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SYMMETRY BREAKING PATTERNS IN SU(5)

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A B S T R A C T

The various symmetry breaking patterns that are possible with a 24-plet and a 5-plet of Higgs fields in SU(5) are classified. Spontaneous breakdown of SU(5) to $SU(3)_{\text{colour}} \times U(1)_{\text{em}}$ can be achieved (excluding the gauge hierarchy problem) without imposing unnatural constraints on the parameters of the Higgs potential.

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The $SU(5)$ model of Georgi and Glashow ¹⁾ provides the simplest example of a theory that unifies the electroweak (à la Weinberg-Salam) ²⁾ and the strong (QCD) interactions. A careful analysis of renormalization effects in the model ³⁾⁻⁵⁾, associated with the electroweak and the QCD couplings, has resulted in a fairly reliable estimate ⁵⁾ for the proton lifetime ($\tau_{\text{proton}} \approx 10^{31 \pm 2}$ yr). This remarkable prediction should be experimentally testable in the near future. Two predictions which have been tested, and agree quite well with experiments, are the values of the weak mixing angle $\theta_W^{(3)}$ and the mass of the b quark ⁴⁾. The last prediction utilizes, in an essential way, the Higgs sector of $SU(5)$, an aspect of the model that has not been explored in detail and deserves further attention.

A heuristic discussion of the Higgs potential [involving one adjoint and one vector representation of $SU(5)$], including radiative corrections, has previously been carried out in Ref. 4). The purpose of this note is to provide a more complete discussion of the Higgs potential in $SU(5)$, mainly in the semi-classical approximation. We classify the various possible symmetry breaking patterns that are possible with a 24-plet and a 5-plet of Higgs fields. Care is required in the choice of values of the parameters of the Higgs potential, in order to obtain the desired symmetry breaking. However, this still leaves one with enough freedom in the range of values of all the parameters, so that no unnatural constraint has to be imposed on them. Although the gauge hierarchy problem remains ⁶⁾, it is hoped that the ensuing analysis shows that, at least, one aspect of the symmetry breaking problem ^{*)}, the naturalness of the symmetry breaking pattern, can be satisfactorily resolved.

As pointed out in Ref. 1), the minimal Higgs system which can be considered involves an adjoint 24 representation ϕ_i^j (regarded as a traceless Hermitian 5×5 matrix) and a fundamental 5-plet H_i ($i, j = 1, \dots, 5$). Imposing a discrete symmetry $\phi_i^j \rightarrow -\phi_i^j$, the most general Higgs potential takes the form ^{**) :}

*) There are two aspects to the symmetry breaking problem. Firstly, to find the conditions on the parameters under which $SU(5)$ can break spontaneously down to $SU(3)_C \times U(1)_{\text{em}}$ and secondly to show that the symmetry breaking can be hierarchical in nature.

**) We use the notation of Ref. 4) in defining the coupling parameters in the Higgs potential.

$$\begin{aligned}
 V(\phi, H) = & -\frac{1}{2}\mu^2 \text{Tr}(\phi^2) + \frac{a}{4} (\text{Tr} \phi^2)^2 + \frac{b}{2} \text{Tr}(\phi^4) \\
 & -\frac{1}{2}\nu^2 H^\dagger H + \frac{\lambda}{4} (H^\dagger H)^2 \\
 & + \alpha (H^\dagger H) \text{Tr}(\phi^2) + \beta H^\dagger \phi^2 H
 \end{aligned} \tag{1}$$

The matrix $\phi_{\perp}^j$ can be diagonalized by a $SU(5)$ gauge transformation :

$$\phi_i^j = \delta_i^j \phi_i \tag{2}$$

$$\sum_{i=1}^5 \phi_i = 0 \tag{3}$$

Simultaneously, the complex five-vector $H_{\perp}$ can be chosen to be real up to an over-all phase factor :

$$H_i = x \hat{H}_i, \quad x \in \mathbb{C} \tag{4}$$

$$\sum_{i=1}^5 \hat{H}_i^2 = 1, \quad \hat{H}_i \in \mathbb{R} \tag{5}$$

In terms of the eleven real variables ϕ_i , $|x|$ and $\hat{H}_i$, the Higgs potential reads as follows :

$$\begin{aligned}
 V(\phi_i, |x|, \hat{H}_i) = & -\frac{\mu^2}{2} \sum_{i=1}^5 \phi_i^2 + \frac{a}{4} \left(\sum_{i=1}^5 \phi_i^2 \right)^2 \\
 & + \frac{b}{2} \sum_{i=1}^5 \phi_i^4 - \frac{1}{2}\nu^2 |x|^2 + \frac{\lambda}{4} |x|^4 \\
 & + \alpha |x|^2 \sum_{i=1}^5 \phi_i^2 + \beta |x|^2 \sum_{i=1}^5 \hat{H}_i^2 \phi_i^2
 \end{aligned} \tag{6}$$

We are interested only in a non-vanishing vacuum expectation value $\langle x \rangle$. Consequently, the value $x = \langle x \rangle$ at the minimum of V is taken as an independent parameter which determines the dimensional quantity v . The precise relation between $\langle x \rangle$ and v will be given later. Our aim now is to display the vacuum expectation values $\langle \phi_i \rangle$ and $\langle H_i \rangle$, and the associated $SU(5)$ breaking patterns, as functions of the six remaining coupling parameters of the Higgs potential (1). Let us choose the following convention for the labelling of ϕ_i :

$$\begin{aligned} \phi_5^2 &\leq \phi_i^2 \quad (1 \leq i \leq 4) \quad , \quad \beta > 0 \\ \phi_5^2 &\gg \phi_i^2 \quad (1 \leq i \leq 4) \quad , \quad \beta < 0 \end{aligned} \quad (7)$$

The potential (6) is minimal with respect to a variation of $\hat{H}_i$ if and only if

$$\langle \hat{H}_i \rangle^2 (\langle \phi_5 \rangle^2 - \langle \phi_i \rangle^2) = 0 \quad (1 \leq i \leq 4) \quad (8)$$

At the minimum (with respect to variation of $\hat{H}_i$), the last term in V becomes

$$\beta |\langle x \rangle|^2 \sum_{i=1}^5 \hat{H}_i^2 \phi_i^2 = \beta |\langle x \rangle|^2 \phi_5^2 \quad (9)$$

Next, we add a Lagrange multiplier term $-\sigma \sum_{i=1}^5 \phi_i$ to the potential (6). The stationarity conditions for variations with respect to the variables ϕ_i are as follows :

$$\left[-\mu'^2 + a \sum_{j=1}^5 \phi_j^2 + 2b \phi_i^2 \right] \phi_i = \sigma \quad (1 \leq i \leq 4) \quad (10)$$

$$\left[-\mu'^2 + 2\beta |\langle x \rangle|^2 + a \sum_{j=1}^5 \phi_j^2 + 2b \phi_5^2 \right] \phi_5 = \sigma \quad (11)$$

where

$$\mu'^2 = \mu^2 - 2\alpha |\langle x \rangle|^2 \quad (12)$$

Comparison of (10) and (11) gives

$$\langle \phi_i \rangle^2 \neq \langle \phi_5 \rangle^2 \quad (1 \leq i \leq 4) \quad (13)$$

Hence, Eq. (8) implies that the 5-plet vacuum expectation value takes the form

$$\langle H_i \rangle = \langle x \rangle \delta_{i5} \quad (1 \leq i \leq 5) \quad (14)$$

Note that the special correlation between the symmetry breakings induced by the vacuum expectation values of the 24-plet and the 5-plet of Higgs fields is valid for all the allowed values of the coupling parameters. Clearly, $\langle x \rangle$ may be made real by a suitable gauge transformation.

From the stationarity equations (10) and (11), and the constraints (3) and (7), one can determine the values $\phi_i = \langle \phi_i \rangle$ which minimize V . Remember that (13) rules out the equality sign in (7). Subtracting repeatedly the cubic equations (10) for ϕ_i ($i \leq 4$) from each other, it is easily seen⁷⁾ that the four eigenvalues $\phi_1, \dots, \phi_4$ can take, at most, three different values $\{\delta_1, \delta_2, \delta_3\}$. Moreover, one has the constraint :

$$\delta_1 + \delta_2 + \delta_3 = 0 \quad (15)$$

This leads to a first classification of all the $SU(5)$ symmetry breaking patterns which are possible with the classical Higgs potential (1).

Depending on the multiplicities of the first four eigenvalues $\langle \phi_i \rangle$ ($i \leq 4$), one distinguishes four different cases [permutations of the first four eigenvalues are also allowed ; the fifth eigenvalue is fixed by the tracelessness condition (3)].

Case (a) : first four eigenvalues equal :

$$\langle \phi_i^j \rangle = \begin{pmatrix} \delta & & & & \\ & \delta & & & \\ & & \delta & & \\ & & & \delta & \\ 0 & & & & -4\delta \end{pmatrix} \quad (16a)$$

Case (b) : first three eigenvalues equal :

$$\langle \phi_i^j \rangle = \begin{pmatrix} \delta & & & & \\ & \delta & & & \\ & & \delta & & \\ & & & \delta(-\frac{3}{2}+\epsilon) & \\ & & & & \delta(-\frac{3}{2}-\epsilon) \end{pmatrix} \quad (16b)$$

Case (c) : the first two and the second two eigenvalues are the same :

$$\langle \phi_i^j \rangle = \begin{pmatrix} \delta & & & & \\ & \delta & & & \\ & & \delta(1+\epsilon) & & \\ & & & \delta(1+\epsilon) & \\ & & & & -2\delta(2+\epsilon) \end{pmatrix} \quad (16c)$$

Case (d) : only the first two eigenvalues are equal :

$$\langle \phi_i^j \rangle = \begin{pmatrix} \delta & & & & \\ & \delta & & & \\ & & \delta(-1+\epsilon) & & \\ & & & -\epsilon\delta & \\ & & & & -\delta \end{pmatrix} \quad (16d)$$

Note that for Case (d), the three different eigenvalues in the set of first four add up to zero, as required by (15).

In order to solve for the system of stationarity conditions (10), (11), we subtract the set of four equations (10) from Eq. (11), thereby eliminating the Lagrange multiplier σ . One obtains

$$(\phi_5 - \phi_i) \left[-\mu'^2 + a \sum_{j=1}^5 \phi_j^2 + 2b(\phi_5^2 + \phi_5\phi_i + \phi_i^2) \right] = -2\beta \langle x \rangle^2 \phi_5$$

$$(1 \leq i \leq 4) \quad (17)$$

For Case (a), the set of four equations (17) are identical. They reduce to just one equation for δ ($\neq 0$).

In Cases (b) to (d), only two of the four equations (17) are linearly independent. Choosing any pair $\{\phi_i, \phi_j\}$ with $\phi_i \neq \phi_j$ ($i, j \leq 4$), the system (10), (11) is equivalent to the following two equations :

$$-\mu'^2 + a \sum_{j=1}^5 \phi_j^2 + 2b (\phi_i^2 + \phi_i \phi_j + \phi_j^2) = 0 \quad (18)$$

$$(\phi_5 - \phi_i)(\phi_5 - \phi_j)(\phi_5 + \phi_i + \phi_j) = -\frac{B}{b} \langle x \rangle^2 \phi_5 \quad (19)$$

Defining

$$\omega = \frac{B}{b} \frac{\langle x \rangle^2}{\delta^2} \quad (20)$$

Eq. (19) reduces to the following quadratic equations for ϵ in the three Cases (b), (c) and (d) :

$$\epsilon \left(\frac{5}{2} + \epsilon \right) = -\frac{\omega}{4} \left(\frac{3}{2} + \epsilon \right) \quad (21b)$$

$$(5 + 2\epsilon)(5 + 3\epsilon) = -2\omega \quad (21c)$$

$$\epsilon(\epsilon - 1) = \frac{\omega}{2} \quad (21d)$$

Equation (18) can be used to solve for δ . Corresponding to the two roots of the quadratic equations (21b-d), each of the three cases (b), (c) and (d) splits into two subcases. Together with Case (a), one has to consider seven distinct possibilities for $\langle \phi_1^j \rangle^*$, each one corresponding to a stationary point of the classical Higgs potential. From among these seven possibilities, we have to identify the absolute minimum.

*) There are seven stationary points plus permutations of the first four eigenvalues of $\langle \phi_1^j \rangle$.

Till now, we did not make any assumption on the ratio $\langle x \rangle^2 / \delta^2$ of the vacuum expectation values of the $\underline{5}$ and the $\underline{24}$ representations. Let us now concentrate on the physically more relevant case

$$\frac{\langle x \rangle^2}{\delta^2} \ll 1 \quad (22)$$

Note, however, that (22) does not have to mean "hierarchically" small. A $(\text{mass})^2$ ratio of the order of g^2 (g is the gauge coupling constant), which can be natural ⁸⁾, would at this stage suffice to carry through our arguments [of course, for $SU(5)$ $\langle x \rangle^2 / \delta^2$ is hierarchically small].

Since $\omega \ll 1$, one may employ the results of Ref. 7) concerning the minimum of the classical Higgs potential when only the 24-plet is present.

For $a > -7/15b$, $b > 0$, it follows from Ref. 7) and continuity that the eigenvalues of $\langle \phi_i^j \rangle$ approach the $SU(3) \times SU(2) \times U(1)$ symmetry breaking pattern as $\omega \rightarrow 0$. For $a > 0$, $b < 0$, $|b| < 10/13a$, the eigenvalues of $\langle \phi_i^j \rangle$ approach the $SU(4) \times U(1)$ symmetry breaking pattern for $\omega \rightarrow 0$. This characterization of $\langle \phi_i^j \rangle$, together with the constraint (7), uniquely picks out the absolute minimum of the Higgs potential from the seven extrema (plus permutations of the first four eigenvalues of $\langle \phi_i^j \rangle$ of the potential V).

Let us list the solutions of Eqs. (21b-d) :

$$\begin{aligned} \epsilon_1 &= -\frac{3}{20}\omega + O(\omega^2) \\ \epsilon_2 &= -\frac{5}{2} - \frac{\omega}{10} + O(\omega^2) \end{aligned} \quad (23b)$$

$$\begin{aligned} \epsilon_1 &= -\frac{5}{2} + \frac{2}{5}\omega + O(\omega^2) \\ \epsilon_2 &= -\frac{5}{3} - \frac{2}{5}\omega + O(\omega^2) \end{aligned} \quad (23c)$$

$$\begin{aligned} \epsilon_1 &= 1 + \frac{\omega}{2} + O(\omega^2) \\ \epsilon_2 &= -\frac{\omega}{2} + O(\omega^2) \end{aligned} \quad (23d)$$

These values of ϵ have to be substituted into Eqs. (16b), (16c) and (16d).

Discussion of the symmetry breaking patterns

Case (a) represents a minimum of V for $a > 0$, $b < 0$ and $\beta < 0$. It corresponds to a strong breaking of $SU(5)$ down to $SU(4) \times U(1)$, and a further weak breaking down to $SU(4)$.

Case (b) The first root in Case (b) $[\epsilon_1$ in (23b)] gives the absolute minimum for $a > 0$, $b > 0$ and $\beta < 0$. Note that the negative sign of ω is just right to make ϕ_5^2 maximal, as required by (7). This is the desired symmetry breaking pattern :

$$SU(5) \xrightarrow{\text{strong}} SU(3) \times SU(2) \times U(1) \xrightarrow{\text{weak}} SU(3) \times U(1)$$

The second root in Case (b) gives the sequence

$$SU(5) \xrightarrow{\text{strong}} SU(4) \times U(1) \xrightarrow{\text{weak}} SU(3) \times U(1)$$

and is not acceptable since it implies a strong breaking of the flavour $SU(2) \times U(1)$ subgroup. Nevertheless, this symmetry breaking pattern describes the ground state of the Higgs potential if $a > 0$, $b < 0$ ($|b| < 10/13a$) and $\beta > 0$.

Case (c) The two roots of Case (c), although they are different, lead to physically equivalent symmetry breakings. The equivalence can be seen by a suitable permutation of the first four eigenvalues of $\langle \phi_i^j \rangle$ and the following redefinition of δ :

$$\delta \rightarrow \delta \left(-\frac{2}{3} - \frac{2}{3}\omega \right)^{-1}$$

The minimum of V belongs to Case (c) if $a > 0$, $b > 0$ and $\beta > 0$. The associated symmetry breaking pattern is

$$SU(5) \xrightarrow{\text{strong}} SU(3) \times SU(2) \times U(1) \xrightarrow{\text{weak}} SU(2) \times SU(2) \times U(1)$$

Case (d) Under the assumption (22), one checks easily that this case never corresponds to the absolute minimum of the classical Higgs potential.

For the convenience of the reader, the above analysis of the symmetry breaking patterns in $SU(5)$ with a 24-plet and a 5-plet of Higgs fields is summarized in the Table.

In our analysis we treated $\langle x \rangle$ as an independent variable which replaces the mass scale v in the Higgs potential (1). Furthermore, we took $\delta \neq 0$. For the desired symmetry breaking pattern, the relations linking $\langle x \rangle$ and δ to the scales μ and v are given by

$$v^2 = \lambda \langle x \rangle^2 + 15 \alpha \delta^2 + \frac{9}{2} \beta \delta^2 + 6 \beta \epsilon_1 \delta^2 + \epsilon_1^2 \delta^2 (4\alpha + 2\beta) \quad (24)$$

$$\mu^2 = 2\alpha \langle x \rangle^2 + a \delta^2 \left(\frac{15}{2} + 2\epsilon_1^2 \right) + 2b \delta^2 \left(\frac{7}{4} - 2\epsilon_1 + \epsilon_1^2 \right) \quad (25)$$

where

$$\epsilon_1 = - \left(\frac{5}{4} + \frac{\omega}{8} \right) + \left[\left(\frac{5}{4} + \frac{\omega}{8} \right)^2 - \frac{3}{8} \omega \right]^{1/2} \quad (26)$$

[for $\omega \ll 1$, ϵ_1 reduces to the first root in (23b)]. Equation (25) is nothing but Eq. (18) evaluated for Case (b). Clearly, a hierarchically small ratio $\langle x \rangle^2 / \delta^2$ implies an unnatural compensation between quantities of order δ^2 in the combined system (24), (25) which gets spoiled, in general ⁸⁾, by higher order radiative corrections. We have nothing new to add to the gauge hierarchy problem. However, we stress that our analysis did not depend on the hierarchy constraint. We made the assumption (22) only to simplify the final step of the analysis.

In conclusion, we have shown that the symmetry breaking of $SU(5)$ down to $SU(3)_c \times U(1)_{em}$ is possible without imposing unnatural constraints on the coupling parameters. Provided one stays away from the transition regions separating the various symmetry breaking patterns (see the Table), the desired symmetry breakdown is guaranteed to be stable under radiative corrections by the Georgi-Glashow theorem ⁹⁾.

NOTE ADDED

After completion of this work, a CERN preprint by J. Ellis et al. (TH. 2696) has appeared in which the $SU(5)$ Higgs potential with no dimensional couplings (such as mass parameters) is discussed.

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	$\beta > 0$	$\beta < 0$
$a > -7/15b, b > 0$	$SU(5)$ $\downarrow$ $SU(3) \times SU(2) \times U(1)$ $\downarrow$ $SU(2) \times SU(2) \times U(1)$	$SU(5)$ $\downarrow$ $SU(3) \times SU(2) \times U(1)$ $\downarrow$ $SU(3) \times U(1)$ Desired breaking
$a > 0, b < 0$ $ b < 10/13a$	$SU(5)$ $\downarrow$ $SU(4) \times U(1)$ $\downarrow$ $SU(3) \times U(1)$	$SU(5)$ $\downarrow$ $SU(4) \times U(1)$ $\downarrow$ $SU(4)$

TABLE - Symmetry breaking patterns in $SU(5)$ with a 24-plet and a 5-plet of Higgs fields. The desired symmetry breakdown is achieved for $a > -7/15b, b > 0$ and $\beta < 0$, where a, b and β are defined in Eq. (1).

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