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APPLICATIONS OF A CLUSTER DECOMPOSITION OF
MANY-DIMENSIONAL PHASE SPACE TO HIGH-ENERGY COLLISIONS[†]

M.M. Block,^{*)}

Department of Physics and Astronomy,

Northwestern University,

Evanston, Illinois, 60201 USA

and

J.D. Jackson,

Department of Physics,

University of California,

Berkeley, California, 94720 USA

ABSTRACT

A high-dimensional phase space is reduced in an approximate way to a much smaller dimensional phase space, by singling out a "cluster" of a smaller number of particles (e.g. two or three). We derive the invariant mass distribution of the cluster, and, for fixed invariant mass, its c.m.s. energy distribution. The basis of the method is a two-cluster decomposition of the original phase space into the cluster of interest and the remainder. Simple, explicit formulas are given for all particles massless, and for one particle with mass and the others massless. A recipe is stated for the application of these easily calculable distributions to practical situations where the masses of particles must be taken into account. Numerical examples illustrating the validity of the approximations and applications to high-energy elementary particle collisions are given.

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1. INTRODUCTION

Phase-space considerations have a long and honorable history, from Dalitz plots for three particles to statistical models of particle production for large numbers of particles [1-3]. In attempts to unravel interaction dynamics or hunt for the production of new particles, the experimenter uses phase space to estimate the shapes of backgrounds in various mass or other distributions. High-speed computers have led to an increasing reliance on Monte Carlo methods to generate these phase-space plots [4]. With present center-of-mass energies, it is necessary to consider large average numbers of particles, typically $\langle N \rangle \approx 10-15$, even though the desired phase-space distribution may be in the invariant mass of only three particles. The generation of such a distribution by Monte Carlo methods can be very expensive in computer time and funds.

The purpose of the present paper is to reduce, in an approximate way, the high-dimensional phase space to a much smaller dimensional phase-space problem, singling out a cluster of a small number of particles and giving its invariant mass distribution, and, for fixed invariant mass, its c.m.s. energy distribution. The emphasis is on simplicity and explicitness in the formulation and ease of computation. Of course, because phase space with particles of arbitrary masses is not generally expressible in "closed" form, some simplifying assumptions are necessary. Most of the particles are taken to be massless. With pions as the most common particles in high-energy hadronic final states, this assumption is not a bad one. One or two particles with mass can be included with little added complexity, as is indicated explicitly below. Furthermore, one might expect the shapes of phase-space distributions for a given number of particles to be roughly similar, provided they are scaled to the same kinematic interval. A simple, *ad hoc* recipe for scaling the distributions for massless particles is shown to give shapes remarkably similar to exact phase space, largely independent of the masses of the individual particles. This recipe permits our simple distributions to be applied directly to experiments.

It should be noted that similar but not the same considerations have been made earlier by others. In particular, the invariant mass squared distribution for a cluster of particles [(11) below] is given by Hagedorn [5].

2. TWO-CLUSTER DECOMPOSITION

The first problem to be considered is the writing of Lorentz-invariant phase space [6] for $N = m + n$ particles with masses μ_i as a two-cluster decomposition, one cluster of m particles having one invariant mass and the other cluster of n particles having another invariant mass. For N particles having a total invariant mass M , the Lorentz-invariant phase space (which can be shown to be a function of M^2 and the μ_i^2) is defined here to be

$$\Phi_N(M^2; \mu_i^2) = \int \prod_{j=1}^N \frac{d^3 p_j}{(2\pi)^3 2E_j} \delta^{(4)}\left(\sum_{j=1}^N p_j - P\right), \quad (1)$$

where the integration is over all N three-momenta, subject to the constraints of energy and momentum conservation, and $P \cdot P = M^2$. It is shown in Appendix A that the N particles may be grouped in clusters of m and n particles, the clusters having variable invariant masses M_1 and M_2 , respectively. The phase space (1) can then be written as [(A.8)]

$$\Phi_N(M^2; \mu_i^2) = \frac{\pi}{2} \int dM_1^2 \int dM_2^2 \Phi_m(M_1^2; \mu_i^2) \mathcal{F}_1\left(\frac{M_1^2}{M^2}, \frac{M_2^2}{M^2}\right) \Phi_n(M_2^2; \mu_j^2). \quad (2)$$

Here Φ_m and Φ_n are the respective m - and n -particle phase spaces, while $\mathcal{F}_1$ is defined by (A.4) and is, apart from a factor, the two-body phase space for particles of masses M_1 and M_2 . This result is the basis of our subsequent discussion. However, before proceeding to that, it is very convenient to introduce dimensionless, scaled variables, as in (A.10):

$$\begin{aligned} x = \frac{M_1^2}{M^2}, \quad u_i = \frac{\mu_i^2}{M^2} & \quad (\text{cluster } m), \\ y = \frac{M_2^2}{M^2}, \quad v_i = \frac{\mu_i^2}{M^2} & \quad (\text{cluster } n). \end{aligned} \quad (3)$$

The scaling property (A.9) of the phase space permits (2) to be transformed to
 [(A.11)]

$$\Phi_N(1; u_i, v_j) = \frac{\pi}{2} \int dx \int dy \Phi_m(x; u_i) \mathcal{F}_1(x, y) \Phi_n(y; v_j) . \quad (4)$$

The integrand in (4) is the joint distribution in x and y , the scaled invariant masses squared of the separated clusters. In the over-all center-of-momentum frame, one of the variables can equally be interpreted as the scaled c.m.s. energy of the other cluster, through the connection $(2E_m/M) = 1+x-y$, for example. Focusing on one cluster, we can therefore view the integrand in (4) as giving the joint distribution in mass and c.m.s. energy of that cluster.

The joint differential distribution of (4) can be exhibited in closed form only in special circumstances, derived in Appendix A. Because of the symmetry in m and n , we enumerated these special cases only for m :

<u>m</u>	<u>Mass(es)</u>	<u>Expression for Φ_m</u>
1	arbitrary	(A.2)
2	both arbitrary	(A.3)
3	2 arbitrary, 1 zero	(A.16)
≥ 2	1 arbitrary, (m-1) zero	(A.13)
≥ 2	all zero	(A.5)

We note that, for more than three particles in a cluster, the phase-space factor Φ_m exists in closed form only for all particles massless, or when only one has mass. Since the double distribution in x and y exists only for $N \geq 4$ (n and m must both be greater than unity), this means that the joint distribution can be integrated in closed form (and therefore explicitly normalized) only for the same two situations.

For all particles massless, the normalized joint differential distribution,

$$p_{mn}^{(2)}(x, y) \equiv \frac{\pi}{2} \frac{\Phi_m(x; 0) \mathcal{F}_1(x, y) \Phi_n(y; 0)}{\Phi_{m+n}(1; 0)} , \quad (5)$$

is found from (4) and (A.5) to be

$$p_{mn}^{(2)}(x,y) = \frac{(m+n-1)!(m+n-2)!}{(m-1)!(m-2)!(n-1)!(n-2)!} x^{m-2} y^{n-2} \mathcal{F}_1(x,y) . \quad (6)$$

This simple, explicit, normalized differential distribution in x and y exemplifies the goal of this paper -- to integrate over a large number of the particles and obtain what can be viewed as the distribution in invariant mass squared and c.m.s. energy of a cluster of m particles among $N = m + n$.

If one of the particles, in cluster m , say, has mass $\mu = \sqrt{u} M$, it can be seen from (A.13) and (A.14) that the normalized joint distribution becomes

$$p_{mn}^{(2)}(x,y; u) = p_{mn}^{(2)}(x,y) \frac{f_{m-1}\left(\frac{u}{x}\right)}{f_{m+n-1}(u)} . \quad (7)$$

The function $f_n(x)$ is defined by (A.12). Its properties are discussed in detail in Appendix B. It is normalized to unity at $x = 0$ for all n and vanishes at $x = 1$. For large n , it falls rapidly to zero for any finite x . It is graphed in Fig. 1 for $n = 1, 2, 3, 5$.

If one particle in cluster m has mass $\mu = \sqrt{u} M$ and one in cluster n has mass $\mu' = \sqrt{v} M$, the joint distribution is

$$p_{mn}^{(2)}(x,y; u; v) \propto x^{m-2} y^{n-2} f_{m-1}\left(\frac{u}{x}\right) f_{n-1}\left(\frac{v}{y}\right) \mathcal{F}_1(x,y) \quad (8)$$

but the proportionality coefficient cannot be calculated in closed form.

Another joint distribution of potential use is the one in which cluster m consists of two particles of masses $\mu_1 = \sqrt{u_1} M$ and $\mu_2 = \sqrt{u_2} M$, while the rest of the particles are massless. The joint differential distribution is

$$p_{2,n}^{(2)}(x,y; u_1, u_2; 0) \propto y^{n-2} \mathcal{F}_1\left(\frac{u_1}{x}, \frac{u_2}{x}\right) \mathcal{F}_1(x,y) . \quad (9)$$

3. NORMALIZED INVARIANT MASS DISTRIBUTIONS

To obtain the single invariant mass squared distribution for, say, cluster m , we integrated over y in (4). Fortunately this can be done formally by using the

concept of the cluster decomposition. Apply (4) to (n+1) particles, one of mass $\sqrt{x}$ M in cluster m, and the rest in cluster n. The use of (A.2) for the single particle shows that

$$\Phi_{n+1}(1; x, v_j) = \frac{1}{(2\pi)^3} \cdot \frac{\pi}{2} \int dy \mathcal{F}_1(x, y) \Phi_n(y, v_j) . \quad (10)$$

Comparison with (4) shows that the integral of $\mathcal{F}_1 \Phi_n$ over y can be expressed in terms of Φ_{n+1} . Thus we have

$$\Phi_N(1; u_i, v_j) = (2\pi)^3 \int dx \Phi_m(x; u_i) \Phi_{n+1}(1; x, v_j) . \quad (11)$$

The integrand here is the (unnormalized) invariant mass-squared distribution for cluster m [5]. The above-listed special circumstances for which the Φ_k have closed forms show that the unnormalized distribution can be exhibited explicitly for $n = 1$ (arbitrary mass), $n = 2$ (one mass zero), $n \geq 3$ (all masses zero), and for the Φ_m 's given in the list.

For normalized distributions in closed form it is necessary to be able to perform the x integration. For $N > 3$ this can be done only for all particles massless, or for one particle in the cluster with mass, as noted earlier. For either case, $v_j = 0$ for all the particles in cluster n. The phase-space factor Φ_{n+1} thus involves one "particle" of mass x, and (A.13) yields

$$\Phi_{n+1}(1; x, 0 \dots) = \Phi_{n+1}(1; 0) f_n(x) . \quad (12)$$

If all the particles in cluster m are massless, the normalized differential distribution in x,

$$p_{mn}(x) \equiv (2\pi)^3 \frac{\Phi_m(x; 0) \Phi_{n+1}(1; x, 0 \dots)}{\Phi_{m+n}(1; 0)} , \quad (13)$$

can be written, by using (A.5) and (12),

$$p_{mn}(x) = (2\pi)^3 \frac{\Phi_m(1; 0) \Phi_{n+1}(1; 0)}{\Phi_{n+m}(1; 0)} x^{m-2} f_n(x) .$$

The normalization constant can be found explicitly from (A.5). The normalized differential distribution in x for all particles massless is therefore

$$p_{mn}(x) = \frac{(m+n-1)!(m+n-2)!}{(m-1)!(m-2)!n!(n-1)!} x^{m-2} f_n(x) . \quad (14)$$

The range of x is $(0,1)$. This invariant mass-squared distribution is the simplest and most useful one. The factorization of the x dependence for cluster m into a power law in m and $f_n(x)$ for the other particles makes the averaging over final-state multiplicities (for fixed m) particularly easy.

If one particle in the cluster m has mass $\mu = \sqrt{u} M$, while all other particles are massless, the normalized differential distribution in x becomes

$$p_{mn}(x; u) = p_{mn}(x) \frac{f_{m-1}(u/x)}{f_{n+m-1}(u)} , \quad (15)$$

as can be inferred from (7). The range of x is now $u \leq x \leq 1$.

As an example of an unnormalized distribution in x of some use, we quote the integral of (9) over y , namely, the invariant mass-squared distribution for a two-particle cluster, both particles having mass, the rest of the particles being massless:

$$p_{2,n}(x; u_1, u_2; 0) \propto f_n(x) \mathcal{F}_1\left(\frac{u_1}{x}, \frac{u_2}{x}\right) . \quad (16)$$

Here the range of x is $(\sqrt{u_1} + \sqrt{u_2})^2 \leq x \leq 1$.

4. MOMENTS OF THE NORMALIZED INVARIANT MASS DISTRIBUTIONS

Of some use in judging the form of the distributions in x are the various moments. For the $m+n$ massless particle distribution in x , the general moment is defined by

$$\langle x^\nu \rangle_{m,n} = \int_0^1 x^\nu p_{mn}(x) dx .$$

For $n = 1$, the result is obtained by elementary integration, and for $n = 2$, by only slightly more work. The recursion formula (B.6) can then be used to obtain a general result,

$$\langle x^v \rangle_{m,n} = \frac{\Gamma(m+n)\Gamma(m+n-1)}{\Gamma(m+n+v)\Gamma(m+n+v-1)} \cdot \frac{\Gamma(m+v)\Gamma(m+v-1)}{\Gamma(m)\Gamma(m-1)} . \quad (17)$$

For integer moments, this expression simplifies considerably. The mean and mean square values are

$$\langle x \rangle_{m,n} = \frac{m(m-1)}{(m+n)(m+n-1)} , \quad (18a)$$

$$\langle x^2 \rangle_{m,n} = \frac{m^2(m^2-1)}{(m+n)^2[(m+n)^2-1]} . \quad (18b)$$

For the normalized distribution (15), where one particle in the cluster has mass, the moments are defined by

$$\langle x^v \rangle_{m,n} = \int_u^1 x^v p_{mn}(x; u) dx . \quad (19)$$

The added complexity of the distribution is such that it is not profitable to display more than the moments of x and x^2 ; that is, the generalizations of (18). These are obtained through judicious use of the derivative formulae (B.8). They are

$$\langle x \rangle_{m,n} = \frac{m-1+nu}{m+n-1} - \frac{n(m-1)f_{m+n}(u)}{(m+n)(m+n-1)f_{m+n-1}(u)} \quad (20a)$$

and

$$\begin{aligned} \langle x^2 \rangle_{m,n} = & \frac{m(m-1) + 2n(m-1)u + n(n+1)u^2}{(m+n)(m+n-1)} \\ & - \frac{2n(m-1)[m + (n+1)u]f_{m+n}(u)}{(m+n)^2(m+n-1)f_{m+n-1}(u)} \\ & + \frac{m(m-1)n(n+1)}{(m+n)^2[(m+n)^2-1]} \cdot \frac{f_{m+n+1}(u)}{f_{m+n-1}(u)} . \end{aligned} \quad (20b)$$

For completeness we give, for x fixed, the mean and mean square c.m.s. energies of the cluster of m particles, where the particles in cluster n are all massless:

$$\langle E_m \rangle_n = \frac{M}{2} \left[1 + x^2 - \left(\frac{n-1}{n+1} \right) \frac{f_{n+1}(x)}{f_n(x)} \right] \quad (21a)$$

$$\begin{aligned} \langle E_m^2 \rangle_n = \frac{M^2}{4} & \left[(1+x^2)^2 - \frac{2(n-1)}{(n+1)} (1+x^2) \frac{f_{n+1}(x)}{f_n(x)} \right. \\ & \left. + \frac{n(n-1)}{(n+2)(n+1)} \frac{f_{n+2}(x)}{f_n(x)} \right]. \end{aligned} \quad (21b)$$

5. APPLICATIONS

5.1 Recipe for including physical masses

The various explicit phase-space distributions of Sections 2 and 3 have the obvious flaw that most of the particles are treated as massless. For final states in which the total rest energy of the particles is small compared to the c.m.s. energy, this approximation is legitimate, but it is desirable to have at least an empirical way of including the effects of mass. It is clear from general arguments that for many particles the phase space depends strongly not only on the total kinematic range available but also on the number of particles. One therefore suspects that the dominant effect of particle masses is the narrowing of the kinematic range, and that, if distributions for a fixed number of particles are scaled appropriately to account for differences in the ranges (of the invariant mass of a cluster, for example), the shapes and even absolute magnitudes will be roughly similar. This expectation promises a simple way of including masses and of being able to apply our largely massless distributions to the real world.

The idea of scaling the intervals to account for masses is to match the end points of our all-massless (or one particle with mass) distributions to the physical end points, determined by the total c.m.s. energy and the specific masses involved. Consider, for example, the massless x distribution [(14)] with its interval of $0 \leq x \leq 1$. If the m particles in the cluster have masses μ_i and the

other n have masses μ_j , then the limits on the invariant mass M of the cluster are

$$\begin{aligned}\mu_{\min} &= \sum_{i=1}^m \mu_i \\ \mu_{\max} &= W - \sum_{j=1}^n \mu_j\end{aligned}\tag{22}$$

where we have used the symbol W for the total c.m.s. energy in order to avoid confusion with the cluster masses. The recipe requires that when $x = 0$, $M = \mu_{\min}$, and when $x = 1$, $M = \mu_{\max}$. For intermediate x values, the question of interpolation arises. We have experimented with quadratic forms in M (linear in x) and linear interpolation in M . The latter works best when compared with exact phase space and will be employed without further justification. For all massless particles, the linear interpolation formula relating x with the cluster mass M is therefore

$$M = (\mu_{\max} - \mu_{\min})\sqrt{x} + \mu_{\min},\tag{23}$$

and its inverse,

$$x = \left(\frac{M - \mu_{\min}}{\mu_{\max} - \mu_{\min}} \right)^2.\tag{24}$$

In practice the distributions in x such as (14) are converted into distributions in $\sqrt{x}$ by multiplication by $2\sqrt{x}$ and then expressed as differential probabilities in the mass M :

$$\frac{dP_{mn}(M)}{dM} = \frac{2(M - \mu_{\min})}{(\mu_{\max} - \mu_{\min})^2} p_{mn}[x(M)],\tag{25}$$

where $x(M)$ is given by (24) and $p_{mn}(x)$ by (14).

If one particle in the cluster m is chosen to have mass, (15) is therefore applicable and a slightly different recipe applies. The same physical limits (22) apply, but now the corresponding range of x is $u_1 \leq x \leq 1$, where $u_1 = \mu_1^2/W^2$. The

linear interpolation formula for the cluster mass is now

$$M = \frac{(\mu_{\max} - \mu_{\min})\sqrt{x} + \mu_{\min} - \sqrt{u_1} \mu_{\max}}{1 - \sqrt{u_1}}, \quad (26)$$

and its inverse,

$$x = \left[\frac{M(1 - \sqrt{u_1}) - \mu_{\min} + \sqrt{u_1} \mu_{\max}}{\mu_{\max} - \mu_{\min}} \right]^2. \quad (27)$$

The analogue of (25) is then

$$\frac{dP'_{mn}(M)}{dM} = \frac{2(1 - \sqrt{u_1})}{(\mu_{\max} - \mu_{\min})} \sqrt{x(M)} p_{mn}[x(M)] \frac{f_{m-1}[u_1/x(M)]}{f_{m+n-1}[u_1/x(M)]} \quad (28)$$

where $x(M)$ is now given by (27).

For single invariant distributions where $k (> 1)$ particles have been assumed to have mass [for example, when $m = 2$ and (16) is being used], the recipe can be written down by inspection since the range of x is now $\left(\sum_{i=1}^k \sqrt{u_i}\right)^2 \leq x \leq 1$. One merely substitutes $\sum_{i=1}^k \sqrt{u_i}$ for $\sqrt{u_1}$ into (27). For a double distribution in x and y , more care must be taken but the concept is the same. For example, with all particles massless, (6) is applicable with the ranges of x and y as $0 \leq x \leq 1$, $0 \leq y \leq (1-\sqrt{x})^2$. There now are two clusters and two sets of mass limits. In practice, one picks M_m and so fixes x . Then the range of y is specified, as is the range for the n cluster mass M_n . Fixing the end points to correspond then determines M_n in terms of $\sqrt{x}$, $\sqrt{y}$, and the kinematic parameters.

The recipe for converting the theoretical phase-space distributions to practical ones is admittedly empirical. To give the reader confidence that the procedure is effective, we present comparisons of exact five-particle phase-space calculations for the invariant mass distributions of a cluster of three particles and compare them with the two recipes, (25) and (28). The examples computed are $\Lambda K \pi \pi \pi$ with $W = 5$ GeV, cluster = $\Lambda \pi \pi$; the same with $W = 10$ GeV; and $p K \bar{K} \pi \pi$ at 10 GeV, cluster = $p K \pi$. In applying (28), the baryon is taken as the particle with

mass. The results of this comparison are shown in Figs. 2, 3, and 4, respectively. The first observation is that the shapes of the distributions are quite similar, verifying our expectation that the kinematic interval is the dominant factor. The second remark is that the use of the theoretical formula with the baryon mass taken as non-zero is a somewhat better approximation, especially at threshold, but not a drastically better one. The simplest formulation, namely, the "all massless" (25), appears adequate for most purposes.

5.2 Examples of mass distributions; averages over multiplicities

Having specified the recipe for getting practical phase-space distributions from our massless or nearly massless theoretical distributions, we return to the distributions themselves and present some examples. These illustrate phase-space mass distributions to be expected in hadronic final states at high energy, and should give the reader some feeling for the shapes of backgrounds anticipated in searches for resonant states or new heavy particles.

We first consider the mass distributions expected for a given multiplicity (charged plus neutral) of particles, for various multiplicities. The illustration is simplest if all the particles are massless, since we can present a single universal curve for each multiplicity (the only scale of mass in the problem is then the c.m.s. energy W). Rather than plot (14), which essentially gives the distribution in mass squared (in reduced units), we introduce $w = \sqrt{x}$, which is the mass of the cluster m divided by W , and calculate dP_{mn}/dw , the normalized differential probability per unit w , given by

$$\frac{dP_{mn}}{dw} = 2wp_{mn}(w^2) . \quad (29)$$

The mass distributions for a three-particle cluster ($m = 3$) for n values of 1, 3, 5, 7, and 9 are shown in Fig. 5. We observe the rapid shift of the peaks to lower masses, as well as a significant narrowing of the distributions as n increases, a result anticipated by examination of the average values of x and x^2 , given by (18a) and (18b). In order to apply (29) to a practical case where all particles

(in both the clusters m and n) have mass, we must replace $w(=\sqrt{x})$ by $(M - \mu_{\min})/(\mu_{\max} - \mu_{\min})$, and multiply dP/dw by the Jacobian $(\mu_{\max} - \mu_{\min})^{-1}$ to obtain dP/dM , as shown earlier in (25).

In many experimental situations of interest, the total multiplicity of any given collision is not known, but rather the average multiplicity $\langle N \rangle$ of charged plus neutral particles for a given c.m.s. energy W . We therefore illustrate the mass distributions to be expected for typical values of $\langle N \rangle$. For this purpose a multiplicity distribution is needed.

The charged-prong multiplicity distributions in high-energy hadronic collisions have been parametrized in various ways [7, 8]. There is also evidence that, for charged multiplicities up to the average and beyond, the neutral pion multiplicity increases in rough proportion [9], the mean number of π^0 's being one half that of charged pions, at least at high energies [10]. We therefore assume that the average total multiplicity $\langle N \rangle$ is given by 1.5 times the charged multiplicity at a given energy. Furthermore, because we are giving only illustrative examples, we use a simpler, but entirely adequate, parametrization of the multiplicities:

$$G_N(\langle N \rangle) = \frac{\pi}{2\langle N \rangle} \frac{N}{\langle N \rangle} \exp \left\{ -\frac{\pi}{4} \left(\frac{N}{\langle N \rangle} \right)^2 \right\}. \quad (30)$$

The normalization is as if the function were continuous in N , but for $\langle N \rangle \geq 5$, $\sum_{N=1}^{\infty} G_N = 1$ (for integer N) to better than $\frac{1}{2}\%$.

For a fixed energy W , we expect a distribution of mass which is a weighted superposition of curves of the type shown in Fig. 5. For a three-particle cluster, we define the normalized distribution function $dP_3(\langle N \rangle)/dw$ as this weighted distribution,

$$\frac{dP_3(\langle N \rangle)}{dw} = \sum_{n=1}^{\infty} \frac{dP_{3,n}(w)}{dw} \frac{G_{n+3}(\langle N \rangle)}{\left[1 - \sum_{N=1}^3 G_N(\langle N \rangle) \right]}, \quad (31)$$

and plot the curves for $\langle N \rangle = 10, 15, 20$, and 25 in Fig. 6. The sharp rise at low mass is caused by the high multiplicity content of (30), whereas the long

tail at high masses is due to the superposition of all the low multiplicity contributions. The application of Fig. 6 to the cases of particles with non-zero mass is identical to the earlier discussion of Fig. 5.

One might also ask, for the mass distribution of a three-particle cluster, for only those collisions with $N > N_{\min} > m+1$, as might occur with an experimental trigger requirement for the study of very high multiplicity events. In this case, (31) is modified by starting the main summation with $n = N_{\min} - m$, and renormalizing the denominator to be $1 - \sum_{N=1}^{N_{\min}-1} G_N$. The qualitative effect is to suppress the high-mass tail (due to low multiplicities) and emphasize the low-mass rise, putting the peak of the curve at lower mass, for a given $\langle N \rangle$.

5.3 Monte Carlo applications

We next consider the use of this mathematical apparatus to make a computer program for generating Monte Carlo events of interest for tracing through an experimental apparatus. We illustrate this with the problem of a diffractive collision, in which a diffractive blob of mass W and laboratory vector momentum $\vec{P}$ "decays" via phase space into n π 's + $\Lambda_c + \bar{D}$. We wish to trace the history of both the charmed baryon Λ_c and a charmed D meson through the experimental apparatus. We can write

$$dP_{2,n}(x) = \text{const} \times f_n(x) \mathcal{F}_1\left(\frac{u_1}{x}, \frac{u_2}{x}\right) dx, \quad (32)$$

where we assume that the n π 's are massless,

$$u_1 = \left(\frac{m_{\Lambda_c}}{W}\right)^2, \quad u_2 = \left(\frac{m_D}{W}\right)^2, \quad \text{and } (\sqrt{u_1} + \sqrt{u_2})^2 \leq x \leq 1.$$

In this case,

$$\mu_{\min} = m_{\Lambda_c} + m_D, \quad \mu_{\max} = W - \sum_{i=1}^n m_{\pi_i}$$

and

$$M = \left(\frac{\mu_{\max} - \mu_{\min}}{1 - \sqrt{x_{\min}}} \right) (\sqrt{x} - \sqrt{x_{\min}}) + \mu_{\min},$$

where $\sqrt{x}_{\min} = \sqrt{u_1} + \sqrt{u_2}$. Since u_1 and u_2 are typically very much less than 1, we can simplify (32) by setting $u_1 = u_2 = 0$ and obtain from (14), with $m = 2$,

$$\frac{dP_{2,n}(x)}{dx} \equiv p_{2,n}(x) = n(n+1)f_n(x), \quad (33)$$

where $0 \leq x \leq 1$, i.e. $x_{\min} = 0$.

A typical method of generating Monte Carlo distributions of the type (32) is to randomly generate x values in the interval $(0, 1)$, and rescale $dP_{2,n}(x)/dx$ so that its maximum value is unity, i.e. define $dP'_{2,n}(x)/dx = f_n(x)$. One then accepts events if, after an x_i is randomly picked, the value of $dP'_{2,n}(x_i)/dx$ is greater than that of a second random number chosen in the interval $(0, 1)$. For the functions $f_n(x)$, which vary rapidly over their argument's range, this method is readily seen to be very inefficient. The efficiency, defined as (accepted events/number of trials), is given in this case by the area under the curve $\int_0^1 f_n(x) dx = 1/n(n+1)$. For $n \approx 15$, the efficiency is only $\sim 5 \times 10^{-3}$. To avoid such a serious loss of efficiency, we have developed in Appendix C a recursion relation (C.4, 6, 7) for the calculation of the indefinite integral, (C.5); $I_{m,n}(x) = \int_0^x p_{mn}(x') dx'$, with $I_{m,n}(0) = 0$ and $I_{m,n}(1) = 1$. If we call $I_{m,n}(x) = z(x)$, and introduce the new variable $z(x)$, $0 \leq z \leq 1$, then the picking of a random z in the interval $(0, 1)$ automatically generates the shape (33), i.e. $dP_{2,n} \propto f_n(x) dx = dz$. This technique can also be applied to the more complex distribution (32), by rewriting (32) as a function of z , i.e.

$$dP_{2,n}(z) \propto \mathcal{F}_1\left(\frac{u_1}{x(z)}, \frac{u_2}{x(z)}\right) dz. \quad (34)$$

When $u_1, u_2 \ll 1$, $\mathcal{F}_1$ is a rather slowly varying function over most of the range of z ; it does vanish for $z = z_{\min} = z(x_{\min}) = z[(\sqrt{u_1} + \sqrt{u_2})^2]$, but rises rapidly towards unity above threshold. We now pick z randomly in the interval $(z_{\min}, 1)$. We determine numerically the maximum value of $dP_{2,n}/dz$, denoted as $dP_{\max}/dz$, for z in the interval $(z_{\min}, 1)$. We then pick z_i randomly in this interval, and keep the event if the product of another random number in the interval $(0, 1)$ times

$dP_{\max}/dz$ is less than $dP_{2,n}(z_i)/dz$. By having removed the most singular portion of (32), $f_n(x)$, by our variable transformation to z in (34), we achieve efficiencies $\gtrsim 80\%$, compared to the fraction of a percent we would have achieved by direct generation of (32).

The calculation can be carried further, again for massless pions. After having picked an x (and hence, the mass M_m of the $\Lambda_c D$ system), we now wish to generate the mass M_n of the n -pion system. This mass M_n (given M_m) specifies the energy of the $\Lambda_c D$ system in the original c.m. system, i.e.

$$E_m^* = \frac{W^2 + M_m^2 - M_n^2}{2W} . \quad (35)$$

We find M_n by choosing the reduced (mass)² variable y (see below). To get the analogue of (32), we use (9), i.e.

$$p_{2,n}^{(2)}(x,y; u_1, u_2, 0) = \text{const} \times y^{n-2} \mathcal{F}_1\left(\frac{u_1}{x}, \frac{u_2}{x}\right) \mathcal{F}_1(x,y) . \quad (36)$$

Having already chosen x , (36) is now only a function of y in the interval $[0, (1 - \sqrt{x})^2]$. It is convenient to rewrite it as

$$\frac{dP_{2,n}}{dy} = \text{const} \times n(n-1)y^{n-2} \mathcal{F}_1(x,y) . \quad (37)$$

Since here x is fixed, (37) applies equally for arbitrary choices of u_1 and u_2 , including zero. The indefinite integral $Z(y) = \hat{f}_n(y,x) = n(n-1) \int_0^y y'^{n-2} \mathcal{F}_1(x,y') dy'$ is defined in Appendix C (C.1) and recursion relations [(C.2 and C.3)] are developed for its calculation. By introducing the variable $Z(y)$, we can rewrite (37) as

$$dP_{2,n} = \text{const} \times dZ . \quad (38)$$

The function $Z(y)$ has its maximum value at $y = (1 - \sqrt{x})^2$, i.e. $Z_{\max} = Z[(1 - \sqrt{x})^2]$, and its minimum value is 0. Hence, by picking Z randomly in the interval $(0, Z_{\max})$, we automatically generate (38) and have the desired distribution (37).

The mass range associated with the variable y is the interval $(\mu_{\min}, \mu_{\max})$, where $\mu_{\min} = \sum_{i=1}^n m_{\pi}$ and $\mu_{\max} = W - M_m$. We associate M_n with y via the relation

$$M_n = \frac{\sqrt{y}(\mu_{\max} - \mu_{\min})}{1 - \sqrt{x}} + \mu_{\min} . \quad (39)$$

We calculate E_m^* from (35), using (37), and the value of M_m obtained from x . All that remains now is to:

- a) Monte Carlo generate the isotropic momentum distribution of the $\Lambda_c D$ system, and Lorentz transform it to the laboratory.
- b) Generate the isotropic decay distribution of the Λ_c (and D) in the rest frame of the $\Lambda_c D$ system and Lorentz transform these two momentum vectors to the laboratory system.

We have now solved our problem of generating (via a very high dimensional space) the laboratory momenta of the Λ_c and D , by averaging over all the unwanted pions without having to generate them.

6. CONCLUSIONS

We have decomposed the Lorentz-invariant phase space of a system of $N = n + m$ particles into a cluster of m particles of interest and another cluster of n particles. The simultaneous mass-squared distributions of the two clusters is equivalent to the joint phase-space probability distribution for the mass of the cluster m of interest, along with its c.m.s. energy distribution. We have demonstrated numerically that by rescaling the kinematics, the approximation of using massless particles, with the resulting joint distribution in terms of elementary functions, is adequate for most purposes. With a slightly more complex formalism, we easily take into account explicitly the heaviest mass in the cluster m . Thus we have effectively been able to reduce the phase-space dimensionality from N , a very large number, to m , which is typically 2 or 3, and have developed a mathematical formalism useful in the Monte Carlo generation of the (much) reduced m -dimensional phase-space problem.

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Figure captions

- Fig. 1 : Examples of the function $f_n(x)$ versus x , for $n = 1, 2, 3$, and 5 .
- Fig. 2 : Normalized mass distributions for the $\Lambda\pi\pi$ system, for the final state $\Lambda K\pi\pi\pi$ at $W = 5$ GeV. The solid curve is exact (by numerically integrating over the phase space); the dashed curve is the prediction when the Λ mass is explicitly taken into account; and the dotted curve is the prediction when all masses are set equal to zero. The predicted curves have been rescaled by (27 and (24), respectively.
- Fig. 3 : Normalized mass distributions for the $\Lambda\pi\pi$ system, for the reaction $\Lambda K\pi\pi\pi$ at $W = 10$ GeV. The legends are the same as for Fig. 2.
- Fig. 4 : Normalized mass distributions for the $pK\pi$ system, for the reaction $pK\bar{K}\pi\pi$ at $W = 10$ GeV. The solid curve is exact; the dashed curve is the prediction when the proton mass is explicitly taken into account; and the dotted curve is the prediction when all masses are set equal to zero. The predicted curves have been rescaled by (27) and (24), respectively.
- Fig. 5 : The normalized mass distributions (in the reduced variable $w = \sqrt{x}$) for a three-particle cluster ($m = 3$), with additional particles $n = 1, 3, 5, 7$, and 9 , versus w . The masses of all particles have been set equal to zero.
- Fig. 6 : The normalized mass distributions (in the reduced variable $w = \sqrt{x}$) for a three-particle cluster ($m = 3$) versus w , averaged [see (30)] over the multiplicity $N = n + m$, for $\langle N \rangle$, the average value of charged + neutral particles, equal to 10, 15, 20, and 25.

Fig. 1

Fig. 2

Fig. 3

Fig. 4

Fig. 5

Fig. 6

TWO-CLUSTER DECOMPOSITION AND OTHER RESULTS

Lorentz-invariant phase space for k particles is defined to be

$$\Phi_k(M^2; \mu_j^2) = \int \prod_{j=1}^k \frac{d^3 p_j}{(2\pi)^3 2E_j} \delta^{(4)}\left(\sum_{j=1}^k p_j - P\right) \quad (\text{A.1})$$

where the integration is over all k three-momenta, subject to the constraints of energy and momentum conservation exhibited by the four-dimensional delta function. The particles have masses μ_j , and M is the invariant mass of the k -particle system ($M^2 = P \cdot P = P_0^2 - \vec{P}^2$). For one and two particles, we find trivially,

$$\Phi_1(M^2; \mu_1^2) = \frac{1}{8\pi^3} \delta(\mu_1^2 - M^2) \quad (\text{A.2})$$

$$\Phi_2(M^2; \mu_1^2, \mu_2^2) = \frac{1}{128\pi^5} \mathcal{F}_1\left(\frac{\mu_1^2}{M^2}, \frac{\mu_2^2}{M^2}\right), \quad (\text{A.3})$$

where

$$\mathcal{F}_1(x, y) = \sqrt{1 - 2(x+y) + (x-y)^2}. \quad (\text{A.4})$$

The quantity $\mathcal{F}_1$ in (A.3) is actually $(2p/M)$, where p is the c.m.s. momentum of each particle, but the notation of (A.3) and (A.4) is more convenient in the following.

For more than two particles, it is not generally possible to exhibit a "closed form" for Φ_k ; that is, an expression in terms of "known" or elementary functions. For three particles with mass, Φ_3 can be written in terms of elliptic integrals. If one of the three is massless, the result, while not simple, can be exhibited explicitly -- see (A.16) below. For $k \geq 4$, the authors are aware of only two special circumstances for which closed expressions can be found, i.e. all particles massless and all but one particle massless. The first of these is well known:

$$\Phi_k(M^2; 0) = \frac{8(M^2)^{k-2}}{(4\pi)^{2k+1} (k-1)! (k-2)!}. \quad (\text{A.5})$$

The second will be derived in due course.

Consider now the phase space of $N = m + n$ particles, for which we are interested in two "clusters" of particles, one with m of the N particles in it, and the other with n in it. If the m -particle cluster is constrained to have invariant mass M_1 and the n -particle cluster invariant mass M_2 , the phase space will exhibit the quasi-two-body character, (A.3). Evidently the full N -particle phase space must be expressible as an integral over the two invariant masses M_1 and M_2 . To exhibit the two-cluster form of the phase space, it is necessary to impose the required constraints on the cluster four-momenta and masses. The identity is written

$$1 = \int dM_1^2 \int dM_2^2 \int d^4 q_1 \int d^4 q_2 \delta^{(4)} \left(\sum_{j=1}^m p_j - q_1 \right) \delta(q_1 \cdot q_1 - M_1^2) \times \\ \times \delta^{(4)} \left(\sum_{j=m+1}^N p_j - q_2 \right) \delta(q_2 \cdot q_2 - M_2^2) .$$

This is then inserted into the integrand of (A.1) for $k = N$ and the orders of integration interchanged. The resulting expression can be written as

$$\Phi_N(M^2; \mu_1^2) = \int dM_1^2 \int dM_2^2 \int d^4 q_1 \int d^4 q_2 \Phi_m(q_1 \cdot q_1; \mu_1^2) \Phi_n(q_2 \cdot q_2; \mu_1^2) \times \\ \times \delta(q_1 \cdot q_1 - M_1^2) \delta(q_2 \cdot q_2 - M_2^2) \times \delta^{(4)}(q_1 + q_2 - P) , \quad (A.6)$$

where M is the invariant mass of all N particles. Apart from the factor $\Phi_m \Phi_n$, the integral over $d^4 q_1 d^4 q_2$ is nothing more than $(2\pi)^6$ times the two-particle phase space. Thus (A.6) can be written as a two-cluster decomposition:

$$\Phi_N(M^2; \mu_1^2) = \int dM_1^2 \int dM_2^2 \Phi_m(M_1^2; \mu_1^2) \Phi_n(M_2^2; \mu_1^2) (2\pi)^6 \Phi_2(M^2; M_1^2, M_2^2) . \quad (A.7)$$

Inserting explicitly the form of Φ_2 , we find

$$\Phi_N(M^2; \mu_1^2) = \frac{\pi}{2} \int dM_1^2 \int dM_2^2 \Phi_m(M_1^2; \mu_1^2) \mathcal{F}_1\left(\frac{M_1^2}{M^2}, \frac{M_2^2}{M^2}\right) \Phi_n(M_2^2; \mu_1^2) . \quad (A.8)$$

Equation (A.8) is our basic result from which all else follows. But before embarking on further derivations, we introduce some useful dimensionless variables. From the form of (A.1) it is clear that a mass scale M_0 can be introduced in order to define a dimensionless phase space in terms of dimensionless variables:

$$\Phi_k(M^2; \mu_i^2) = M_0^{2k-4} \Phi_k\left(\frac{M^2}{M_0^2}; \frac{\mu_i^2}{M_0^2}\right). \quad (\text{A.9})$$

For (A.8) we choose $M_0 = M$ and define the dimensionless variables,

$$\begin{aligned} x &= \frac{M_1^2}{M^2}, & u_i &= \frac{\mu_i^2}{M^2} & (\text{cluster } m) \\ y &= \frac{M_2^2}{M^2}, & v_i &= \frac{\mu_i^2}{M^2} & (\text{cluster } n). \end{aligned} \quad (\text{A.10})$$

Then the two-cluster form of the $N = m + n$ particle phase space (A.8) becomes

$$\Phi_N(1; u_i, v_j) = \frac{\pi}{2} \int dx \int dy \Phi_m(x; u_i) \mathcal{F}_1(x, y) \Phi_n(y; v_j). \quad (\text{A.11})$$

We now use (A.11) to derive various special cases.

Phase space for k particles, one with mass

To obtain the phase space for k particles, one of which has mass, we use (A.11) with $N = k$, $m = k - 1$, $n = 1$, putting all the masses in cluster m equal to zero and the mass of the other particle equal to μ . Then by substituting (A.2) and (A.5) into (A.11), we find

$$\Phi_k(1; v, 0, \dots) = \frac{8(k-1)(k-2)}{(4\pi)^{2k+1} (k-1)! (k-2)!} \int_0^{x_{\max}} x^{k-3} \mathcal{F}_1(x, v) dx,$$

where $v = \mu^2/M^2$ and $x_{\max} = (1 - \sqrt{v})^2$. It is convenient to define the function,

$$f_k(z) = k(k-1) \int_0^{(1-\sqrt{z})^2} x^{k-2} \mathcal{F}_1(x, z) dx. \quad (\text{A.12})$$

Explicit forms, recursion formulas, and other properties of $f_k(z)$ are given in Appendix B. It is defined on the interval (0,1) and consists of a polynomial in z plus a polynomial in z times $z \ln z$. It is straightforward to show from (A.12) that $f_k(0) = 1$, $f_k(1) = 0$, for all k . In terms of $f_k(z)$, the phase space for k particles, one with mass, is

$$\Phi_k(1; v, 0, \dots) = \Phi_k(1; 0) f_{k-1}(v) . \quad (\text{A.13})$$

The first factor is the phase space (A.5) for k massless particles. The factor $f_{k-1}(v)$ can be thought of as a reduction factor, causing the phase space to be smaller because one of the particles has mass.

In employing (A.13) as one of the factors in (A.11) it is necessary to appreciate that v in (A.13) is the square of the mass μ divided by the square of the mass of the cluster, not the total mass of the system. Thus, with the variables defined by (A.10), the n -particle phase space in (A.11) would be

$$\Phi_n(y; v, 0, \dots) = \Phi_n(y; 0) f_{n-1}\left(\frac{v}{y}\right) , \quad (\text{A.14})$$

where one particle in the cluster n has mass μ .

Phase space for three particles, one massless and two with mass

The phase space for three particles, one massless and the other with masses $\mu_1 = \sqrt{u}M$ and $\mu_2 = \sqrt{v}M$, can be obtained from (A.11) by choosing the cluster n to be the two particles with mass and cluster m to be the single massless particle. Then, with (A.2) and (A.3), we find

$$\Phi_3(1; u, v, 0) = \Phi_3(1; 0) 2 \int_0^1 \frac{(1-y)}{(\sqrt{u}+\sqrt{v})^2} \mathcal{F}_1\left(\frac{u}{y}, \frac{v}{y}\right) dy . \quad (\text{A.15})$$

The result of elementary integration is

$$\begin{aligned} \Phi_3(1; u, v, 0) = \Phi_3(1; 0) & \left\{ (1+u+v) \mathcal{F}_1(u, v) + (u+v+|u-v|-2uv) \ln(4uv) \right. \\ & + 2(2uv-u-v) \ln | \mathcal{F}_1(u, v) + 1-u-v | \\ & \left. - 2|u-v| \ln \left| (u-v)^2 - (u+v) + |u-v| \mathcal{F}_1(u, v) \right| \right\}. \end{aligned} \quad (\text{A.16})$$

Closely related forms were evaluated 20 years ago by Yamaguchi [11] in connection with the leptonic decays of hyperons. For $u = 0$, (A.16) reduces to

$$\Phi_3(1; 0, v, 0) = \Phi_3(1; 0) \{1 - v^2 + 2v \ln v\},$$

the same as follows from (A.13).

To utilize (A.16) as a cluster component in (A.11) it is only necessary to scale u and v as in (A.14) and use the appropriate massless three-particle phase space in front.

APPENDIX B

PROPERTIES OF THE FUNCTIONS $f_k(z)$

The central function for our purposes is $f_k(z)$, defined by (A.12) for $k > 1$. For $k = 1$ this definition is useless, but comparison of (A.13) for $k = 2$ with (A.3) for $\mu_2 = 0$ shows that

$$f_1(z) = \mathcal{F}_1(z, 0) = 1 - z. \quad (\text{B.1a})$$

Direct computation with (A.12) for $k = 2, 3, 4$ yields

$$f_2(z) = 1 - z^2 + 2z \ln z \quad (\text{B.1b})$$

$$f_3(z) = 1 + 9z - 9z^2 - z^3 + 6z(1+z) \ln z \quad (\text{B.1c})$$

$$f_4(z) = 1 + 28z - 28z^3 - z^4 + 12z(1+3z+z^2) \ln z. \quad (\text{B.1d})$$

The integral (A.12) can be transformed into some factors times a special hypergeometric function. We find the alternative representations,

$$f_k(z) = \frac{2(k!)^2}{(2k)!} (1-z)^{2k-1} {}_2F_1(k-1, k; 2k; 1-z) \quad (\text{B.2})$$

$$f_k(z) = \frac{2^{2k-1}(k!)^2}{(2k)!} (1+\sqrt{z})(1-\sqrt{z})^{2k-1} {}_2F_1\left[-\frac{1}{2}, k-1; k+\frac{1}{2}; \left(\frac{1-\sqrt{z}}{1+\sqrt{z}}\right)^2\right] \quad (\text{B.3})$$

$$f_k(z) = \frac{2^{2k}(k!)^2}{(2k)!} z^{1/4} (1-\sqrt{z})^{2k-1} {}_2F_1\left[-\frac{1}{2}, \frac{3}{2}; k+\frac{1}{2}; -\frac{(1-\sqrt{z})^2}{4\sqrt{z}}\right] \quad (\text{B.4})$$

These various forms follow from the special quadratic transformation properties of $F(a, b; 2b; z)$ [12]. An explicit derivative formula [13] for $k > 1$ is

$$f_k(z) = \frac{(1-z)^{2k-1}}{(k-1)!(k-2)!} \frac{d^{k-1}}{dz^{k-1}} \left[z^k \frac{d^{k-1}}{dz^{k-1}} \left(\frac{\ln z}{1-z} \right) \right]. \quad (\text{B.5})$$

It is this expression or equivalently those in (B.1) that permit us to speak of "closed" forms or elementary functions, in spite of the presence of a hypergeometric function.

The relations of Gauss for contiguous hypergeometric functions [or direct manipulation of (A.12)] provide a recursion relation for the $f_k(z)$, valid for $k \geq 2$:

$$f_{k+1}(z) = \left(\frac{2k-1}{k-1} \right) (1+z) f_k(z) - \frac{k}{k-1} (1-z)^2 f_{k-1}(z) . \quad (\text{B.6})$$

Given f_1 and f_2 , all higher f_k can be generated with (B.6).

In numerical applications that involve k values as large as 10 or 20, the recursion formula (B.6) proves very convenient, especially as one usually wishes the individual f_k for all $k \leq k_{\text{max}}$. In practice, the recursion relation loses accuracy rapidly for large k if z is too close to unity. For all z except very small values ($z < 0.1$), the representation (B.4) converges very rapidly, especially for large k . It is an alternating series that converges absolutely for $z > (3-2\sqrt{2})^2 = 2.944 \times 10^{-2}$. Because factors involving k appear only in the denominator, the convergence for any z is better the larger is k . The first four terms in the expansion of (B.4) are

$$f_k(z) = \frac{2^{2k} (k!)^2}{(2k)!} z^{1/4} (1-\sqrt{z})^{2k-1} \left[1 + \frac{3}{2(2k+1)} \frac{(1-\sqrt{z})^2}{4\sqrt{z}} - \frac{15}{8(2k+1)(2k+3)} \frac{(1-\sqrt{z})^4}{16z} + \dots \right] . \quad (\text{B.7})$$

The rapid convergence is illustrated by the fact that even for $k = 1$ at $z = 0.25$ the three terms in (B.7) approximate $f_1(z) = 1-z$ to 1 part in 10^4 .

The other expressions, (B.2) and (B.3), converge for all z values on the interval $(0,1)$ and can be employed to generate $f_k(z)$, if desired. The presence of factors of k in the numerator of each term means poorer convergence than (B.4) for any given z , at least for large k and z near unity.

In the calculation of various moments and recursion relations for the indefinite integral of $z^{m-2} f_n(z)$ (see Appendix C) the following derivative relations are useful:

$$\frac{df_k}{dz} = \frac{k}{2z} \left[f_k - (1-z)f_{k-1} \right], \quad (\text{B.8a})$$

$$\frac{d^2 f_k}{dz^2} = \frac{k(k-1)}{z} f_{k-1}(z). \quad (\text{B.8b})$$

These equations can be combined to yield a second-order differential equation in $(1-z)$ that leads directly to the solution (B.2).

INDEFINITE INTEGRALS OF CERTAIN FUNCTIONS

In numerical calculations via Monte Carlo methods, indefinite integrals whose values can be generated via recursion formulas permit efficient use of the computer. Random values are chosen in the interval defined by the indefinite integral, rather than in a variable with a weighted distribution.

Indefinite integral corresponding to $f_k(z)$

We define the indefinite integral,

$$\hat{f}_k(t, z) = k(k-1) \int_0^t x^{k-2} \mathcal{F}_1(x, z) dx . \quad (C.1)$$

For $t = t_{\max} = (1-\sqrt{z})^2$ this is equal to $f_k(z)$. By direct calculation,

$$\hat{f}_2(t, z) = f_2(z) + (t-1-z) \mathcal{F}_1(t, z) - 4z \ln \left| \frac{1+z-t-\mathcal{F}_1(t, z)}{2\sqrt{z}} \right| . \quad (C.2)$$

For higher k values, manipulation of the integral (C.1) leads to the recursion relation, for $k \geq 2$,

$$\begin{aligned} \hat{f}_{k+1}(t, z) = & \frac{(2k-1)(1+z)}{(k-1)} \hat{f}_k(t, z) - \frac{k(1-z)^2}{(k-1)} \hat{f}_{k-1}(t, z)(1-\delta_{k,2}) + \\ & + k \left[t^{k-2} [\mathcal{F}_1(t, z)]^3 - \delta_{k,2} (1-z)^3 \right] . \end{aligned} \quad (C.3)$$

The recursion relation (B.6) follows when $t = t_{\max}$, since $\mathcal{F}_1(t_{\max}, z) = 0$.

Indefinite integral of the normalized M^2 distribution

For $N = m+n$ massless particles, the normalized distribution in $x = M_1^2/M^2$ for the invariant mass squared of cluster m is given by (14):

$$p_{mn}(x) = \frac{(m+n-1)!(m+n-2)!}{(m-1)!(m-2)!n!(n-1)!} x^{m-2} f_n(x) . \quad (C.4)$$

The range of x is $0 \leq x \leq 1$. The indefinite integral of $p_{mn}(x)$ is

$$I_{m,n}(z) = \int_0^z p_{mn}(x) dx. \quad (C.5)$$

By definition, $I_{mn}(0) = 0$, $I_{mn}(1) = 1$. The special case of $n = 1$, with $f_1(z) = 1-z$, is

$$I_{m,1}(z) = mz^{m-1} - (m-1)z^m. \quad (C.6)$$

By means of the derivative relations (B.8) and various integrations by parts, we can obtain a recursion relation in n , for fixed m , to be used for $n = 1, 2, 3, \dots$:

$$I_{m,n+1}(z) = I_{m,n}(z) + \frac{(2m+3n-1)}{2(m+n)(m+n-1)} zp_{m,n+1}(z) + \frac{z(1-z)}{2n} p_{mn}(z). \quad (C.7)$$

There is also a similar recursion relation in m , for fixed n , useful for $m = 2, 3, \dots$:

$$\begin{aligned} I_{m+1,n}(z) = I_{m,n}(z) + \frac{(n+1)n(n-1)}{2(m+n+1)(m+n)^2(m+n-1)} p_{m+1,n+1}(z) \\ - \frac{(2m+n-1)}{2(m+n)(m+n-1)} (1-z)p_{m+1,n}(z). \end{aligned} \quad (C.8)$$

To use this relation $I_{2,n}(z)$ must be generated by iteration of (C.7), beginning with $I_{2,1}(z) = 2z-z^2$.

WJ