Neutral currents in gauge theories

R. Gatto

Département de Physique Théorique, Université de Genève, Switzerland

F. Strocchi

CERN, Geneva, Switzerland, Scuola Normale Superiore, Pisa, Italy

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The structure of the low-energy neutral current is discussed in unified gauge theories and the results of Georgi and Weinberg are generalized.

The structure of low-energy neutral-current interactions in a gauge theory based on a group G of the form $G = G_1 \times G_2 \times U(1)$, broken by a Higgs mechanism which leaves only the electric charge unbroken, has been analyzed by Georgi and Weinberg.¹ They have shown in particular that, under some general conditions on the vacuum expectation values of the Higgs fields, the neutral-current interactions, at zero momentum transfer, of any fermion f_0 which is electrically neutral and singlet under G_2 , are the same as if the gauge group were $G_1 \times U(1)$. Thus, any extension $SU(2) \times U(1) \times G_2$ of the standard $SU(2) \times U(1)$ weak group will give rise to the same low-energy neutral-current interactions of left-handed neutrinos.

In this note we extend the analysis to a gauge group of the form $G = G_1 \times G_2 \times K$ and we discuss under which general conditions the low-energy neutral-current interactions of neutrinos reduce to those of the standard $SU(2) \times U(1)$ model. This may explain why the standard model has proved to be so successful. On the other hand, our results show that if large departures from the standard theory are observed at low momentum transfer, one has to look for a nontrivial extension of the standard $SU(2) \times U(1)$ group.

Let G be the gauge group of the theory, which is broken by a Higgs mechanism. Let $\mathcal{G} \subset G$ be the unbroken subgroup of G and Q^I be the generators of $\mathcal{G}$. If T_α are the generators of G , the Q^I will be linear combinations of them,

$$Q^I = \sum c_\alpha^I T_\alpha, \quad (1)$$

and we will say that a generator T_α is "neutral" if there exists at least one $c_\alpha^I \neq 0$ in Eq. (1), i.e., if there is at least one Q^I to which T_α gives a nontrivial contribution. Since $\mathcal{G}$ is unbroken the complete mass matrix $M_{\alpha\beta}^2$ (α, β running over all the generators of G) is block diagonal,

$$M_{\alpha\beta}^2 = \begin{pmatrix} \mu_{\alpha\beta}^2 & 0 \\ 0 & m_{\alpha\beta}^2 \end{pmatrix},$$

where μ^2 denotes the mass matrix of the "neutral" intermediate vector bosons.

The diagonalization of $\mu_{\alpha\beta}^2$ defines the masses of the massive vector bosons

$$\mu_{\alpha\beta}^2 u_\beta^k = m_k^2 u_\alpha^k, \quad k=1, \dots, N \quad (2)$$

and the massless vector bosons

$$\mu_{\alpha\beta}^2 p_\beta^l = 0, \quad l=1, \dots, L. \quad (3)$$

The part of the Hamiltonian involving the neutral generators is

$$H_{\text{neutral}} = \sum_\alpha g_\alpha j_\alpha^\mu A_{\alpha\mu}, \quad (4)$$

where

$$j_\alpha^\mu = \bar{\psi} \gamma^\mu t_\alpha \psi,$$

t_α being the matrix representation of T_α on the fermions (α neutral index) and A_α^μ the neutral vector fields.

The Hamiltonian (4) gives rise to the following effective interaction:

$$\begin{aligned} H^{\text{eff}} &= g_\alpha g_\beta j_\alpha^\mu j_\beta^\nu \langle A_{\alpha\mu} A_{\beta\nu} \rangle \\ &= g_\alpha g_\beta j_\alpha^\mu j_\beta^\nu \sum_{l,l'} \langle B_\mu^l B_\nu^{l'} \rangle p_\alpha^l p_\beta^{l'} \\ &\quad + g_\alpha g_\beta j_\alpha^\mu j_\beta^\mu \sum_{k,k'} \langle Z_\mu^k Z_\nu^{k'} \rangle u_\alpha^k u_\beta^{k'}, \end{aligned} \quad (5)$$

where

$$B_\mu^l = \sum_\alpha p_\alpha^l A_{\alpha\mu}, \quad (6)$$

$$Z_\mu^k = \sum_\alpha u_\alpha^k A_{\alpha\mu} \quad (7)$$

are the neutral fields associated with the massless vector bosons and the massive vector bosons, respectively.

The second term in Eq. (5) defines the (effective) neutral-current interaction, which in the low-energy limit becomes

$$\begin{aligned} \mathcal{H}_{\text{neutral}} &= g_\alpha g_\beta j_\alpha j_\beta \sum_k u_\alpha^k u_\beta^k \frac{1}{m_k^2} \\ &\equiv g_\alpha g_\beta j_\alpha j_\beta \Delta_{\alpha\beta}. \end{aligned} \quad (8)$$

$\Delta_{\alpha\beta}$ is uniquely characterized by the following properties:

$$\Delta_{\alpha\beta} p_\beta^I = 0 = \Delta_{\alpha\beta} p_\alpha^I, \quad (9)$$

$$\begin{aligned} \mu_{\alpha\beta}^2 \Delta_{\beta\gamma} &= \sum_k u_\alpha^k u_\gamma^k \\ &= \delta_{\alpha\gamma} - \sum_i p_\alpha^I p_\gamma^I \equiv \zeta_{\alpha\gamma}. \end{aligned} \quad (10)$$

Let us consider the submatrix $\bar{\mu} = \{\mu_{mn}^2\}$ of $\mu_{\alpha\beta}^2$ obtained by eliminating a set of rows and columns corresponding to a choice of neutral generators of G , such that they contribute in an essential way to the charges (that is, in such a way that the determinant of the coefficients c_α^I restricted to such α is different from zero). We call C the set of such generators. The number of massless vector bosons is the same as the number of neutral generators belonging to C .

In the following a, b, a', b' are indices which take values in C , and m, n, m', n' are indices which correspond to the remaining neutral generators of G . We then have the following proposition 1.

Proposition 1. The matrix $\bar{\mu}_{mn} = \mu_{mn}^2$ defined above has an inverse.

Proof. Let us suppose that $\bar{\mu}_{mn}$ has one eigenvector q corresponding to the eigenvalue zero

$$\bar{\mu}_{mn} q_n = \mu_{mn}^2 q_n = 0.$$

Then if we complete q by adding zero components we also obtain

$$\mu_{m\alpha}^2 q_\alpha = 0.$$

On the other hand, $\mu_{\alpha\beta}^2$ has L zero eigenvalues and therefore L rows can be expressed as linear combinations of the others; this implies

$$\mu_{\beta\alpha}^2 q_\alpha = 0,$$

i.e., the so completed vector q is an eigenvector of $\mu_{\alpha\beta}^2$ with zero eigenvalue. This would mean that there exists a linear combination of the conserved charges Q^I which does not involve any generator belonging to C , contrary to the assumptions. We can now give the following proposition 2.

Proposition 2. $R_{\alpha\beta} = \zeta_{\alpha m} \zeta_{\beta n} \bar{\mu}^{-1}_{mn}$ satisfies Eqs. (9) and (10) and therefore

$$R_{\alpha\beta} = \Delta_{\alpha\beta}. \quad (11)$$

Proof. Equations (9) are trivially satisfied. On the other hand, if m is an index corresponding to generators which do not belong to C

$$\mu_{m\gamma}^2 \zeta_{\gamma n'} \zeta_{\beta n} \bar{\mu}^{-1}_{n'n} = \mu_{mn'}^2 \zeta_{\beta n} \bar{\mu}^{-1}_{n'n} = \zeta_{\beta m},$$

and if a is an index corresponding to a generator of C

$$\begin{aligned} \mu_{\alpha\gamma}^2 \zeta_{\gamma n'} \zeta_{\beta n} \bar{\mu}^{-1}_{n'n} &= \mu_{an'}^2 \zeta_{\beta n} \bar{\mu}^{-1}_{n'n} \\ &= \mu_{n'a}^2 \zeta_{\beta n} \bar{\mu}^{-1}_{n'n}. \end{aligned}$$

Now since

$$0 = \mu_{n'\beta}^2 p_\beta^I = \mu_{n'a}^2 p_a^I + \mu_{n'm}^2 p_m^I,$$

one has

$$\mu_{n'a}^2 = \frac{\mu_{n'm}^2 p_m^I (\text{ad } p)_{aI}}{\det p}.$$

Hence

$$\begin{aligned} (\mu^2 R)_{a\beta} &= -\mu_{n'm}^2 p_m^I \frac{(\text{ad } p)_{aI}}{\det p} \bar{\mu}^{-1}_{n'n} \zeta_{\beta n} \\ &= -\frac{p_m^I (\text{ad } p)_{aI}}{\det p} \zeta_{\beta n}. \end{aligned}$$

On the other hand,

$$p_m^I \zeta_{\beta m} + p_a^I \zeta_{\beta a} = 0 \quad (12)$$

and therefore

$$(\mu^2 R)_{a\beta} = p_a^I \frac{(\text{ad } p)_{aI}}{\det p} \zeta_{\beta b} = \zeta_{\beta a}. \quad (13)$$

This completes the proof of proposition 2.

We may now fix the normalization of the Q^I 's, or equivalently of the coefficients c_α^I in Eq. (1), in such a way that for a definite elementary fermion $\bar{f}$ the matrix $Q^I = \sum_\alpha c_\alpha^I t_\alpha$ takes the value $+1$. Then

$$\left[\sum_\alpha \left(\frac{c_\alpha^I}{g_\alpha} \right)^2 \right]^{-1/2} = q^I,$$

the l th charge of $\bar{f}$, and

$$\mathcal{H}_{\text{neutral}} = \frac{1}{2} \sum_{mn} (\bar{\psi} \gamma^\mu n_m \psi) (\bar{\psi} \gamma_\mu n_n \psi) (\bar{\mu}^{-1})_{mn},$$

$$\begin{aligned} n_j &= \sum_\alpha g_\alpha \zeta_{\alpha j} t_\alpha \\ &= g_j t_j - \sum_{i \neq j} p_\alpha^I p_j^I g_\alpha t_\alpha \\ &= g_j t_j - \sum_i q^I Q^I p_j^I \\ &\equiv \sum_i G_{ji} \left[\frac{c_j^I t_j}{L_j} - q_i^2 \left(\frac{c_j^I}{g_j} \right)^2 Q^I \right], \end{aligned} \quad (14)$$

where

$$\begin{aligned} G_{ii} &\equiv g_i / c_i^I \quad \text{if } c_i^I \neq 0, \\ &\equiv 0 \quad \text{if } c_i^I = 0 \end{aligned}$$

and L_i is the number of c_i^I different from zero.

We now consider the case in which there is a decomposition of G , $G_1 \times G_2 \times \cdots \times G_n \times K$, such that the vacuum expectation values of the Higgs fields can be grouped together into multiplets $\langle \varphi_j \rangle$ which transform nontrivially under $G_j \times K$ and are singlets with respect to the other groups G_k , $k \neq j$. This is typically the situation which occurs when

the vacuum expectation values $\langle \varphi \rangle$ define a chain of breakings of G .

We assume that the neutral generators of K contribute in an essential way to the conserved charges in the sense specified before, so that we can identify them with the elements of C .

Since the commutator of two unbroken charges is an unbroken charge, the unbroken charges generate a group, $\mathcal{G}$. Thus the elements of C are generators of a group. For simplicity in the following we will choose a basis Q^I for the conserved charges such that the matrix connecting C and $\{Q^I\}$ is diagonal.

We can now explore the physical consequences of proposition 2. Since by assumption the vacuum expectation values $\langle \varphi_j \rangle$ of the Higgs fields link G_j with C , but not the G_k among themselves, the effective neutral-current interaction may be written as a sum of terms:

$$\mathcal{H}_{\text{neutral}} = \sum_k g_\alpha g_\beta j_\alpha j_\beta \Delta_{\alpha\beta}^{(k)} = \sum_k H_{\text{neutral}}^{(k)}, \quad (15)$$

where

$$\Delta_{\alpha\beta} = \sum_k \Delta_{\alpha\beta}^{(k)} \quad (16)$$

is the decomposition of $\Delta_{\alpha\beta}$ arising from the fact that μ_{mn}^2 (and therefore also $\bar{\mu}_{mn}^{-1}$) is block diagonal and $\Delta_{\alpha\beta}^{(k)}$ is obtained by restricting the sum in $\Delta_{\alpha\beta} = \zeta_{\alpha m} \zeta_{\beta n} \bar{\mu}_{mn}^{-1}$ to the k th block.

We consider now the case of a fermion f_0 which is neutral under all the G_k 's except one (say $G_{\bar{k}}$) and neutral under all the Q^I 's, then all the currents $\bar{\psi} n_m f_0$ vanish for all T_m in $G_{\bar{k}}$ ($k \neq \bar{k}$) and the effective neutral-current interactions reduce to $H_{\text{neutral}}^{(\bar{k})}$, i.e., the same neutral-current interaction as in the theory based on the gauge group

$G_{\bar{k}} \times K$, where the neutral generators of K are defined as

$$T_I = c^I T_I^{(C)} + \sum_{\substack{i \in G_k \\ k \neq \bar{k}}} c_i^I T_i. \quad (17)$$

The corresponding coupling constants g'_I are

$$g_I'^{-2} = \sum'_{\substack{i \in G_k \\ k \neq \bar{k}}} \left(\frac{c_i^I}{g_i} \right)^2, \quad (18)$$

where, if K is decomposed as a direct product of simple groups, the sum $\sum'$ means that one has to average over the simple group to which T_I belongs. Hence

$$\frac{1}{g_I'^2} = \frac{1}{g_I'^2} + \sum_{i \in G_{\bar{k}}} \left(\frac{c_i^I}{g_i} \right)^2. \quad (19)$$

As an example we discuss the application to the case in which the gauge group is $SU(4)_c \times SU(2)_L \times SU(2)_R$ with unconfined color.² In this case the electric charge is the only conserved charge. In our notation $K = SU(4)_c$ and the neutral generator of K is the 15th generator of $SU(4)$. The neutral generators of $SU(2)_L$ and $SU(2)_R$ are T_3^L and T_3^R , respectively. In the case of confined color³ the conserved charges are the electric charge and generators of the color $SU(3)_c$. Again $K = SU(4)_c$ but the neutral generators of C are $SU(3)_c$ and the 15th generator of $SU(4)$.

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