integrals have been given before. We therefore present the results only in terms of diagrams. The transverse momentum diagrams for the s_{ab} , s_{abc} , s_{bc} , s_{bcd} , and s_{cd} discontinuities are given in Fig. 18a, b, c, d, e, respectively. For example, Fig. 18a stands for

$$- \pi s \left\{ [g H_{VV}] \beta(t_1) [g \vec{\Gamma}(q_1 - q_2)] \frac{1}{t_2 - M^2} [g \vec{\Gamma}(q_2 - q_3)] \frac{1}{t_3 - M^2} [g H_{VV}] + [g H_{VV}] K(q_1 - q_2) [g \vec{\Gamma}(q_2 - q_3)] \frac{1}{t_3 - M^2} [g H_{VV}] \right\} \quad (4.56)$$

The group structure of all expressions is that of Fig. 7c.

Our (unsigned) multiregge ansatz for $T_{2 \rightarrow 4}$ is¹⁴⁾

$$\begin{aligned} T_{2 \rightarrow 4} = & \frac{(-s)^{\alpha_1}}{T_1} \frac{(-s_{bcd})^{\alpha_2 - \alpha_1}}{B_{21}} \frac{(-s_{cd})^{\alpha_3 - \alpha_2}}{B_{32}} F_{LL} \\ & + \frac{(-s)^{\alpha_2}}{T_2} \frac{(-s_{ab})^{\alpha_1 - \alpha_2}}{B_{12}} \frac{(-s_{cd})^{\alpha_3 - \alpha_2}}{B_{32}} F_{RL} \\ & + \frac{(-s)^{\alpha_3}}{T_3} \frac{(-s_{abc})^{\alpha_2 - \alpha_3}}{B_{23}} \frac{(-s_{ab})^{\alpha_1 - \alpha_2}}{B_{12}} F_{RR} \\ & + \frac{(-s)^{\alpha_3}}{T_3} \frac{(-s_{abc})^{\alpha_1 - \alpha_3}}{B_{13}} \frac{(-s_{bc})^{\alpha_2 - \alpha_1}}{B_{21}} F_{LR}^{(1)} \\ & + \frac{(-s)^{\alpha_1}}{T_1} \frac{(-s_{bcd})^{\alpha_3 - \alpha_1}}{B_{31}} \frac{(-s_{bc})^{\alpha_2 - \alpha_3}}{B_{23}} F_{LR}^{(2)} \end{aligned} \quad (4.57)$$

The discontinuity structure of these terms is illustrated in Fig. 19. In order to find the expressions for F_{LL} , etc., we compute, to lowest order, the discontinuities in s_{ab} , s_{bc} , s_{cd} , s_{abc} , s_{bcd} from (4.57) and compare them with the results of Fig. 18*). This gives us five equations and allows to determine all five partial wave functions in (4.57). The results are (Fig. 20):

*) In order to compare our unitarity integrals [Fig. 18, Eq. (4.56)] with the discontinuities taken from (4.57), Eq. (4.56) has to be multiplied by $1/4$. When deriving (4.56) we used the tree approximation (3.10) for $T_{2 \rightarrow 4}$, which represents the full signed amplitude. Equation (4.57), on the other hand, is only part of the $2 \rightarrow 4$ amplitude with the right-hand singularities in all energy variables.

$$\begin{aligned} \mathcal{F}_{LL} = \frac{1}{4} [g H_{VV}] & [g \vec{\Gamma}(q_1 - q_2) \beta_2 - T_1 K(q_1 - q_2)] \\ & \cdot [g \vec{\Gamma}(q_2 - q_3) \beta_3 - T_2 K(q_2 - q_3)] [g H_{VV}] \end{aligned} \quad (4.58)$$

$$\begin{aligned} \mathcal{F}_{RL} = \frac{1}{4} [g H_{VV}] & [g \beta_1 \vec{\Gamma}(q_1 - q_2) - K(q_1 - q_2) T_2] \\ & \cdot [g \vec{\Gamma}(q_2 - q_3) \beta_3 - T_2 K(q_2 - q_3)] [g H_{VV}] \end{aligned} \quad (4.59)$$

$$\begin{aligned} \mathcal{F}_{RR} = \frac{1}{4} [g H_{VV}] & [g \beta_1 \vec{\Gamma}(q_1 - q_2) - K(q_1 - q_2) T_2] \\ & \cdot [g \beta_2 \vec{\Gamma}(q_2 - q_3) - K(q_2 - q_3) T_3] [g H_{VV}] \end{aligned} \quad (4.60)$$

$$\begin{aligned} \mathcal{F}_{LR}^{(1)} = \frac{1}{4} [g H_{VV}] & \left\{ [g \vec{\Gamma}(q_1 - q_2)] [g \vec{\Gamma}(q_2 - q_3)] \beta_1 \beta_2 + K(q_1 - q_2) \cdot \right. \\ & \cdot [K(q_2 - q_3) T_2 T_3 - K(q_1 - q_2) T_1 \beta_1 [g \vec{\Gamma}(q_2 - q_3)] \\ & \left. - [g \vec{\Gamma}(q_1 - q_2)] K(q_2 - q_3) T_3 \beta_1 - K(q_1 - q_2 - q_3) T_3 \beta_2] \right\} [g H_{VV}] \end{aligned} \quad (4.61)$$

$$\begin{aligned} \mathcal{F}_{LR}^{(2)} = \frac{1}{4} [g H_{VV}] & \left\{ [g \vec{\Gamma}(q_1 - q_2)] [g \vec{\Gamma}(q_2 - q_3)] \beta_2 \beta_3 + K(q_1 - q_2) \cdot \right. \\ & \cdot [K(q_2 - q_3) T_1 T_2 - K(q_1 - q_2) T_1 \beta_3 [g \vec{\Gamma}(q_2 - q_3)] \\ & \left. - [g \vec{\Gamma}(q_1 - q_2)] K(q_2 - q_3) T_3 \beta_3 - K(q_1 - q_2 - q_3) T_1 \beta_{23}] \right\} [g H_{VV}]. \end{aligned} \quad (4.62)$$

Here we have defined:

$$K(q_1, -q_2, -q_3) = g^2 \int \frac{d^2 k_\perp}{(2\pi)^3} \frac{1}{-k_\perp^2 - M^2} \frac{1}{-(q_1 - k)_\perp^2 - M^2} [g \vec{\Gamma}(q_1 - k, -q_2 + k)] \cdot \frac{1}{-(q_2 - k)_\perp^2 - M^2} [g \vec{\Gamma}(q_2 - k, -q_3 + k)] \frac{1}{-(q_3 - k)_\perp^2 - M^2} . \quad (4.63)$$

The group structure of (4.58)-(4.62) is always that of Fig. 7c. The partial waves F_{LL} , F_{RL} , and F_{RR} have the expected factorization properties. In $F_{LR}^{(1)}$ and $F_{LR}^{(2)}$, however, the last piece which is proportional to $K(q_1, -q_2, -q_3)$ violates factorization: as we shall see later, this term belongs to a two-reggeon cut in the t_2 -channel.

As a test of our ansatz (4.57), we calculate from (4.57) with (4.58)-(4.62) the discontinuity in s . After many cancellations we find:

$$- \pi s \frac{i}{4} [g H_{VV}] K(q_1, -q_2, -q_3) [g H_{VV}] . \quad (4.64)$$

The same result is obtained if we compute the s discontinuity from the unitarity equation (see footnote on p. 23).

With the ansatz (4.57) we now follow the same line of arguments as we used for $T_{2 \rightarrow 3}$ in the previous subsection. We simultaneously consider all $T_{2 \rightarrow 4}$ and $T_{3 \rightarrow 3}$ amplitudes which are connected via crossing. These eight amplitudes are shown in Fig. 21. Each of them has only right-hand energy cuts. For the first four $T_{2 \rightarrow 4}$ amplitudes the form (4.57) with appropriately defined energy variables holds. The remaining four $T_{3 \rightarrow 3}$ amplitudes in Fig. 21 are somewhat different, according to their energy discontinuity structure (Fig. 22) which is different from Fig. 19. However, it is possible to find an expression analogous to (4.57), and the partial wave functions can be computed in the same way. We are not going to demonstrate this in more detail, but only mention the following. In the previous subsection we showed for $T_{2 \rightarrow 3}$ that in the weak coupling limit (and finite η) a simple factorizing form (4.49) emerges. Similarly, one finds that when combining all five terms in (4.57) all pieces of the partial waves which contain loop integrals cancel. As a result, (4.49) simply becomes:

$$T_{2 \rightarrow 4} = \frac{s}{4} [g H_{VV}] \frac{(-s_{ab})^{\alpha_1 - 1}}{T_1} [g \vec{\Gamma}(q_1, -q_2)] \frac{(-s_{bc})^{\alpha_2 - 1}}{T_2} \cdot [g \vec{\Gamma}(q_2, -q_3)] \frac{(-s_{cd})^{\alpha_3 - 1}}{T_3} [g H_{VV}] \cdot \text{group structure of Fig. 7c.} \quad (4.65)$$

An analogous simple expression can be derived for $T_{3 \rightarrow 3}$, and it is this form of the amplitudes in Fig. 21 that we use in order to find the full signatred amplitude. The result is:

$$T_{2 \rightarrow 4} = 2s [g H_{VV}] \frac{s_{ab}^{\alpha_1-1}}{T_1} [g \vec{F}(q_1, q_2)] \frac{s_{bc}^{\alpha_2-1}}{T_2} \cdot [g \vec{F}(q_2, q_3)] \frac{s_{cd}^{\alpha_3-1}}{T_3} [g H_{VV}] \cdot \text{group structure Fig. 7c.} \quad (4.66)$$

4.4 Order g^7

The calculation of the one-loop contribution to the $2 \rightarrow 5$ amplitude is very similar to that of the $2 \rightarrow 4$ amplitude. We therefore only mention that the result, when written in the factorized form, generalizes (4.66). The other amplitude that we have to calculate in this order of perturbation theory is the two-loop contribution to the $2 \rightarrow 3$ amplitude. We shall demonstrate the validity of (4.41) [with (4.44) and (4.46) for F_L and F_R] by calculating the s_{bc} discontinuity.

The three contributions to the unitarity equation are shown in Figs. 23a-c. Let us start with the two-particle intermediate state of Fig. 23a. The $2 \rightarrow 3$ amplitude of order g^5 is taken from (4.49):

$$T_{2 \rightarrow 3}^{(s)} = \left\{ 2s [g H_{VV}] \beta_1 \epsilon s_{ab} [g \vec{F}(q_1, q_2)] \frac{1}{T_2} [g H_{VV}] + 2s [g H_{VV}] \frac{1}{T_1} [g \vec{F}(q_1, q_2)] \beta_2 \epsilon s_{bc} [g H_{VV}] \right\} \quad (4.67)$$

· group structure of Fig. 6b.

The transverse momentum structure, together with the $\ln s$ factors, is illustrated in Fig. 24a. Transforming the production vertex into the centre-of-mass system of the (bc)-channel leads us to Fig. 24b. The multiplication with the $2 \rightarrow 2$ amplitude in the (bc)-channel follows our discussion in Section 4.2. The only new element is the phase-space integral for the term proportional to $\ln s_{ab}$ in (4.67). One finds:

$$(2\pi)^2 \int \frac{d^4 k}{(2\pi)^4} \delta^+(q_1 - k)^2 - M^2 \delta^+(p_2 + k)^2 - M^2 \epsilon (p_1 - k)^2 - M^2 \\ \sim \frac{\pi}{2s} \epsilon(s/s_{bc}) \cdot \int \frac{d^2 k_\perp}{(2\pi)^3} \quad (4.68)$$

The result is then shown in Fig. 25a. After transformation to the over-all centre-of-mass system one arrives at Fig. 25b. This diagram contains a new element: the circle of the four-particle vertex denotes a factor $(t-M^2)$ (Fig. 25c). The group structure of Figs. 25a and b is the same as in Fig. 6b.

For the other two-particle intermediate state contribution in Fig. 23b we simply repeat the operations of Section 4.2. We only replace (4.18) by:

$$T_{2 \rightarrow 2}^{(4)} = 2 s_{bc} \ell u s_{bc} [g \mathcal{D}_{bc}(q) H_{VV} \mathcal{D}_{bc}(q')^T] \beta(k^2) [g H_{VV}]. \quad (4.69)$$

The result is then shown in Fig. 26 (in the over-all centre-of-mass system). The group structure is again that of Fig. 6b.

For the contribution of the three-particle intermediate state in Fig. 23c we first transform the $2 \rightarrow 4$ amplitude to the $(b'd'c')$ centre-of-mass system, and for the production vertex of the line d we use the multiplication law (4.51) (Fig. 16). The result is shown in Fig. 27a and, after transformation back to the over-all centre-of-mass system, in Fig. 27b.

We finally sum all contributions. After several cancellations we find (Fig. 28):

$$\begin{aligned} \text{disc}_{s_{bc}} T_{2 \rightarrow 3}^{(7)} = & -\pi s \left(\beta_1 T_1 \ell u^5 s_{bc} + \beta_2 T_2 \ell u s_{bc} \right) \\ & \cdot \left\{ [g H_{VV}] \frac{1}{T_1} [g \bar{F}(q_1 - q_2)] \beta_2 [g H_{VV}] \right. \\ & \left. - [g H_{VV}] K(q_1 - q_2) [g H_{VV}] \right\} \cdot \text{group structure Fig. 6b.} \end{aligned} \quad (4.70)$$

The same result is obtained if we take the s_{bc} discontinuity of (4.41) and compute the term proportional to g^7 . This then proves that (4.41) is correct up to this order. We still mention that as a further test the s discontinuity of (4.41) in seventh-order perturbation theory agrees with what we obtain from a direct evaluation of the unitarity integral.

4.5 Order g^8

Let us first calculate the $2 \rightarrow 2$ amplitude. The contribution from the four-particle intermediate state is shown in Fig. 29. Again we have used the multiplication law of the production vertex [Eq. (4.51) and Fig. 16], and the group structure is the same as in Fig. 4b. As to the three-particle intermediate state, we use the fifth-order result of $T_{2 \rightarrow 3}$ [Eq. (4.67) and Fig. 24a] and the following phase-space integral:

$$\begin{aligned}
 & (2\pi)^3 \int \frac{d^4 k_1 d^4 k_2}{(2\pi)^8} \delta^+(p_1 - k_1^2 - M^2) \delta^+(k_1 - k_2^2 - M^2) \\
 & \quad \cdot \delta^+(p_2 + k_2^2 - M^2) \mathcal{L}u((p_1 - k_2)^2) \\
 & \sim \frac{\pi}{5} \frac{\mathcal{L}u^2}{2} \int \frac{d^2 k_{1\perp} d^2 k_{2\perp}}{2(2\pi)^3 2(2\pi)^3} .
 \end{aligned} \tag{4.71}$$

The result for this contribution is illustrated in Fig. 30. Finally, for the two-particle intermediate state we need $T_{2 \rightarrow 2}$ in second, fourth, and sixth order [Eqs. (3.1), (4.13), and (4.55)]. This leads to the contribution shown in Fig. 31. The group structure is always that of Fig. 4b. When summing up the contributions of Figs. 29, 30, and 31, the necessary cancellations occur in order to make the result simple:

$$\begin{aligned}
 T_{2 \rightarrow 2}^{(\beta)} &= \frac{s \mathcal{L}u^3(-s) - u \mathcal{L}u^3(-u)}{3!} [g H_{\mu\nu}] \frac{[(t - M^2)\beta(t)]^3}{t - M^2} [g H_{\mu\nu}] \\
 &\quad \cdot \text{group structure Fig. 4b.}
 \end{aligned} \tag{4.72}$$

This agrees with (4.15).

Finally, we demonstrate the validity of (4.66) in eighth order. The procedure is the same as used in Section 4.3 for $T_{2 \rightarrow 4}$ in sixth order: from the unitarity equations we calculate the discontinuities in s_{ab} , s_{abc} , and s_{bc} (those for s_{cd} and s_{bcd} then follow by symmetry arguments). The results are shown in Fig. 32; the group structure is always that of Fig. 7c. The s_{ab} discontinuity is easy to compare with Eq. (4.57), F_{LL} , etc., being given in (4.58)-(4.60). The other discontinuities of (4.57) contain the partial waves $F_{LR}^{(1)}$ and $F_{LR}^{(2)}$. Already the sixth-order results [Eqs. (4.61) and (4.62) and Fig. 20] contain a non-factorizing piece $K(q_1, -q_2, -q_3)$, indicating that these partial waves have a more complicated structure than the other ones. As we shall see in the following section, $K(q_1, -q_2, -q_3)$ is the lowest-order term of a two-reggeon cut contribution in the t_2 -channel. Now, in eighth order, we therefore expect the next higher-order term of this cut contribution. In order to verify this, in (4.57) we replace the last two terms by a Sommerfeld-Watson integral for the t_2 -channel:

$$\begin{aligned}
 & \frac{1}{2\pi i} \int d\omega_2 \left[\frac{(-s)^{\alpha_3}}{T_3} \frac{(-s_{abc})^{\alpha_1 - \alpha_3}}{B_{13}} \frac{(-s_{bc})^{\omega_2 - (\alpha_1 - 1)}}{\omega_2 - \beta_1 T_1} F_{LR}^{(1)} \right. \\
 & \quad \left. + \frac{(-s)^{\alpha_1}}{T_1} \frac{(-s_{bcd})^{\alpha_3 - \alpha_1}}{B_{31}} \frac{(-s_{bc})^{\omega_2 - (\alpha_3 - 1)}}{\omega_2 - \beta_3 T_3} F_{LR}^{(2)} \right] .
 \end{aligned} \tag{4.73}$$

The partial waves $F_{LR}^{(1)}$ and $F_{LR}^{(2)}$ contain singularities in $\omega_2 = j_2 - 1$. We now take the s_{bc} , s_{abc} , and s_{bcd} discontinuities of (4.57) where the last two terms are replaced by the integral (4.73), and compare them with the results shown in Fig. 32. One finds that $F_{LR}^{(1)}$ and $F_{LR}^{(2)}$ are the seventh-order expansion terms of

$$\begin{aligned} \mathcal{F}_{LR}^{(1)} = & \frac{1}{\omega_2 - \beta_2 T_2} [g H_{VV}] \left\{ [g \vec{T}(q_1 - q_2)] [g \vec{T}(q_2 - q_3)] \beta_1 \beta_2 \right. \\ & + K(q_1 - q_2) K(q_2 - q_3) T_2 T_3 - K(q_1 - q_2) [g \vec{T}(q_2 - q_3)] T_1 \beta_1 \\ & \left. - [g \vec{T}(q_1 - q_2)] K(q_2 - q_3) T_3 \beta_1 \right\} [g H_{VV}] \\ & + T_3 (\omega_2 - \beta_2 T_3) [g H_{VV}] \mathcal{F}_{LR}^{cut} [g H_{VV}]. \end{aligned} \quad (4.74)$$

A graphical illustration of this expression is given in Fig. 33a. The last term, $\mathcal{F}_{LR}^{cut}$, which we have not written out in (4.74), will be discussed in more detail in the following section. For $F_{LR}^{(2)}$ one finds a result similar to (4.74), and we show it in Fig. 33b. If we compare this with (4.61) and (4.62), we find agreement except for the piece $\mathcal{F}_{LR}^{cut}$: $K(q_1, -q_2, -q_3)$ appears as the lowest-order expansion term of $\mathcal{F}_{LR}^{cut}$.

We have thus established that (4.57) holds except for the extra piece $\mathcal{F}_{LR}^{cut}$ in $F_{LR}^{(1)}$ and $F_{LR}^{(2)}$ which has to be written as a Sommerfeld-Watson integral. But, as we have discussed in Section 4.3, (4.57) can be written in the somewhat simpler form of (4.65), and the piece $K(q_1, -q_2, -q_3)$ which in fifth order represents $\mathcal{F}_{LR}^{cut}$ drops out. The same cancellation also works in seventh order, and (4.65) remains correct. The transition to the signed amplitude (4.66) works in the same way as in Section 4.3.

4.6 Summary of Section 4

We conclude this rather long section with a brief summary. Using the method we have outlined in Section 2, we have constructed, order by order perturbation theory up to eighth order, the amplitudes $T_{2 \rightarrow 2}$, $T_{2 \rightarrow 3}$, $T_{2 \rightarrow 4}$ in the leading- $\ln s$ approximation. In this approximation, all exchange channels carry isospin 1. Based on the one-loop corrections to these amplitudes, we guessed Regge pole exchange behaviour, and confirmed this in the two-loop approximation (for $T_{2 \rightarrow 2}$ also in the three-loop approximation). For $T_{2 \rightarrow 2}$, the result is contained in (4.15). For $T_{2 \rightarrow 3}$, the use of dispersion relations requires a representation which exhibits the full analytic structure in the energy variables. This is the form (4.41), and we have calculated the partial waves F_L and F_R in lowest-order perturbation

theory. It is, however, convenient to rewrite this into another form (4.49), which has the nice factorization properties of multiregge amplitudes. For $T_{2 \rightarrow 4}$, the use of dispersion relations again leads to the representation (4.57) which exhibits the analytic structure. However, rather than discussing the full signatured partial waves of this representation, we only presented the factorizing form (4.66) which will be needed for future calculations. The $T_{3 \rightarrow 3}$ amplitude, in its factorized form, follows from (4.66) by crossing.

Based on these results for $T_{2 \rightarrow 2}$, $T_{2 \rightarrow 3}$, $T_{2 \rightarrow 4}$, $T_{3 \rightarrow 3}$ we make the assumption that an analogous Regge behaviour also holds for $T_{2 \rightarrow n}$ and $T_{n \rightarrow m}$. We then summarize this section with the following multiregge formula for $T_{2 \rightarrow n}$ (Fig. 34)

$$T_{2 \rightarrow n} = 2s [g H_{VV}] \frac{s_{12}^{\alpha_1-1}}{T_1} [g \vec{F}(q_1, q_2)] \frac{s_{23}^{\alpha_2-1}}{T_2} \dots$$

$$\cdot [g \vec{F}(q_{n-2}, q_{n-1})] \frac{s_{n-1,n}^{\alpha_{n-1}-1}}{T_{n-1}} [g H_{VV}] \cdot \quad (4.75)$$

• group structure Fig. 35.

The expression for $T_{n \rightarrow m}$ follows from this via crossing.

5. UNITARITY CONTENT OF THE LEADING- $\ln s$ APPROXIMATION

The way we have constructed the high-energy behaviour of the $T_{n \rightarrow m}$ amplitudes in the leading- $\ln s$ approximation implies that s-channel unitarity is satisfied not only in the over-all s variable but also in the subenergy variables, at least up to that order of perturbation theory we have been considering. In this section we now demonstrate that all s-channel unitarity equations are satisfied if we consider the set of (infinite-order perturbation theory) amplitudes $T_{n \rightarrow m}$ (4.75). This will serve two purposes. Firstly, it shows the consistency of our extrapolation from finite-order perturbation theory results to infinite order. The highly non-linear character of the unitarity equations puts a strong constraint on the $T_{n \rightarrow m}$ amplitudes, and the fact that the $T_{n \rightarrow m}$ of (4.75) obey these constraints is almost a proof that (4.75), which originally was guessed from low-order results, is true to all orders. Secondly, an illustration of the unitarity content of the leading- $\ln s$ approximation is necessary in order to understand why the leading- $\ln s$ approximation for the vacuum exchange channel (Pomeron) fails. Based on this observation we shall, in a forthcoming paper, discuss how a satisfactory description of the Pomeron might be obtained.

The basic ingredient for the demonstration of unitarity of the set of $T_{n \rightarrow m}$ amplitudes is the bootstrap equation for the reggeization of the vector particle, first found by Lipatov²⁾ for the $2 \rightarrow 2$ amplitude. We therefore start with the unitarity equation for $T_{2 \rightarrow 2}$:

$$\text{disc}_s T_{2 \rightarrow 2} = \sum_{n=2}^{\infty} \int d\Omega_n T_{2 \rightarrow n} T_{2 \rightarrow n}^* . \quad (5.1)$$

For $T_{2 \rightarrow n}$ we use (4.75), but we write it as a multiple Sommerfeld-Watson integral. Again it is useful to introduce a diagrammatic notation (see Fig. 34 for $T_{2 \rightarrow n}$): for the wavy line with momentum k there is a reggeon propagator

$$\frac{1}{-k_{\perp}^2 - M^2} \cdot \frac{1}{\omega + 1 - \alpha(k)} \quad (5.2)$$

$[\omega = j - 1, \text{ and } \alpha(k) \text{ is given in (4.16)}]$, for the reggeon particle-particle vertex

$$[g H_{VV}] \quad \text{or} \quad [-ig H_{VV}] , \quad (5.3)$$

and for the production vertex

$$[g \vec{F}(q_1, -q_2)] \quad \text{or} \quad [g M] . \quad (5.4)$$

For the group structure we always use a separate diagram (for $T_{2 \rightarrow n}$ see Fig. 35). For the evaluation of the r.h.s. of (5.1) (after switching to the partial wave language) it is convenient to define a two-particle $\rightarrow$ two-reggeon amplitude $C(k, q-k, \omega)$ (Fig. 36a) such that the r.h.s. of (5.1) is

$$- \pi [H_{VV} g] \int \frac{d^2 k_{\perp}}{(2\pi)^3} C(k, q-k, \omega) \frac{1}{-k_{\perp}^2 - M^2} \frac{1}{-(q-k)_{\perp}^2 - M^2} \frac{1}{\omega + 2 - \alpha(k) - \alpha(q-k)} \cdot [g H_{VV}] \cdot \text{group structure Fig. 4b} \quad (5.5)$$

with the group structure of Fig. 4b. An integral equation is easily found by squaring Fig. 35 and using the square of the production vertex (4.51). When closing the group structure diagram on itself (Fig. 35), we project out the anti-symmetric part, the $I = 1$ exchange [cf. (4.6) and (4.7)]. All this leads to:

$$C(k, q-k, \omega) = g^2 + g^2 \int \frac{d^2 k'_{\perp}}{(2\pi)^3} C(k', q-k', \omega) \frac{1}{-k'_{\perp}^2 - M^2} \frac{1}{-(q-k')_{\perp}^2 - M^2} \cdot \frac{1}{\omega + 2 - \alpha(k') - \alpha(q-k')} \cdot \left\{ (q^2 - M^2) - \frac{(-k'_{\perp}^2 - M^2)(-(q-k')_{\perp}^2 - M^2) + (-k_{\perp}^2 - M^2)(-(q-k)_{\perp}^2 - M^2)}{-(k-k')_{\perp}^2 - M^2} \right\} \quad (5.6)$$

The expression in the curly brackets defines a quartic reggeon vertex which is illustrated in Fig. 36b. The solution to (5.6) is:

$$C(k, q-k, \omega) = g^2 \frac{\omega + 2 - \alpha(k) - \alpha(q-k)}{\omega + 1 - \alpha(q)} . \quad (5.7)$$

By inserting this solution into (5.5) and computing the s discontinuity from

$$T_{2 \rightarrow 2} = \frac{-1}{2\pi i} \int dj \frac{e^{-i\pi j} - 1}{t - M^2} s^j [g H_{\nu\nu} \epsilon_{a,cb_1}] \frac{1}{j - \alpha(t)} \cdot [g H_{\nu\nu} \epsilon_{a_2 b_2 c}] \quad (5.8)$$

one finds that (5.1) is satisfied.

Equation (5.1) says that the square of reggeizing $2 \rightarrow n$ production amplitudes reproduces the Regge pole in the $2 \rightarrow 2$ amplitude, i.e. the Regge pole bootstraps. In view of the difficulties which have plagued earlier attempts¹⁵⁾ at constructing such a bootstrap model it might be useful to briefly show how the present model takes care of the cancellation à la Amati, Fubini, Stanghellini (AFS) and yet avoids the two-reggeon cut. We illustrate this by considering that part of the r.h.s. of Eq. (5.1) which is the analogue of the Mandelstam diagram. The transverse momentum structure is shown in Fig. 37a. There are two types of contributions to these diagrams. One, obviously, comes from squaring the $2 \rightarrow 2$ amplitude (Fig. 37b), whereas the other one corresponds to a three-particle intermediate state (Fig. 37c). For the latter it is essential that the full structure of the production vertex is taken into account: starting from the right-hand reggeon diagram in Fig. 37c, we take the particle pole exchange in the upper rapidity gap. The production vertex, as shown in Fig. 11b, has a contribution which represents the production of the vector particle from the upper line (Fig. 37d), and it is this configuration which leads to transverse momentum structure in Fig. 37c. One can now show that, quite similar to the situation in ϕ^3 theory models¹⁶⁾, the planar part of Fig. 37d cancels against Fig. 37b. Hence, only the non-planar structure of Fig. 37d survives, which is of the Mandelstam type. Thus the AFS cancellation works in the same way as in ϕ^3 theory¹⁶⁾. Nevertheless, the two-reggeon cut of Fig. 37a does not contribute to the $I = 1$ exchange in the t -channel: this simply comes from signature conservation. The reggeon (bubble string in Fig. 37a) has odd signature, and the $I = 1$ combination of Fig. 37a just projects out the odd signature piece of this diagram. Thus the two-reggeon cut only appears in the even ($I = 0$ or $I = 2$) signature combinations but not in the $I = 1$ exchange. The same argument applies to the reggeon particle exchange of Fig. 37e. It is only the two-particle exchange of Fig. 37f to which the AFS cancellation does not apply: hence this transverse momentum diagram survives in the $I = 1$ t -channel configuration and, when iterated, leads to the Regge pole amplitude on the l.h.s. of Eq. (5.1).

Next we check unitarity of $T_{2 \rightarrow 3}$, say, in the (bc)-channel. The result for the s_{bc} discontinuity, obtained from the unitarity equation, is shown in Fig. 38a: at the production vertex of particle b we repeated exactly the procedure of Section 4.3, and for the lower half of the diagram we use the simple form of C (Fig. 36). The result (Fig. 38a) reads:

$$\text{disc}_{s_{bc}} T_{2 \rightarrow 3} = 2\pi s \iint \frac{d\omega_1 d\omega_2}{(2\pi i)^2} \frac{s_{ab}^{\omega_1}}{T_1} s_{bc}^{\omega_2} \frac{1}{\omega_1 + 1 - \alpha(t_1)} F_L \frac{1}{\omega_2 + 1 - \alpha(t_2)} \quad (5.9)$$

and F_L is the same as in (4.44). For comparison, we take the s_{bc} discontinuity of (4.41) and again find (5.9). This proves that (4.41) satisfies unitarity in the (bc)-channel:

$$\text{disc}_{s_{bc}} T_{2 \rightarrow 3} = \sum_{n=2}^{\infty} \int d\Omega_n T_{2 \rightarrow n+1} T_{2 \rightarrow n}^* \quad (5.10)$$

With the same methods, one can show that unitarity also holds in the s_{ab} - and the s-channel.

As a final example, we demonstrate unitarity for $T_{2 \rightarrow 4}$. For simplicity we will restrict ourselves to that part of $T_{2 \rightarrow 4}$ which has only right-hand energy cuts, and we shall use the representation (4.57), rewritten as a multiple Sommerfeld-Watson integral. Unitarity in the (ab) and (cd) subsystems is verified in the same manner as for $T_{2 \rightarrow 3}$. We first write down the unitarity integral:

$$\sum_{n=2}^{\infty} \int d\Omega_n T_{2 \rightarrow n+2} T_{2 \rightarrow n}^* \quad , \quad (5.11)$$

and then use the identity for $C(k, q-k, \omega)$, depicted in Fig. 36a. The result agrees with what we obtain directly from (4.57), and F_{LL} , F_{RL} , and F_{RR} from (4.58)-(4.60) are correctly reproduced.

Discontinuities across energy variables which contain the central rapidity gap of $T_{2 \rightarrow 4}$, the (bc) subsystem, require a few more explanations. If we calculate the unitarity integrals for the s_{bc} and s_{abc} discontinuities, we have the following situation. Whenever the upper or lower end of a "reggeon ladder" is a "local" vertex, i.e. a point or circle in our diagrammatic language, we can use the identity of Fig. 36a, and the reggeon ladder reduces to a simple Regge pole. For both the s_{bc} and s_{abc} discontinuities, however, we find a piece where both the upper and the lower end consist of the "non-local" piece of the production vertex for the

reggeon ladder in the central rapidity gap, and the identity of Fig. 36a cannot be applied. We are thus left with the results shown in Fig. 39a and b, and the last terms are those for which the two-reggeon cut in the central rapidity gap is not spurious. The reason for this can be understood from the following signature argument. For the odd signature particle-reggeon amplitude C , the decoupling of the two-reggeon cut is simply due to signature conservation (as we have discussed before). Such an argument, however, does not apply to a reggeon + particle $\rightarrow$ reggeon + reggeon amplitude. In addition to the signature associated with the angular momentum in the t_2 -channel, there is an internal signature of the reggeon + particle $\rightarrow$ reggeon + reggeon vertex, and the signature conservation law is more complicated. As a result of this, negative signature in the t_2 -channel, which is relevant for the leading- $\ln s$ approximation, does not exclude a two-reggeon cut in this exchange channel.

All this leads to the result that the partial waves $F_{LR}^{(1)}$ and $F_{LR}^{(2)}$ must have the form given in Eq. (4.74) and Figs. 33a and b. [What we had found in sixth- and eighth-order perturbation theory were the first terms of the power expansion in g^2 of (4.74). In particular, the term $K(q_1, -q_2, -q_3)$ in (4.61) and (4.62) belongs to the two-reggeon cut piece F_{LR}^{cut} .] But as we mentioned before, (4.57) can be rewritten in the factorized form (4.65), and the F_{LR}^{cut} term in $F_{LR}^{(1)}$ and $F_{LR}^{(2)}$ drops out. This then completes the proof that our result for $T_{2 \rightarrow 4}$ [either (4.57) with (4.58)-(4.60) and (4.74), or (4.65)] is unitary in the subenergy variables $s_{ab}, s_{bc}, s_{cd}, s_{abc}, s_{bcd}$. Unitarity in the total energy s can be checked by the same methods.

In order to complete our demonstration of unitarity of $T_{2 \rightarrow 4}$ in all energy variables, we have to consider discontinuities in variables such as $s_{ac} = (p_a + p_c)^2$. This has to be done by studying unitarity of the $3 \rightarrow 3$ amplitude (first diagram of Fig. 22) and analytically continuing. One starts with an ansatz similar to (4.57) which contains the five pieces illustrated in Fig. 22. The partial wave coefficients are then determined by comparison with the calculation of the five unitarity integrals, and finally one shows that the factorized form analogous to (4.65) holds.

This completes our demonstration of unitarity of the amplitudes $T_{2 \rightarrow 2}, T_{2 \rightarrow 3}, T_{3 \rightarrow 3}$ and $T_{2 \rightarrow 4}$. Based on the fact that these unitarity equations put a strong constraint not only on $T_{2 \rightarrow 2}, T_{2 \rightarrow 3}, T_{2 \rightarrow 4}$ and $T_{3 \rightarrow 3}$, but on $T_{n \rightarrow m}$ for arbitrary n and m , we feel confident in believing that complete unitarity holds for all $T_{n \rightarrow m}$. The fact that the leading- $\ln s$ approximation leads to such a unitary set of amplitudes $T_{n \rightarrow m}$ gives rise to the hope that spontaneously broken non-Abelian gauge theories are closer to real hadron physics than other theories such as ϕ^3 theory or quantum electrodynamics (QED).

6. SUMMARY AND CONCLUSIONS

Let us now briefly summarize the main results of this paper. We have outlined a calculational scheme which allows calculation of the high-energy behaviour in the leading- $\ln s$ approximation of $2 \rightarrow 2$, $2 \rightarrow 3$, $2 \rightarrow 4$, etc., amplitudes. It only uses tree approximations, which have to be extracted from the underlying field theory, and dispersion relations together with unitarity equations. In addition to the dispersion relations, the analytic structure of multiregge amplitudes is extensively used. The calculations start with lowest-non-trivial-order perturbation theory, and then go step by step to higher and higher order.

As to explicit calculations, we have carried through this program up to eighth order, which includes one- and two-loop corrections to $T_{2 \rightarrow 2}$, $T_{2 \rightarrow 3}$ and $T_{2 \rightarrow 4}$. The results of these low-order calculations allowed us to generalize to arbitrary order perturbation theory and to general $T_{n \rightarrow m}$ as well. As a check of this extrapolation we have shown, for a few examples, that the set of amplitudes $T_{n \rightarrow m}$ satisfies unitarity in all subchannels. The resulting amplitudes have non-zero quantum number exchange in *all* t -channels and show Regge behaviour with only pole exchange.

The implications of this result, with respect to our aim to find a satisfactory high-energy description of the field theory, are the following. If we restricted ourselves to a world which knows only scattering processes with quantum number exchange, then the leading- $\ln s$ approximation would be a perfectly acceptable high-energy theory with complete (asymptotic) s -channel unitarity. However, as soon as vacuum quantum number exchange amplitudes are constructed, the leading- $\ln s$ approximation becomes insufficient and new methods have to be developed in order to find a unitary description.

Let us first briefly indicate how vacuum exchange amplitudes are found. When constructing in a given order perturbation theory, the leading- $\ln s$ approximation of, say, the $2 \rightarrow 2$ amplitude, we evaluated the unitarity equation (2.5), inserting on both sides the leading- $\ln s$ approximation for the $2 \rightarrow n$ production amplitudes which had been computed at some earlier stage. The group structure of the theory then allows to form combinations which are either even or odd under $s \leftrightarrow u$ crossing. In order to obtain the maximal power of $\ln s$ one takes the odd combination which corresponds to non-zero quantum number exchange. Vacuum quantum number exchange belongs to the symmetric combination of group tensors and, hence, is down by one power of $\ln s$. Since we know the leading- $\ln s$ approximation for $T_{2 \rightarrow n}$ to all orders (Fig. 34), it follows that the vacuum exchange amplitude in this approximation is given by the diagrams of Fig. 1b, where the group tensors of both production amplitudes (Fig. 34) are combined to zero quantum number in the t -channel. This result is the leading- $\ln s$ approximation for the vacuum exchange channel and leads to a fixed j -cut above one^{3,6)}, which violates the Froissart bound.

The reason for this is the lack of s-channel unitarity. For a $2 \rightarrow 2$ scattering process we now have two types of amplitudes -- non-zero quantum number exchange in Fig. 1a and vacuum exchange in Fig. 1b. When we take the unitarity cut of any of these amplitudes, on both sides we find only amplitudes with non-zero quantum number exchange and no vacuum quantum number exchange. This clearly violates unitarity.

Any attempt to restore s-channel unitarity must necessarily go beyond the leading- $\ln s$ approximation. A simple argument⁸⁾ shows that it cannot be sufficient to calculate the next-to-leading- $\ln s$ terms, but full unitarity has to go infinitely far beyond the leading- $\ln s$ approximation. In the absence of being able to calculate in any given order perturbation theory *all* non-leading- $\ln s$ coefficients (such a calculation, of course, would guarantee full unitarity), one can try to find, at least, all those non-leading- $\ln s$ terms which are required for unitarity. The appropriate method seems to be, again, the use of dispersion relations.

A possible way of arriving at a unitary high-energy description might be the following perturbative scheme. The first step is defined by the "reggeon tree amplitudes" $T_{n \rightarrow m}$ which we have calculated in this paper. They have non-zero quantum numbers in all t-channels, and when an energy discontinuity is taken, again one only finds amplitudes with non-zero quantum exchange on both sides of the cut. At the same time, these amplitudes have only one reggeon in any t-channel. Next one uses dispersion relations and unitarity and forms $n \rightarrow m$ amplitudes with two reggeons in the t-channels. These are the amplitudes with vacuum quantum number exchanges (and also $I = 2$ exchanges). If any energy discontinuity is taken, there is still only non-zero quantum number exchange on both sides of the discontinuity cut. Figure 1b is the $T_{2 \rightarrow 2}$ member of this set of amplitudes. The following step puts together amplitudes from the first and from the second step: this leads to amplitudes with, at most, three reggeons in the t-channels. The unitarity content is such that on one side of the unitarity integral one finds vacuum exchange (or $I = 2$), on the other side $I = 1$ exchange. Carrying on this scheme to higher and higher order (the expansion parameter is the number of reggeons in the t-channel), s-channel unitarity becomes better and better. At the same time, a reggeon calculus emerges with arbitrarily high-order reggeon interaction vertices. This implies that t-channel unitarity is satisfied, in terms of reggeon unitarity.

Work along these lines is under progress and will be published in a forthcoming paper. The third, and possibly most difficult, part of this program then remains the solution of the resulting reggeon calculus. As we have explained elsewhere^{8,17)}, the Pomeron in this reggeon calculus appears as a bound state of (at least) two reggeons, and it is a non-trivial part of our program to extract a Pomeron field theory from this reggeon calculus. First results on such a Pomeron field theory are published elsewhere¹⁸⁾.

Acknowledgements

Very helpful discussions with A.R. White and I.G. Halliday are gratefully acknowledged.

Note added in proof

After this paper was completed we learnt about two papers by H. Cheng, J. Dickinson, C.Y. Lo, K. Olanssen and P.S. Yeung, in which a unitarization program similar to ours is outlined. The basic difference between their approach and ours lies in the treatment of s-channel unitarity: they incorporate unitarity only in the total energy variable s , whereas an essential element of our scheme is to satisfy unitarity in *all* subenergy variables.

APPENDIX

In this appendix we calculate the transformation which takes us from the overall centre-of-mass system of the 2→3 amplitude in Fig. 5a to the centre-of-mass system of the (bc)-subchannel. Under a homogeneous Lorentz transformation Λ an amplitude with spinning external particles transforms as:

$$A_{\lambda_1 \dots \lambda_4}(p_1, \dots, p_4) \rightarrow D_{\lambda_1 \lambda'_1}^{(1)} \dots D_{\lambda_4 \lambda'_4}^{(4)} A_{\lambda'_1 \dots \lambda'_4}(\Lambda p_1, \dots, \Lambda p_4). \quad (\text{A.1})$$

The matrix $D^{(i)}$ depends on spin and momentum of particle i , as well as on Λ . These are the matrices that we have to compute. Rather than writing down the D 's in their general form and then taking the appropriate high-energy limit, we simply compute one particular Feynman diagram in both reference frames and derive the transformation matrices. Throughout these calculations we disregard $SU(2)$ group tensors.

The Feynman diagram we chose is shown in Fig. 3b. Let us first compute it in its centre-of-mass system. The external momenta are $p_1 + p_2 \rightarrow p'_1 + p'_2$, and their components $(p_0, p_Z, \vec{p}_\perp)$ are:

$$\begin{aligned} p_1 &= \left(\frac{\sqrt{s}}{2}, \frac{\sqrt{s-4M^2}}{2}, 0, 0 \right) \\ p_2 &= \left(\frac{\sqrt{s}}{2}, -\frac{\sqrt{s-4M^2}}{2}, 0, 0 \right) \\ p'_1 &= \left(\frac{\sqrt{s}}{2}, \frac{\sqrt{s}}{2} - \frac{M^2-t}{\sqrt{s}}, -\vec{k}_\perp \right) \\ p'_2 &= \left(\frac{\sqrt{s}}{2}, -\frac{\sqrt{s}}{2} + \frac{M^2-t}{\sqrt{s}}, \vec{k}_\perp \right) \\ k &= p_1 - p'_1 = \left(0, \frac{-t}{\sqrt{s-4M^2}}, \vec{k}_\perp \right) \end{aligned} \quad (\text{A.2})$$

(k with $t = k^2$ is the momentum transfer). For the polarization vectors we chose "1" and "2" to be the transverse, "3" the longitudinal polarization. They satisfy:

$$\begin{aligned} e_\mu^i(p) e^{k\mu}(p) &= -\delta_{ik} \\ \sum_i e_\mu^i(p) e_\nu^i(p) &= -\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{M^2} \right) \\ e_\mu^i(p) p^\mu &= 0 \end{aligned} \quad (\text{A.3})$$

For the momenta p_1 and p_1' these polarization vectors are:

$$\begin{aligned} e^1(p_1) &= (0, 0, 1, 0) & e^1(p_1') &= (0, \frac{2k_x}{\sqrt{s}}, 1 - \frac{2k_x^2}{s}, 0) \\ e^2(p_1) &= (0, 0, 0, 1) & e^2(p_1') &= (0, \frac{2k_y}{\sqrt{s}}, 0, 1 - \frac{2k_y^2}{s}) \\ e^3(p_1) &= \frac{1}{M} \left(\frac{\sqrt{s-4M^2}}{2}, \frac{\sqrt{s}}{2}, 0, 0 \right) & e^3(p_1') &= \frac{1}{M} \left(\frac{\sqrt{s-4M^2}}{2}, \frac{\sqrt{s}}{2} + \frac{t}{\sqrt{s}}, -\vec{k}_\perp \right). \end{aligned} \quad (A.4)$$

With the Feynman rules of Fig. 2 and the approximation

$$\frac{g_{\mu\nu} - \frac{k_\mu k_\nu}{M^2}}{k^2 - M^2} \sim \frac{2 p_{3\mu} p_{4\nu}}{s} \frac{1}{k^2 - M^2} \quad (A.5)$$

we find for the high-energy behaviour of Fig. 3b:

$$2s \cdot \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 + \frac{t}{2M^2} \end{pmatrix} \cdot \frac{g^2}{t - M^2} \cdot \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 + \frac{t}{2M^2} \end{pmatrix}. \quad (A.6)$$

The matrices in (A.6) refer to the polarizations of incoming and outgoing particles. [Note the difference between (A.6) and (3.1): they differ in the matrix element for the "3" polarization. The change from (A.6) to (3.1) is the result of adding the other diagrams of Fig. 3.] Replacing s by s_{bc} and t by k^2 in (A.6), for the process $q + p_2' \rightarrow q' + p_2''$ in Fig. 10 in the (bc) centre-of-mass system, we find:

$$2s_{bc} g^2 \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 + \frac{k^2}{2M^2} \end{pmatrix} \cdot \frac{1}{k^2 - M^2} \cdot \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 + \frac{k^2}{2M^2} \end{pmatrix}. \quad (A.7)$$

Next we compute the same amplitude in the over-all centre-of-mass system of Fig. 10. With the Feynman rules of Fig. 2 and polarization vectors for p_2 and p_2' , chosen similar to (A.4), we first find that the right-hand-side vertex of the amplitude remains unchanged:

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 + \frac{k^2}{2M^2} \end{pmatrix}. \quad (A.8)$$

This implies that the D matrices in (A.1), which refer to the r.h.s. incoming and outgoing vector particles, are unity. For the left-hand-side vertex the momenta q , q' and their polarization vectors are:

$$\begin{aligned}
 q &= \frac{S_{ab} + S_{bc}}{2s} (p_1 + p_2) + \frac{S_{bc} - S_{ab}}{2s} (p_1 - p_2) + \vec{q}_\perp \\
 &= \left(\frac{S_{ab} + S_{bc}}{2\sqrt{s}}, \frac{S_{bc} - S_{ab}}{2\sqrt{s}}, \vec{q}_\perp \right) \quad (\eta = \frac{S_{ab} S_{bc}}{s} = M^2 + q_\perp^2) \quad (A.9) \\
 q' &= \left(\frac{S'_{ab} + S_{bc}}{2\sqrt{s}}, \frac{S_{bc} - S'_{ab}}{2\sqrt{s}}, \vec{q}'_\perp \right) \quad (\eta' = \frac{S'_{ab} S_{bc}}{s} = M^2 + q'^2_\perp)
 \end{aligned}$$

and

$$\begin{aligned}
 e^1(q) &= \frac{1}{|\vec{q}|} (0, -|\vec{q}_\perp|, \frac{q_z q_x}{|\vec{q}_\perp|}, \frac{q_z q_y}{|\vec{q}_\perp|}) \\
 e^2(q) &= \frac{1}{|\vec{q}_\perp|} (0, 0, -q_y, q_x) \\
 e^3(q) &= \frac{1}{M} (|\vec{q}|, \frac{(S_{ab} + S_{bc})(S_{bc} - S_{ab})}{4s|\vec{q}|}, \frac{S_{ab} + S_{bc}}{2\sqrt{s}|\vec{q}|} \vec{q}_\perp)
 \end{aligned} \quad (A.10)$$

After some algebra we find the following matrix elements for the l.h.s. vertex:

$$\begin{aligned}
 (1,1) \quad & \frac{1}{|\vec{q}| |\vec{q}'| |\vec{q}_\perp| |\vec{q}'_\perp|} \left\{ \frac{1}{4} (\eta - M^2)(\eta - \eta' + k^2)(S_{bc} - S'_{ab}) + \frac{1}{4} (\eta' - M^2) \right. \\
 & \quad (\eta' - \eta - k^2)(S_{bc} - S_{ab}) - S_{bc} (\eta - M^2)(\eta' - M^2) \\
 & \quad \left. - S_{bc} \frac{(S_{bc} - S_{ab})(S_{bc} - S'_{ab})}{8s} (\eta + \eta' + k^2 - 2M^2) \right\} \\
 (2,2) \quad & \frac{S_{bc}}{2|\vec{q}_\perp| |\vec{q}'_\perp|} [\eta + \eta' + k^2 - 2M^2] \\
 (3,3) \quad & \frac{s}{2M^2 S_{bc} |\vec{q}| |\vec{q}'|} \left[-M^4 + \left(-M^2 + \frac{S_{bc}}{2s} (S_{ab} + S_{bc}) \right) \left(-M^2 + \frac{S_{bc}}{2s} (S'_{ab} + S_{bc}) \right) \right] \\
 (1,2) \quad & \frac{\sqrt{s} \sqrt{-\lambda(q_\perp^2, q'^2_\perp, k^2)}}{2|\vec{q}'| |\vec{q}_\perp| |\vec{q}'_\perp|} \left[-M^2 + \frac{S_{bc}}{2s} (S'_{ab} + S_{bc}) \right]
 \end{aligned} \quad (A.11)$$

$$\begin{aligned}
 (1,3) &= \frac{(s_{ab} + s_{bc}) |q'_\perp|}{4 |\vec{q}'| |\vec{q}| M} (\eta' - \eta + k^2) + \frac{(s_{bc} - s'_{ab}) (\eta - \eta' + k^2)}{4 |\vec{q}'| |\vec{q}'| |q'_\perp| M} \left(-M^2 + \frac{s_{bc}}{2s} (s_{ab} + s_{bc}) \right) \\
 &+ \frac{s_{bc} (s_{bc} + s_{ab})}{4 |\vec{q}'| |\vec{q}| |q'_\perp| M s} \left(|q'_\perp|^2 (s_{bc} - s_{ab}) - \vec{q}_\perp \cdot \vec{q}'_\perp (s_{bc} - s'_{ab}) \right) \\
 (2,3) &= \frac{\sqrt{s} \sqrt{-\lambda(q_\perp^2, q_\perp'^2, k_\perp^2)}}{2 |q'_\perp| |\vec{q}'|} M
 \end{aligned} \tag{A.11}$$

Here we have set: $s'_{ab} = (p'_\perp + q'_\perp)^2$, $\eta' = (s'_{ab} s_{bc})/s = M^2 + q_\perp'^2$.

We now compare (A.11) with the l.h.s. matrix in (A.7). If we call $T(q, q')$ the matrix consisting of the elements (A.11), then our task is to find matrices D_{bc} with:

$$T(q, q') = D_{bc}(q) \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 + \frac{k^2}{2\pi^2} \end{pmatrix} D_{bc}(q')^T \tag{A.12}$$

The result is that $D_{bc}(q)$ has the form

$$D_{bc}(q) = L_{bc}(q) R(q_\perp, k_\perp) \tag{A.13}$$

with

$$L_{bc}(q) = \begin{pmatrix} A & 0 & -B \\ 0 & 1 & 0 \\ B & 0 & A \end{pmatrix} \tag{A.14}$$

$$R(q_\perp, k_\perp) = \begin{pmatrix} \sin \vartheta & -\cos \vartheta & 0 \\ \cos \vartheta & \sin \vartheta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{A.15}$$

$$\begin{aligned}
 A &= \frac{\sqrt{s}}{|\vec{q}'| s_{bc}} \left(-M^2 + \frac{s_{bc}}{2s} (s_{ab} + s_{bc}) \right) \\
 B &= \frac{|q_\perp| M \sqrt{s}}{|\vec{q}'| s_{bc}}
 \end{aligned} \tag{A.16}$$

$$\cos \vartheta = \frac{\vec{k}_\perp \cdot \vec{q}_\perp}{|\vec{k}_\perp| |\vec{q}_\perp|}, \quad \sin \vartheta = \frac{\sqrt{-\lambda(q_\perp^2, q_\perp'^2, k_\perp^2)}}{2 |\vec{k}_\perp| |\vec{q}_\perp|}. \quad (\text{A.17})$$

θ denotes the angle between $\vec{k}_\perp$ and $\vec{q}_\perp$. A similar result holds for $D(q')$. We note that because of

$$A^2 + B^2 = 1, \quad (\text{A.18})$$

both L_{bc} and R are orthogonal.

Finally, we compute the transformed production vertex $L_{bc}^T \vec{\Gamma}$. The components of $\vec{\Gamma}$ are given in (3.6); they are obtained from Γ_σ in (3.7) and the polarization vectors (A.10). By applying L_{bc} from (A.14) on $\vec{\Gamma}$, we find the simple result:

$$i L_{bc}^T \vec{\Gamma} = \begin{pmatrix} 2|\vec{q}_\perp| \cos(\vartheta_{1\perp}, \vartheta_\perp) \\ 2|\vec{q}_\perp| \sin(\vartheta_{1\perp}, \vartheta_\perp) \\ -M \end{pmatrix} - \frac{q_\perp^2 - M^2}{-q_\perp^2 - M^2} \begin{pmatrix} 2|\vec{q}_\perp| \\ 0 \\ -2M \end{pmatrix}. \quad (\text{A.19})$$

This leads to Eq. (4.29) and completes our calculations.

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Figure captions

- Fig. 1 : Elastic vector-vector scattering in the leading- $\ln s$ approximation: (a) Regge pole exchange in the isospin-one exchange channel, (b) two-reggeon cut diagrams for the isospin-zero and isospin-two exchange. (c) A non-leading- $\ln s$ contribution to the vacuum exchange channel with a triple Pomeron (two reggeon $\rightarrow$ four reggeon) vertex. In all three diagrams, a wavy line denotes the reggeizing vector particle.
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- Fig. 19 : The five terms of the $T_{2 \rightarrow 4}$ amplitude (4.57). The dotted lines denote the subenergies in which the amplitude has singularities.
- Fig. 20 : Transverse momentum structure of the two partial waves F_{LL} and $F_{LR}^{(1)}$ of Eqs. (4.58) and (4.61).
- Fig. 21 : The eight six-point amplitudes which form the signatred 2 $\rightarrow$ 4 amplitude.
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- Fig. 34 : The $2 \rightarrow n$ amplitude.
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- Fig. 38 : The s_{bc} discontinuity of $T_{2 \rightarrow 3}$.
- Fig. 39 : The s_{bc} discontinuity of $T_{2 \rightarrow 4}$ (a) and the s_{abc} discontinuity of $T_{2 \rightarrow 4}$ (b).

Fig. 1

$$(-i) \left[g_{\mu\nu} - \frac{k_\mu k_\nu}{M^2} \right] \frac{1}{k^2 - M^2} \delta_{ab}$$

$$\frac{i}{k^2 - M^2}$$

$$g \epsilon_{abc} \left[(r-p)_\mu g_{\lambda\nu} + (p-q)_\nu g_{\lambda\mu} + (q-r)_\lambda g_{\mu\nu} \right]$$

$$-ig^2 \left[\epsilon_{abf} \epsilon_{cdf} (g_{\lambda\nu} g_{\mu\sigma} - g_{\lambda\sigma} g_{\mu\nu}) \right]$$

$$-ig^2 \left[\epsilon_{acf} \epsilon_{bdf} (g_{\lambda\mu} g_{\nu\sigma} - g_{\lambda\sigma} g_{\mu\nu}) \right]$$

$$-ig^2 \left[\epsilon_{adf} \epsilon_{cbf} (g_{\lambda\mu} g_{\nu\sigma} - g_{\lambda\nu} g_{\mu\sigma}) \right]$$

$$ig M \delta_{ab} g_{\mu\nu}$$

Fig. 2

Fig. 3

Fig. 4

+ 5 other diagrams
(b)

Fig. 5

Fig. 6

Fig. 7

Fig. 8

Fig. 9

Fig. 10

Fig. 11

Fig. 12

Fig. 13

$$\text{disc } T_{2 \rightarrow 2}^{(6)} = 2 \left(\text{diagram with } g^4 \text{ and } g^2 \text{ vertices} \right) + \left(\text{diagram with } g^3 \text{ vertices} \right)$$

Fig. 14

Fig. 15

$$\begin{aligned} & \text{diagram with } k_1, k_2, q-k_1, q-k_2 \text{ external lines} \\ &= -2g^2 \left(\begin{aligned} & \text{diagram with } (q^2 - \frac{3}{2}M^2) \text{ factor} \\ & - \text{diagram with } (k_1^2 - M^2) \text{ and } (q-k_2)^2 - M^2 \text{ factors} \\ & - \text{diagram with } (k_2^2 - M^2) \text{ and } (q-k_1)^2 - M^2 \text{ factors} \end{aligned} \right) \end{aligned}$$

Fig. 16

Fig. 17

Fig. 18

Fig. 19

$$\begin{aligned}
 F_{LL} &= \text{diagram with inputs a, b, c and output d, containing two loops with labels } \beta_2, T_1, \beta_3, T_2 \\
 F_{LR}^{(1)} &= \text{diagram with inputs a, b, c and output d, containing two loops with labels } \beta_2, \beta_1, T_2, T_3 \\
 &\quad - T_1 \beta_1 \text{ (loop with b)} - T_3 \beta_1 \text{ (loop with c)} \\
 &\quad - T_1 B_{21} \text{ (loop with b, c)}
 \end{aligned}$$

Fig. 20

Fig. 21

Fig. 22

$$\text{disc}_{s_{bc}} T_{2 \rightarrow 3}^{(7)} = \begin{array}{c} \text{diagram (a)} \\ \text{(a)} \end{array} + \begin{array}{c} \text{diagram (b)} \\ \text{(b)} \end{array} + \begin{array}{c} \text{diagram (c)} \\ \text{(c)} \end{array}$$

The diagrams (a), (b), and (c) show a large ellipse connected to a smaller circle. In (a), the large ellipse has label 'a' at the top and 'g⁵' at the bottom; the small circle has labels 'b' and 'c' on the right and 'g²' at the bottom. In (b), the large ellipse has label 'a' at the top and 'g³' at the bottom; the small circle has labels 'b' and 'c' on the right and 'g⁴' at the bottom. In (c), the large ellipse has label 'a' at the top and 'g⁴' at the bottom; the small circle has labels 'b' and 'c' on the right and 'g³' at the bottom.

Fig. 23

$$\ln s_{ab} \left[\text{diagram 1} \right] + \ln s_{bc} \left[\text{diagram 2} \right]$$

(a)

$$\ln s_{ab} \left[\text{diagram 1} - T_1 \text{diagram 2} \right] + \ln s_{bc} \left[\text{diagram 3} - \text{diagram 4} \right]$$

(b)

Fig. 24

$$\ln (s/s_{bc}) \left[\text{diagram 1} - T_1 \text{diagram 2} \right] + \ln s_{bc} \left[\text{diagram 3} - \text{diagram 4} \right]$$

(a)

$$\ln (s/s_{bc}) \left[\text{diagram 1} - \text{diagram 2} \right] + \ln s_{bc} \left[\text{diagram 3} - \text{diagram 4} \right]$$

(b)

$$\begin{array}{c} k_1 \quad k_2 \\ \diagdown \quad \diagup \\ \text{---} \text{---} \\ \diagup \quad \diagdown \\ q-k_1 \quad q-k_2 \end{array} = (q^2 - M^2) \cdot \text{diagram 5}$$

(c)

Fig. 25

$$\ln s_{bc} \left(\text{diagram 1} - \text{diagram 2} \right)$$

Fig. 26

$$\begin{aligned}
 & \begin{array}{cc}
 \begin{array}{c} \diagup \quad \diagdown \\ | \quad | \\ \bullet \quad \circ \quad \bullet \\ | \quad | \\ \diagdown \quad \diagup \end{array} & \begin{array}{c} \diagup \quad \diagdown \\ | \quad | \\ \bullet \quad \circ \quad \bullet \\ | \quad | \\ \diagdown \quad \diagup \end{array} \\
 2 & \qquad +2 \\
 \begin{array}{c} \diagup \quad \diagdown \\ | \quad | \\ \bullet \quad \circ \quad \bullet \\ | \quad | \\ \diagdown \quad \diagup \end{array} & \begin{array}{c} \diagup \quad \diagdown \\ | \quad | \\ \bullet \quad \circ \quad \bullet \\ | \quad | \\ \diagdown \quad \diagup \end{array}
 \end{array} \\
 &= 2 \left(\begin{array}{c} \diagup \quad \diagdown \\ | \quad | \\ \bullet \quad \circ \quad \bullet \\ | \quad | \\ \diagdown \quad \diagup \end{array} - 2 \begin{array}{c} \diagup \quad \diagdown \\ | \quad | \\ \bullet \quad \circ \quad \bullet \\ | \quad | \\ \diagdown \quad \diagup \end{array} - 2 \begin{array}{c} \diagup \quad \diagdown \\ | \quad | \\ \bullet \quad \circ \quad \bullet \\ | \quad | \\ \diagdown \quad \diagup \end{array} + \begin{array}{c} \diagup \quad \diagdown \\ | \quad | \\ \bullet \quad \circ \quad \bullet \\ | \quad | \\ \diagdown \quad \diagup \end{array} \right)
 \end{aligned}$$

Fig. 30

$$2 \begin{array}{c} \diagup \quad \diagdown \\ | \quad | \\ \bullet \quad \circ \quad \bullet \\ | \quad | \\ \diagdown \quad \diagup \end{array} + 2 \begin{array}{c} \diagup \quad \diagdown \\ | \quad | \\ \bullet \quad \circ \quad \bullet \\ | \quad | \\ \diagdown \quad \diagup \end{array}$$

Fig. 31

$$\begin{aligned}
& \ln s_{ab} \left(\text{diagram 1} - \text{diagram 2} \right) \\
& + \ln s_{bc} \left(\text{diagram 3} - \text{diagram 4} \right) \\
& + \ln s_{cd} \left(\text{diagram 5} - \text{diagram 6} \right)
\end{aligned}
\tag{a}$$

$$\begin{aligned}
& \ln s_{ab} \left(\text{diagram 1} - \text{diagram 2} \right) \\
& - \text{diagram 3} + \text{diagram 4} \\
& + \ln s_{bc} \left(\text{diagram 5} - \text{diagram 6} \right) \\
& - \text{diagram 7} + \text{diagram 8} \\
& + \text{diagram 9} + \text{diagram 10} \\
& - \text{diagram 11} - \text{diagram 12} \\
& + \ln s_{cd} \left(\text{diagram 13} - \text{diagram 14} \right) \\
& - \text{diagram 15} + \text{diagram 16}
\end{aligned}
\tag{b}$$

$$\begin{aligned}
& \ln s_{ab} \left(\text{diagram 1} - \text{diagram 2} \right) \\
& + \ln s_{bc} \left(- \text{diagram 3} - \text{diagram 4} \right) \\
& + \text{diagram 5} - \text{diagram 6} \\
& + \text{diagram 7} + \text{diagram 8} \\
& + \ln s_{cd} \left(\text{diagram 9} - \text{diagram 10} \right)
\end{aligned}
\tag{c}$$

Fig. 32

$$\begin{aligned}
 & \text{Diagram 1} \cdot \beta_2 \beta_1 + \text{Diagram 2} \cdot T_2 T_3 - \text{Diagram 3} \cdot T_1 \beta_1 \\
 & \text{Diagram 4} \cdot T_3 \beta_1 + T_3 (\omega_2 - \beta_1 T_1) \cdot \text{Diagram 5} \dots \text{Diagram 6}
 \end{aligned}$$

(a)

$$\begin{aligned}
 & \text{Diagram 1} \cdot \beta_2 \beta_3 + \text{Diagram 2} \cdot T_2 T_1 - \text{Diagram 3} \cdot T_1 \beta_3 \\
 & \text{Diagram 4} \cdot T_3 \beta_3 + T_1 (\omega_2 - \beta_3 T_3) \cdot \text{Diagram 5} \dots \text{Diagram 6}
 \end{aligned}$$

(b)

Fig. 33

Fig. 34

Fig. 35

$$\begin{aligned}
 & \text{Diagram: A circle with two external lines on the left and two on the right. The top-right external line is labeled } k, \text{ and the bottom-right external line is labeled } q-k. \\
 & = \text{Diagram: A vertex with four external lines} + \text{Diagram: A vertex with a loop and two external lines} + \text{Diagram: A vertex with two loops and two external lines} + \dots \\
 & = \text{Diagram: A vertex with three external lines} \cdot (\omega + 2 - \alpha(k) - \alpha(q-k)) \\
 & \text{(a)}
 \end{aligned}$$

$$\begin{aligned}
 & \text{Diagram: A vertex with four external lines. Top-left is } k_1, \text{ top-right is } k_2, \text{ bottom-left is } q-k_1, \text{ bottom-right is } q-k_2. \\
 & = \text{Diagram: A vertex with four external lines} \cdot (q^2 - M^2) - \text{Diagram: A vertex with two external lines} \cdot (k_1^2 - M^2) \cdot ((q-k_2)^2 - M^2) - \text{Diagram: A vertex with two external lines} \cdot (k_2^2 - M^2) \cdot ((q-k_1)^2 - M^2) \\
 & \text{(b)}
 \end{aligned}$$

Fig. 36

Fig. 37

Fig. 38

Fig. 39