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CONSTRAINTS ON THE DECAY RATES OF QUARKONIUM

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A B S T R A C T

We derive inequalities between the leptonic decay rates of 1S and 2S states of quarkonium, when the binding potential is an increasing concave (convex) function of the inter-quark distance in a framework where some relativistic corrections have been made to the Van Royen - Weisskopf formula for these rates. Experimental decay rates of the γ and γ' rule out the convex increasing potential.

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During the last few years one of us (A.M.) with co-workers has carried through a study of properties of quarkonium systems which can be deduced from rather general properties of the interquark potential in the framework of a non-relativistic quark model. In particular it has been shown that the ψ' wave function is greater (less) than the J/ψ wave function at the origin if the binding potential is a convex (concave) function of the inter-particle distance ¹⁾.

The value of the wave function at the origin is important since it appears in the Van Royen-Weisskopf formula ²⁾ for the leptonic width and to fit the respective experimental widths the magnitude of $\psi_{2S}(0)$, (ψ') must be 1.27 times smaller than the magnitude of $\psi_{1S}(0)$, (J/ψ). The purely convex binding potential is therefore ruled out.

Recently Bergstrom et al. ³⁾ have derived a new formula for the decay rates of quarkonium in which some relativistic corrections are made to the standard formula. Their expression for the decay rate of a vector meson V into a lepton pair is

$$\Gamma_{V \rightarrow \ell^+ \ell^-} = m_V f_V^2 \frac{4\pi\alpha^2}{3} \left[1 + \left(\frac{m_\ell}{m_V} \right)^2 \right] \left[1 - 4 \left(\frac{m_\ell}{m_V} \right)^2 \right]^{1/2} \quad (1)$$

where m_ℓ and m_V are the lepton and vector meson masses, respectively, and

$$f_V = 2 m_V^{-3/2} e_V \left| \psi_{nS}(0) - A_{nS}/3 \right| \quad (2)$$

In this last expression, e_V is the effective charge which takes into account the quark content of different vector mesons, $\psi_{nS}(0)$ is the value of the quark-antiquark wave function (assumed an S state) at the origin and

$$A_{nS} = \int \frac{d^3p}{(2\pi)^3} \left(1 - \frac{m_q}{E(p)} \right) \tilde{\psi}_{nS}(p) \quad (3)$$

with $E(p) = (m_q^2 + p^2)^{1/2}$, m_q the quark mass and $\tilde{\psi}_{nS}(p)$ the Fourier transform of $\psi_{nS}(r)$. If A_{nS} is omitted in (2), then it reduces to the Van Royen-Weisskopf formula. A_{nS} is therefore the correction term to this formula and it has been shown in Ref. 3) that it gives corrections of up to 24% in the decay rates of charmonium states.

We will now generalize the earlier results of one of us (A.M.) by deriving inequalities between $|\psi_{nS}(0) - A_{nS}/z|$ where $n=1,2$ for concave and convex potentials which are non-decreasing. A preliminary version of this work has been presented by one of us (A.K.C.)⁴⁾. To start we note that it is straightforward to write A_{nS} as an integral over co-ordinate space⁵⁾, i.e.,

$$A_{nS} = \frac{1}{2\sqrt{\pi}} u'_n(0) - \frac{m_q^2}{\pi^{3/2}} \int_0^\infty u_n(r) K'_0(m_q r) dr, \quad (4)$$

where $u_n(r) = r\psi_{nS}(r)\sqrt{4\pi}$ and K_0 is the modified Bessel function.

Our method is to use properties of $u_1(r)$ and $u_2(r)$ proved in Ref. 1) for concave (convex) potentials. Consider first the case when $V(r)$ is a concave function of r . Then choosing the over-all sign of $u_1(r)$ such that it is positive close to $r=0$, then

$$u'_1(0) > u'_2(0) > 0. \quad (5)$$

Also it is proved in Section 2 of the above work that

$$u_1^2(r) - u_2^2(r) \neq 0, \quad r_0 \geq r > 0 \quad (6)$$

where $r=r_0$ is the unique zero of $u_2(r)$ in $(0, \infty)$. Remembering that the ground state has no nodes, it follows that

$$u_1(r) - u_2(r) > 0, \quad 0 < r < \infty. \quad (7)$$

Therefore,

$$\begin{aligned} \psi_{1S}(0) - \psi_{2S}(0) &= (A_{1S} - A_{2S})/3 \\ &= \frac{2}{6\sqrt{\pi}} [u'_1(0) - u'_2(0)] + \frac{m_q^2}{3\pi^{3/2}} \int_0^\infty |K'_0(m_q r)| [u_1(r) - u_2(r)] dr > 0 \quad (8) \end{aligned}$$

In Appendix I, we will show that if dV/dr for $r > 0$, then

$$u_2'(r) > -u_2'(0) \quad r > 0. \quad (9)$$

Therefore

$$\begin{aligned} & \psi_{1s}(0) + \psi_{2s}(0) - (A_{1s} + A_{2s})/3 \\ &= \frac{1}{3\sqrt{\pi}} [u_1'(0) + u_2'(0)] + \frac{m_q^2}{3\pi^{3/2}} \int_0^\infty K_0(m_q r) [u_1(r) + u_2(r)] dr \\ &\geq \frac{2}{3\sqrt{\pi}} u_2'(0) + \frac{m_q}{3\pi^{3/2}} \int_0^\infty K_0(m_q r) u_2'(r) dr \\ &\geq \frac{2}{3\sqrt{\pi}} u_2'(0) - \frac{1}{6\sqrt{\pi}} u_2'(0) > 0. \end{aligned} \quad (10)$$

Combining (8) with (10),

$$|\psi_{1s}(0) - A_{1s}/3| > |\psi_{2s}(0) - A_{2s}/3| \quad (11)$$

In the limit case of a linear potential, the right-hand side of (8) is still strictly positive.

For the case of convex potentials we have not been able to prove the reverse inequality to (11) but only the weaker inequality ^{*)}:

$$\frac{3}{2} |\psi_{2s}(0) - A_{2s}/3| > |\psi_{1s}(0) - A_{1s}/3|. \quad (12)$$

*) This improves by a factor of 2 the corresponding result proved by one of us (A.K.C.) in an earlier work ⁴⁾.

This inequality is equivalent to the requirement that

$$\frac{3}{2} \left[\frac{1}{3\sqrt{\pi}} u_2'(0) + \frac{m_q}{3\pi^{3/2}} \int_0^{\infty} \frac{du_2}{dr} K_0(m_q r) dr \right] \\ > \frac{1}{3\sqrt{\pi}} u_1'(0) + \frac{m_q}{3\pi^{3/2}} \int_0^{\infty} \frac{du_1}{dr} K_0(m_q r) dr, \quad (13)$$

since the right-hand side of this inequality is positive. We show in Appendix I that

$$u_1'(r) < u_1'(0) \quad (14)$$

and in Appendix II that

$$\int_0^{\infty} \frac{du_2}{dr} K_0(m_q r) dr > 0. \quad (15)$$

Using (9), (14) and (15) and the integral $m_q \int_0^{\infty} K_0(m_q r) dr = \pi/2$, we see that (13) and hence (12) is indeed satisfied.

As mentioned earlier experimental decay rates ⁷⁾ for the charmonium system would, if we believe the results of Ref. 4), imply that

$$1.27 \left| \psi_{2S}(0) - A_{2S}/3 \right| \simeq \left| \psi_{1S}(0) - A_{1S}/3 \right|, \quad (16)$$

and neither (11) and (12) are violated ; which means that we cannot in this way exclude a purely convex potential. We must nevertheless keep in mind that (12) is not really optimal : one gets (12) by letting $m_q \rightarrow 0$ in the left-hand side of (13) and $m_q \rightarrow \infty$ in the right-hand side of (13).

However, for bottomium, $\Gamma_{\Upsilon \rightarrow e^+e^-} = 1.2 \pm 0.2$ keV and $\Gamma_{\Upsilon' \rightarrow e^+e^-} = 0.33 \pm 0.12$ keV. Taking $M_{\Upsilon} = 9.46$ GeV, $M_{\Upsilon'} = 10.02$ GeV and the central values for the above decay rates, then corresponding to (16), we find

$$1.70 \left| \psi_{2S}(0) - A_{2S}/3 \right| \simeq \left| \psi_{1S}(0) - A_{1S}/3 \right| \quad (17)$$

so that in this case (12) is violated and purely convex increasing potentials are ruled out for bottomium, with more than two standard deviations.

Using the computed values of the decay rates for J/ψ and ψ' given in Table 1 of Bergström et al.⁴⁾ for the harmonic oscillator potential considered by Jackson⁷⁾, we have

$$\left| \psi_{2s}(0) - A_{2s}/3 \right| = 1.16 \left| \psi_{1s}(0) - A_{1s}/3 \right| \quad (18)$$

so that the inequality (12) is far from being saturated in this case and there is a possibility that it can be improved still further.

In conclusion, though there may be quantitative changes to the Van Royen-Weisskopf formula, the qualitative features of the quark-antiquark potential that one can deduce rigorously by applying it to experimental data are not drastically changed and one can expect to show, by working harder, that these changes are indeed small.

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APPENDIX I

We prove here the inequalities

$$u_2'(r) > -u_2'(0) \quad ; \quad u_1'(r) < u_1'(0) \quad , \quad r > 0$$

used earlier.

If as we are assuming $dV/dr > 0$, then

$$u_2''(r) = - [V(r) - E_2] u_2(r) \tag{A.1}$$

is zero once when $V(r) = E_2$ and once when $u_2(r) = 0$, i.e., at $r = r_0$. Now $u_2''(r) = 0$ for r between the minimum of u_2 and ∞ [corresponding to a local maximum of $u_2'(r)$] occurs when $V(r) = E_2$. Therefore the other zero ($r = r_0$) corresponds to the minimum value of $u_2'(r)$, i.e.,

$$u_2'(r) > u_2'(r_0) = - |u_2'(r_0)| \tag{A.2}$$

But

$$\begin{aligned} [u_2'(r_0)]^2 - [u_2'(0)]^2 &= 2 \int_0^{r_0} u_2'(r) u_2''(r) dr \\ &= 2 \int_0^{r_0} u_2'(r) u_2(r) V(r) dr \\ &= - \int_0^{r_0} u_2^2(r) \frac{dV}{dr} dr < 0 \end{aligned} \tag{A.3}$$

Therefore

$$u_2'(r) \geq u_2'(r_0) > -u_2'(0) \quad r > 0 \quad \text{Q.E.D.}$$

Finally since $(dV/dr) > 0$ for all $r > 0$, $u_1''(r) = -[E_1 - V(r)]u_1(r)$ has a unique zero at $r = r_2$, say, and is negative for $0 < r < r_2$. Therefore

$$u_1'(0) > u_1'(r)$$

for all $0 < r < r_2$, and since $u_1'(r)$ is negative for $r > r_2$, inequality (A.4) holds for all $0 < r < \infty$.

APPENDIX II

We prove here that for convex increasing potentials

$$\int_0^{\infty} \frac{du_2(r)}{dr} K_0(m_q r) dr = m_q \int_0^{\infty} u_2(r) K_1(m_q r) dr > 0. \quad (\text{A.4})$$

Let

$$\omega = k_1(\lambda r) \equiv (\lambda r)^{1/2} K_1(\lambda r) \quad (\text{A.5})$$

with $\lambda > 1$. Then,

$$\omega'' - \left(\lambda^2 - \frac{3}{4r^2}\right) \omega = 0 \quad (\text{A.6})$$

and

$$k_1(r) \frac{d}{dr} k_1(\lambda r) - k_1(\lambda r) \frac{d}{dr} k_1(r) = (1 - \lambda^2) \int_r^{\infty} k_1(r') k_1(\lambda r') dr' < 0. \quad (\text{A.7})$$

Therefore

$$K'_0(M_q r) / K'_0(m_q r) = K_1(M_q r) / K_1(m_q r)$$

is a decreasing function of r for $M_q > m_q$ so that

$$\int_0^{\infty} u_2(r) \left[\frac{K_1(M_q r)}{K_1(M_q r_0)} - \frac{K_1(m_q r)}{K_1(m_q r_0)} \right] dr > 0. \quad (\text{A.8})$$

As a result (A.4) holds for all $m_q > 0$, if we can prove it to be true for $m_q \rightarrow 0$, i.e., if we can prove

$$\int_0^{\infty} \frac{u_2(r)}{r} dr > 0 \quad (\text{A.9})$$

To prove this we compare $u_2(r)$ with $v(r)$ which is the solution of the Schrödinger equation corresponding to a potential which is constant for $0 \leq r \leq r_0$ and infinite for $r > r_0$, i.e., $v(r) \sim \sin(\pi r / r_0)$. By standard Wronskian arguments it is easy to prove that u_2/v decreases in $(0, r_0)$ if the potential $V(r)$ is increasing. Hence for $0 \leq r \leq r_0$

$$u_2(r) > \frac{r_0}{\pi} |u'_2(r_0)| \sin\left(\frac{\pi r}{r_0}\right). \quad (\text{A.10})$$

In $r > r_0$, we compare u_2 to the solution w of the Schrödinger equation for a linear potential which is an Airy function. One can choose the scale such that $r_0 = x_1 - x_2$ where x_1 and x_2 are the first two zeros of the Airy function. Then

$$w'' - (r + x_2) w = 0 \quad (\text{A.11})$$

By using the Wronskians of u_2 and w , and from the convexity of the potential the fact that $V - E_2 - (r + x_2)$ vanishes only once for $r > r_0$, it follows that for these values of r ,

$$|u_2(r)| < \frac{|u_2'(r_0)|}{w'(r_0)} w(r) = \frac{|u_2'(r_0)|}{A_i'(x_1)} A_i(r + x_2). \quad (\text{A.12})$$

Using (A.10) and (A.12),

$$\int_0^\infty \frac{u_2(r)}{r} dr > |u_2'(r_0)| \left[\frac{x_1 - x_2}{\pi} \int_0^\pi \frac{\sin r}{r} dr - \frac{1}{A_i'(x_1)} \int_{x_1}^\infty \frac{A_i(x) dx}{x - x_2} \right] \quad (\text{A.13})$$

and it is straightforward to check numerically that the right-hand side is positive so that (A.9) and hence (A.4) follows.

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