Ref.TH.2696-CERN

A HIERARCHY OF GAUGE HIERARCHIES

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ABSTRACT

We investigate the possibility that there is no intrinsic mass scale below the Planck mass m_P , and that both stages of symmetry breaking in grand unified theories may be generated by radiative corrections. In such a scheme it is plausible that $1 \gg \gg m_X/m_P \gg m_W/m_X$, where the subscript X denotes the superheavy bosons in grand unified theories. This hierarchy of gauge hierarchies arises because the X bosons get their masses from Higgs representations with larger non-Abelian charges than the Higgs responsible for the W and Z masses. In a specific SU(5) model, we find that m_X/m_P is naturally $O(10^{-5})$, whereas m_W/m_X can be much smaller. However, it can only be achieved in SU(5) at the expense of one unnatural condition on the Higgs potential.

Ref.TH.2696-CERN

10 July 1979

1. INTRODUCTION

Our present physical theories contain many mass scales. QCD has an intrinsic scale $\Lambda = O(1)$ GeV at which the strong interactions become strong. The weak interactions have a scale of $O(10^2)$ GeV associated with the spontaneous symmetry breaking which determines the weak vector boson masses m_W, m_Z . In addition, the normal pattern of spontaneous symmetry breaking involves the introduction of an as yet unknown mass parameter μ which determines the Higgs boson mass: $m_H = \sqrt{2}\mu$ in the Weinberg-Salam model [1]. The fermions involved in the strong and weak interactions have masses m_f which are already known to vary over a wide range: $O(1)$ MeV for m_e to $O(10)$ GeV for m_t . People now speculate about the grand unification of strong, weak and electromagnetic interactions [2], and even the minimal such theories introduce one new mass scale [2,3] of grand unification of $O(10^{14}$ to $10^{15})$ GeV associated with superheavy gauge bosons X . Beyond all these scales we have the intrinsic scale of gravitational interactions, the Planck mass $m_P \approx 2 \times 10^{19}$ GeV. Many other mass scales may also exist: perhaps a mass associated with supersymmetry or supergravity which is sometimes thought to be $O(m_P)$ [4]: perhaps other gauge interactions whose symmetries are broken either by a Higgs mechanism or dynamically [5].

Our attempts to understand this proliferation of mass scales and their interrelations must proceed in the context of gauge theories, since these provide our models of low energy ($< m_P$) phenomena. But the mass scales are probably injected by considerations which are not purely gauge-theoretical. After all, a great strength of gauge theories is that their crucial property of renormalizability [6] survives the introduction of mass scales by any amount of spontaneous symmetry breaking. The key to many of the low-energy mass scales is provided by the couplings of Higgs fields, either to themselves or to fermions, and these are not constrained by the gauge principle. For example, in the minimal Weinberg-Salam [1] model

$$m_H^2/m_W^2 = 8 \left(\frac{\lambda}{g^2} \right) = O(1)? \quad (1.1)$$

where λ is the $|H|^4$ coupling, and

$$\frac{m_f^2}{m_W^2} = 4 \left(\frac{g_{ffH}^2}{g^2} \right) = O(1)? \quad (1.2)$$

If we accept the approximate validity of relations like (1.1) or (1.2), then the minimal set of scales to be understood is (Λ, m_W, m_X, m_P) . These scales are grossly disparate. We immediately notice that

$$\left(\frac{\Lambda}{m_X} \right) = O(10^{-15}) = O(\exp - \frac{C}{\alpha})? \quad (1.3)$$

$$\left(\frac{m_W}{m_X} \right) = O(10^{-13}) = O(\exp - \frac{C'}{\alpha})? \quad (1.4)$$

where C and C' should be constants of order 1. Grand unified theories [2] actually give a calculational basis [3] for (1.3), since they start off with a grand unified coupling constant $\alpha_{GUM} = O(\alpha)$ at a scale $O(m_X) \approx 10^{15}$ GeV, whose SU(3) subgroup coupling evolves logarithmically according to the renormalization group, increasing to $O(1)$ at a scale $O(\Lambda) \sim 1$ GeV

$$\ln \left(\frac{m_X^2}{\Lambda^2} \right) \approx \frac{12\pi}{(33-2N_q)} \frac{1}{\alpha_{GUM}} : \alpha_{GUM} \approx \frac{1}{40} \quad (1.5)$$

where N_q is the number of quark flavours. Indeed, this argument is used in reverse to make a phenomenological estimate of $m_X \approx (10^{14} \text{ to } 10^{15})$ GeV [7]. It is natural to suspect that the small ratio (1.4), commonly called a gauge hierarchy, may arise similarly from the logarithmic variation of some renormalizable coupling constant. In order for such an extreme hierarchy $O(10^{-13})$ to be attained, it is clear that the Higgs mass scale μ must be microscopic on the scale of grand unification. This is something that can be arranged [8], although it seems very unnatural at our present level of understanding. Weinberg [9] has recently shown that this arbitrary "miniaturization" of μ is

not only necessary but also sufficient to permit a hierarchy like (1.4) to be attained at the one-loop level. The logarithmic variation this time is that of the $|H|^4$ coupling λ , which bounds the possible magnitude of hierarchy. Indeed, it is natural to suspect [10] that this bound may be almost saturated, by dint of μ being zero or at any rate negligible. In this case the weak interaction spontaneous symmetry breakdown and the Higgs boson mass are determined by radiative corrections [11], so that if there is just one weak isodoublet of Higgs fields

$$m_H^2 = \frac{3\alpha^2}{8\sqrt{2}G_F} \left\{ \frac{2 + \sec^2 \Theta_W}{\sin^4 \Theta_W} - O\left(\frac{m_f}{m_W}\right)^4 \right\} \quad (1.6)$$

in the Weinberg-Salam model, yielding $m_H \approx 10$ GeV if $\sin^2 \Theta_W \approx 0.20$. The phenomenology of a Higgs boson with mass $O(10)$ GeV is amusing [10], and it seems possible that such a particle might be found quite soon if it exists in this mass range. This possibility is suggested by grand unified theories, but the two are logically independent. It could be that grand unified theories are correct, but $10^3 \text{ GeV} \gtrsim \mu \gg 10 \text{ GeV}$, so that symmetry breaking and m_H are not determined by radiative corrections. It could be that $m_H \approx 10$ GeV, but that this has nothing to do with symmetry breaking by radiative corrections. It could be that m_H is determined by radiative corrections but is different from 10 GeV, e.g. if there are massive fermions or other massive scalar fields in the theory.

What of the ratio m_X/m_P , which we have not discussed up to now? It is sometimes blithely asserted that m_X is of order m_P , so that there is no other hierarchy to explain, and the grand unification of strong, weak, and electromagnetic interactions is inextricably linked with gravitation. However, we take the point of view that calculations [3,7,8,12] in grand unified theories have established that the unification takes place between 10^{14} and 10^{15} GeV, so that

$$m_X/m_P = O(10^{-5}) \quad (1.7)$$

Two problems then arise: why is the ratio (1.7) so much less than one, and why is it much larger than the hierarchy m_W/m_X ?

$$1 \gg \frac{m_X}{m_P} \gg \frac{m_W}{m_X} \quad (1.8)$$

We seek to show in this paper that such a hierarchy of gauge hierarchies is to be expected in the context of grand unified theories, and that a ratio like (1.7) is natural if all symmetry breaking in grand unified theories is achieved dynamically by radiative corrections. In this case one would have a theory in which there was only one "intrinsic" mass scale $O(m_P)$, associated with gravity and perhaps supersymmetry or supergravity, and that all other mass scales are generated dynamically in a manner calculable using the renormalization group.

Before discussing how this may occur, let us briefly recall the generic structure of grand unified theories. There is a high-energy symmetry which is broken down first to an intermediate symmetry $G_1 [= SU(3) \otimes SU(2) \otimes U(1)?]$ at a scale of $O(10^{15})$ GeV, and secondly to a low-energy symmetry $G_2 [= SU(3) \otimes U(1)]$ at a scale of $O(10^2)$ GeV

$$G \xrightarrow{m_X \sim 10^{14} \text{ to } 10^{15} \text{ GeV}} G_1 \xrightarrow{m_{W,Z} \sim 10^2 \text{ GeV}} G_2 \quad (1.9)$$

In general, at least two multiplets of Higgs fields are needed to realize this two-stage symmetry breaking. Those associated with the 10^{15} GeV scale often form an adjoint Higgs representation {24 of $SU(5)$ [2,8], 45 of $SO(10)$ [13], 78 of $E(6)$ [14]}, while those associated with the 10^2 GeV scale are often in fundamental representations {5 of $SU(5)$ [2,8], 10 of $SO(10)$ [13]}. We would like to understand if possible why the adjoint Higgs breaks first, and on such a different scale from the fundamental representation.

We suppose that at a scale of $O(10^{19})$ GeV, where the next piece of physics occurs (gravity, supersymmetry [4]?) the quartic Higgs couplings are at most of order g^2 , as found in several different theoretical frameworks such as

supersymmetry [4] or dimensional reduction [5]. As we evolve the couplings down to lower energies using the renormalization group, we notice that the adjoint Higgs couplings decrease much faster than those involving the fundamental representation, basically because of their larger non-Abelian charge (Casimir operator value). We find that an initial adjoint coupling of $O(g^2)$ may evolve to zero when the energy scale changes by $O(10^5)$, yielding symmetry breaking and a hierarchy like (1.7). Meanwhile the slowly evolving fundamental Higgs couplings are non-zero at m_X , and subsequently evolve even more slowly at energy scales $< m_X$, because many Higgs and gauge bosons have acquired masses and no longer contribute to the renormalization group equations. This qualitative behaviour is illustrated in fig. 1. We therefore find that the hierarchy of hierarchies (1.8) is not only possible but even plausible in such a framework of dynamical symmetry breaking by radiative corrections [9,11].

Our remarks are illustrated mostly by calculations in the SU(5) model of Georgi and Glashow [2] though they should have more general applicability. An example of the possible evolution of the quartic Higgs coupling in this model is shown in fig. 2. In this model the hierarchy of hierarchies still requires "miniaturizing" μ^2 arbitrarily, and we have no progress to report on the fundamental problem of finding a natural way to do this. We suspect that this is a general problem in grand unified theories which is likely to be more acute in models with more Higgs representations.

In section 2 we discuss the SU(5) Higgs potential in the tree approximation, establishing the conditions necessary for a gauge hierarchy to be obtainable, and deriving positivity bounds on the tree Higgs potential. In section 3 we discuss radiative corrections to the Higgs potential, verifying that the gauge hierarchy can be maintained in the one-loop approximation, still at the expense of an unnatural condition on the coupling constants. Technical details of this study are relegated to two appendices. Section 4 discusses the solution of the renormalization group [15] equations for the SU(5) Higgs potential. Section 5 summarizes the outlook.

2. HIGGS POTENTIAL IN THE TREE APPROXIMATION

Let us briefly recall the essentials of Weinberg's [9] discussion of gauge hierarchies as it applies to grand unified theories [2]. We consider a Higgs potential $V_t(\Phi)$ in the tree approximation which only contains quartic terms:

$$V_t(\Phi) = \lambda_{ijkl} \Phi_i \Phi_j \Phi_k \Phi_l \quad (2.1)$$

which must be ≥ 0 when $|\Phi| \rightarrow \infty$ in any direction. If $V_t(\Phi) = 0$ in some direction Φ_0 then the radiative corrections $V_r(\Phi)$ become important. They ensure that $|\Phi| = 0$ is a local maximum of $V(\Phi) = V_t + V_r$, and that the absolute minimum of $V(\Phi)$ will be at $|\Phi_{\min}| \neq 0$, with Φ approximately in the direction Φ_0 . In general λ is a function of the scale at which the effective potential is renormalized: $|\Phi| = M$, and the zero of λ will occur at some scale M_0 :

$$\lambda_{ijkl}(M_0) \Phi_{0i} \Phi_{0j} \Phi_{0k} \Phi_{0l} = 0 \quad (2.2)$$

with

$$|\Phi_{\min}| = O(M_0) \quad (2.3)$$

The evolution of $\lambda(M) : M > M_0$ can be calculated using the appropriate renormalization group [15] equations up to the scale M_1 [$= O(M_p)$?] at which the next stage of unification (gravity, supersymmetry [4]?) occurs. If $\lambda(M_1) = O(g^2)$ then a hierarchy

$$\frac{M_0}{M_1} = O\left(\exp - \frac{c}{g^2}\right) \quad (2.4)$$

can be established. When we look at scales $M < M_0$ then we want to be left with a subset $\hat{\Phi}$ of Higgs fields which have mass parameters $\hat{\mu} \ll M_0$, at least some of which will be responsible for another scale of symmetry breaking on a scale $M_{-1} \ll M_0$. This can be given by another hierarchy

$$\frac{M_{-1}}{M_0} = O\left(\exp - \frac{\hat{c}}{g^2}\right) \quad (2.5)$$

if $\hat{\mu}$ is sufficiently small that the symmetry breaking is generated by radiative corrections $\hat{V}_r(\hat{\Phi})$ in the neighbourhood of a zero $\hat{\lambda}(M_{-1}) = 0$ of the Higgs potential for the effective low-energy Higgs system $\hat{\Phi}$. We hope to find that

$$1 \gg \frac{M_0}{M_1} \gg \frac{M_{-1}}{M_0} \quad (2.6)$$

corresponding to $C \gg \hat{C}$, so that the desired hierarchy of hierarchies can be established.

When applying this philosophy to grand unified theories [2] we must first examine in the tree approximation the general form of $V_t(\Phi)$ and determine for what values of the λ_{ijkl} the tree potential has a zero in some direction of the Higgs fields Φ_{0i} . One of these will hopefully correspond to the desired pattern of symmetry breaking [e.g. $SU(5) \rightarrow SU(3) \otimes SU(2) \otimes U(1)$ with a left-over massless isodoublet of Higgs fields]. We must then verify that the radiative corrections [11] generate symmetry breaking exactly in the desired direction, and not in some other direction differing from it in some low power of g^2 , which would preclude the establishment of the second hierarchy (2.5). Finally we must compute the renormalization of λ at scales M larger than M_0 and ensure that it does not encounter one of the other zeros of λ making it impossible to establish the first hierarchy (2.4). We hope that $C \gg \hat{C}$ so that the desired hierarchy of hierarchies (2.6) can be obtained.

In this section we start on this programme by studying the zeros of the tree Higgs potential $V_t(\Phi)$. In the case of the $SU(5)$ grand unified model [2,8], the minimal set of Higgs fields contains an adjoint representation ϕ (a 5×5 traceless, symmetric tensor of dimension 24) and a complex 5-dimensional vector representation H . The most general form of Higgs potential can be written in the form

$$\begin{aligned} V_t(\Phi) = & \lambda_1 \text{Tr} \phi^4 + \lambda_2 (\text{Tr} \phi^2)^2 + \lambda_3 \text{Tr} \phi^3 \\ & + \lambda_4 |H|^2 \text{Tr} \phi^2 + \lambda_5 \bar{H} \phi^2 H + \lambda_6 \bar{H} \phi H \\ & + \lambda_7 |H|^4 + \lambda_8 \text{Tr} \phi^2 + \lambda_9 |H|^2 \end{aligned} \quad (2.7)$$

In order to implement our philosophy that there should be no intrinsic mass parameters in the theory, we set

$$\lambda_3 = \lambda_6 = 0 \quad ; \quad \lambda_8 = \lambda_9 = 0 \quad (2.8)$$

The parameters λ_3 and λ_6 could equally well be set to zero by requiring the discrete symmetry $\phi \leftrightarrow -\phi$, but no such symmetry exists to require $\lambda_8 = \lambda_9 = 0$. However, if we make these arbitrary assumptions we are left with five parameters $(\lambda_1, \lambda_2, \lambda_4, \lambda_5, \lambda_7)$ in the Higgs potential.

In order to have a vacuum in which $|\Phi|$ is finite, it is necessary that $V_t(\Phi)$ be ≥ 0 as $|\phi| \rightarrow \infty$. This implies certain conditions on the λ_i , one of which is obtained by considering the possibility $|\phi| = 0, |H| \rightarrow \infty$:

$$\lambda_7 \geq 0 \quad (2.9)$$

Hereafter we will write $\lambda_7 = \lambda_D$.

When one considers the limit $|\phi| \rightarrow \infty, |H| = 0$ it is easy to satisfy oneself that there are bounds on the ratio $\text{Tr } \phi^4 / (\text{Tr } \phi^2)^2$:

$$\frac{7}{30} \leq \frac{\text{Tr } \phi^4}{(\text{Tr } \phi^2)^2} \leq \frac{13}{20} \quad (2.10)$$

The upper and lower limits correspond to the directions

$$\phi \propto (1, 1, 1, -\frac{3}{2}, -\frac{3}{2}) \quad ; \quad \phi \propto (1, 1, 1, 1, -4) \quad (2.11)$$

respectively for the elements of the adjoint representation taken to be diagonal. These directions correspond to $SU(5) \rightarrow SU(3) \otimes SU(2) \otimes U(1)$ and $SU(5) \rightarrow SU(4) \otimes U(1) \otimes U(1)$, respectively. It is then convenient to rewrite the parts of (2.7) which are quartic in ϕ in the form

$$V_t(\phi) = \lambda_m \left[\text{Tr } \phi^4 - \frac{7}{30} (\text{Tr } \phi^2)^2 \right] + \lambda_c \left[\frac{13}{20} (\text{Tr } \phi^2)^2 - \text{Tr } \phi^4 \right] \quad (2.12)$$

The bounds (2.10) then imply particularly simple conditions on the couplings λ_m and λ_c of eq. (2.12):

$$\lambda_m \geq 0 \quad , \quad \lambda_c \geq 0 \quad (2.13)$$

with the potential $V_0(\Phi)$ being zero in the directions

$$\left. \begin{array}{l} \Phi_0^{(3,2)} : \phi \propto (1, 1, 1, -\frac{3}{2}, -\frac{3}{2}), |H| = 0 \quad \text{if } \lambda_c = 0 \\ \Phi_0^{(4,1)} : \phi \propto (1, 1, 1, 1, -4), |H| = 0 \quad \text{if } \lambda_m = 0 \end{array} \right\} \quad (2.14)$$

If one or the other of these conditions is satisfied, radiative corrections may or may not generate a first stage of symmetry breaking in a similar direction. In order for the second stage of symmetry breaking to take the desired form and be separated from this first stage by the appropriate hierarchy (2.5), we must require that some of the Higgs fields have essentially zero (mass)², while others have positive (mass)². The necessary conditions on the λ_i are easily seen to be

$$\begin{aligned} \text{for } SU(5) \rightarrow SU(3) \times SU(2) \times U(1) \rightarrow SU(3) \times U(1) : \\ \lambda_c = 0, \quad \lambda_5 = -\frac{10}{3} \lambda_4 \quad : \quad \lambda_4 > 0 \end{aligned} \quad (2.15)$$

$$\begin{aligned} \text{for } SU(5) \rightarrow SU(3) \times SU(2) \times U(1) \rightarrow SU(2) \times SU(2) \times U(1) \times U(1) : \\ \lambda_c = 0, \quad \lambda_5 = -\frac{15}{2} \lambda_4 \quad : \quad \lambda_4 < 0 \end{aligned} \quad (2.16)$$

$$\begin{aligned} \text{for } SU(5) \rightarrow SU(4) \times U(1) \times U(1) \rightarrow SU(4) \times U(1) : \\ \lambda_M = 0, \quad \lambda_5 = -\frac{5}{4} \lambda_4 \quad : \quad \lambda_4 > 0 \end{aligned} \quad (2.17)$$

$$\begin{aligned} \text{for } SU(5) \rightarrow SU(4) \times U(1) \times U(1) \rightarrow \begin{cases} SU(2) \times SU(2) \times U(1) \times U(1) \\ \text{or } SU(3) \times U(1) \times U(1) \times U(1) \end{cases} \\ \lambda_M = 0, \quad \lambda_5 = -20 \lambda_4 \quad : \quad \lambda_4 < 0 \end{aligned} \quad (2.18)$$

Even if one of the conditions (2.15) to (2.18) is obtained, this is not sufficient for symmetry breaking to be in the desired direction. We must also ensure that $V_t(\Phi) \geq 0$ for all other directions of Φ , including those where $|H| \neq 0$. Simple algebra tells us that the necessary conditions are:

$$(2.15) \quad \oplus \quad \lambda_M \geq \frac{5}{3} \lambda_4^2 / \lambda_D \quad (2.19)$$

$$(2.16) \quad \oplus \quad \lambda_M \geq 15 \lambda_4^2 / \lambda_D \quad (2.20)$$

$$(2.17) \quad \oplus \quad \text{no new condition necessary} \quad (2.21)$$

$$(2.18) \quad \oplus \quad \lambda_c \geq 15 \lambda_4^2 / \lambda_D \quad (2.22)$$

The conditions (2.15) to (2.18) define four classes of potentials that satisfy $V_t(\phi, H) = 0$ in some direction $|\phi| \neq 0, |H| = 0$. The conditions (2.19) to (2.22) ensure, respectively, that the potentials defined by (2.15) to (2.18) are positive semi-definite in all directions. One should also consider the positivity bounds on the general potential; if one of these were saturated we would get $V_t(\phi, H) = 0$ in a direction $|\phi| \neq 0$ and $|H| \neq 0$, i.e. spontaneous symmetry breaking with no gauge hierarchy. We will only present the results for the case $\lambda_M \geq \lambda_c$ which

contains the solution (2.19) in which we are ultimately interested for a realistic SU(5) model [2,8]. We find that to keep $V_t(\phi, H) \geq 0$

for $\lambda_M \geq \lambda_C$ and $\lambda_5 < 0$

$$\begin{aligned} -\lambda_4/\lambda_5 &> \frac{4}{5} \quad \text{needs no new condition} \\ \frac{4}{5} > -\lambda_4/\lambda_5 &> \frac{3}{10} \quad \text{needs } \lambda_M \geq \frac{3}{5} \frac{(\lambda_4 + \frac{4}{5}\lambda_5)^2}{\lambda_D} \end{aligned} \quad (2.23a)$$

$$\frac{3}{10} > -\frac{\lambda_4}{\lambda_5} \quad \text{needs either } (\lambda_M - \lambda_C) \leq \frac{3}{20} \frac{\lambda_5^2}{\lambda_D} \quad \text{and } \lambda_M \geq \frac{3}{5} \frac{(\lambda_4 + \frac{4}{5}\lambda_5)^2}{\lambda_D} \quad (2.23b)$$

$$\text{or } (\lambda_M - \lambda_C) > \frac{3}{20} \frac{\lambda_5^2}{\lambda_D} \quad \text{and } \lambda_M \geq \frac{3}{5} \frac{(\lambda_4 + \frac{3}{10}\lambda_5)^2}{\lambda_D} \quad (2.23c)$$

for $\lambda_M \geq \lambda_C$ and $\lambda_5 > 0$

$$-\frac{\lambda_4}{\lambda_5} > \frac{2}{15} \quad \text{and } \frac{15}{16} (\lambda_M - \lambda_C) > \frac{\lambda_5^2}{4\lambda_D} \quad \text{and } \frac{1}{4} (\lambda_M - \lambda_C) + \frac{\lambda_4\lambda_5}{2\lambda_D} > 0$$

$$\text{needs } \frac{-\left[\frac{1}{4}(\lambda_M - \lambda_C) + \frac{\lambda_4\lambda_5}{2\lambda_D}\right]^2}{4\left[\frac{15}{16}(\lambda_M - \lambda_C) - \frac{\lambda_5^2}{4\lambda_D}\right]} + \frac{1}{4} (\lambda_M - \lambda_C) + \left(\frac{13}{20}\lambda_C - \frac{7}{30}\lambda_M\right) - \frac{\lambda_4^2}{4\lambda_D} > 0 \quad (2.24a)$$

$$\begin{aligned} -\frac{\lambda_4}{\lambda_5} &> \frac{2}{15} \quad \text{and } \frac{15}{16} (\lambda_M - \lambda_C) > \frac{\lambda_5^2}{4\lambda_D} \quad \text{and } \frac{1}{4} (\lambda_M - \lambda_C) + \frac{\lambda_4\lambda_5}{2\lambda_D} < 0 \\ \text{or } 0 < -\frac{\lambda_4}{\lambda_5} \quad \text{and } \frac{15}{16} (\lambda_M - \lambda_C) < \frac{\lambda_5^2}{4\lambda_D} \\ \text{needs } \frac{1}{60} \lambda_M + \frac{2}{5} \lambda_C - \frac{\lambda_4^2}{4\lambda_D} &> 0 \end{aligned} \quad (2.24b)$$

with no new condition required for other values of the coupling constants. The conditions (2.23) and (2.24) represent the general positivity bounds on the tree potential $V_t(\phi)$ when $\lambda_M \geq \lambda_C$. We see that they reduce to the previous special cases (2.19) and (2.20) when the special conditions (2.15) and (2.16), respectively, are obeyed. Since we will be particularly interested in these two conditions as starting points for our discussion of renormalization of the potential parameters λ_i at high scales $M > M_0$, it will be convenient to rewrite the potential $V_t(\phi)$ (2.7) in terms of two new variables

$$\begin{aligned}\lambda_A &= \frac{9}{5} \lambda_4 + \frac{6}{25} \lambda_5 \\ \lambda_J &= \frac{4}{5} \lambda_4 + \frac{6}{25} \lambda_5\end{aligned}\quad (2.25)$$

as well as the previously introduced variables λ_M and λ_C of eq. (2.12) and $\lambda_D \equiv \lambda_7$.

The potential then takes the form

$$\begin{aligned}V_t(\Phi) &= \lambda_M \left[\text{Tr} \phi^4 - \frac{7}{30} (\text{Tr} \phi^2)^2 \right] + \lambda_C \left[\frac{13}{20} (\text{Tr} \phi^2)^2 - \text{Tr} \phi^4 \right] \\ &+ \lambda_A \left[|H|^2 \text{Tr} \phi^2 - \frac{10}{3} \bar{H} \phi^2 H \right] + \lambda_J \left[\frac{15}{2} \bar{H} \phi^2 H - |H|^2 \text{Tr} \phi^2 \right] \\ &+ \lambda_D |H|^4\end{aligned}\quad (2.26)$$

In terms of the variables $\{\lambda_M, \lambda_C, \lambda_A, \lambda_J, \lambda_D\}$ the starting conditions (2.19) and (2.20) can be written as

$$\lambda_C = 0, \lambda_M \geq 0; \lambda_J = 0, \lambda_A \geq 0; \lambda_M \geq \frac{5}{3} \frac{(\lambda_A - \lambda_J)^2}{\lambda_D} \quad (2.15')$$

$$\lambda_C = 0, \lambda_M \geq 0; \lambda_A = 0, \lambda_J \geq 0; \lambda_M \geq \frac{15}{\lambda_D} \frac{(\lambda_A - \lambda_J)^2}{\lambda_D} \quad (2.16')$$

These are the starting points of the renormalization group [15] analysis of section 4. Bounds on the gauge hierarchy M_0/M_1 (2.4) will be imposed by the requirements that

- a) the potential be ≥ 0 [conditions (2.23) and (2.24)],
- b) that all the $\{\lambda_M, \lambda_C, \lambda_A, \lambda_J, \lambda_D\}$ be $\leq 0(g^2)$.

However, before embarking on such an analysis we should first check that the radiative corrections to $V_t(\Phi)$ in fact implement the appropriate pattern of symmetry breaking indicated in eqs. (2.15) to (2.18). This we do in section 3.

3. RADIATIVE CORRECTIONS TO THE HIGGS POTENTIAL

So far we have only considered the tree contribution to the Higgs potential which can, as we have seen, be constructed to give the required pattern of symmetry breaking $SU(5) \rightarrow SU(3) \otimes SU(2) \otimes U(1) \rightarrow SU(3) \otimes U(1)$ at the price of imposing the hierarchy conditions (2.15'). The second equality $\lambda_J = 0$ in (2.15') ensures the masslessness of that part of the 5-dimensional Higgs fields which transforms as a (1,2) under the initially unbroken $SU(3) \times SU(2)$ subgroup of $SU(5)$. However, at the tree level the shifted (1,1) component of the 24 adjoint Higgs representation also remains massless, and the magnitudes of the vacuum expectation values (v.e.v.) are undetermined.

When radiative corrections to the Higgs potential are considered, one must specify the scale at which the effective potential is renormalized, and hierarchy conditions such as (2.15') are not in general renormalization group invariant. As we will see in section 4, it is then quite natural that if a theory is specified with the quartic Higgs couplings $\lambda_i = O(g^2)$ at the Planck mass, then λ_C will go to zero, causing the spontaneous breakdown of SU(5), at some scale of the Higgs fields

$$[|\phi|^2 + |H|^2]^{\frac{1}{2}} = \mu = O(10^{-5}) m_P$$

The magnitudes of the Higgs v.e.v. are then determined, and we wish to ensure that the hierarchy established by (2.15') at the tree graph level is retained when radiative corrections are taken into account. So we must study their contributions for values of $[|\phi|^2 + |H|^2]^{\frac{1}{2}}$ close to μ . They can in principle give $O(g^4)$ modifications of the symmetry-breaking pattern which would destroy the hierarchy $O(\exp -C/\alpha)$ that we wish to establish. We shall show that in the one-loop approximation to the effective potential the desired hierarchy can still be maintained at the expense of one unnatural condition. This is a radiative correction to the (2.15') condition $\lambda_J = 0$ which takes the form of $\lambda_J(\mu) =$ a complicated expression involving logarithms of coupling constants of order g^4 or λ^2 . We could recover the simple constraint $\lambda_J(\mu) = 0$ by a redefinition of the renormalization procedure. However, the appropriate renormalization specification would then itself take a somewhat baroque form, and we have found no simple formulation of the desired condition.

The conclusion of this investigation of radiative corrections is that the idea of generating all symmetry breaking in a grand unified theory, and in particular in SU(5), is a viable possibility, but that this picture offers no further insight into the well-known gauge hierarchy problem.

For the following discussion we write the effective scalar potential, including one-loop radiative corrections [11], in the form:

$$V(\phi, H) = V_t(\mu) + V_r(\mu) :$$

$$V_t(\mu) = \lambda_m(\mu) \left[\text{Tr} \phi^4 - \frac{7}{30} (\text{Tr} \phi^2)^2 \right] + \lambda_c(\mu) \left[\frac{13}{20} (\text{Tr} \phi^2)^2 - \text{Tr} \phi^4 \right] \\ + \lambda_A(\mu) \left[|H|^2 \text{Tr} \phi^2 - \frac{10}{3} \bar{H} \phi^2 H \right] + \lambda_J(\mu) \left[\frac{15}{2} \bar{H} \phi^3 H - |H|^2 \text{Tr} \phi^2 \right] \\ + \lambda_D(\mu) |H|^4,$$

$$V_r(\mu) = \frac{1}{64\pi^2} \sum_i \eta_i \text{Tr} (m_i^4 \ln m_i^2/\mu^2). \quad (3.1)$$

where $m_i^2(\phi, H)$ is the squared mass acquired by the i^{th} field in the tree approximation if $\langle \phi \rangle, \langle H \rangle = \phi, H$, and $\eta_i = 3, 1, -4$ for $i = \text{vector, scalar, fermion}$, respectively. Equations (3.1) in fact define the renormalization prescription: a shift of any of the λ_i in V_t by an amount of order g^4 or λ_i^2 can be absorbed in V_r and thus regarded as a different prescription. We shall neglect fermion loop contributions to V_r under the (phenomenologically supported) assumption that fermion masses are relatively small. [In the standard SU(5) model these give contributions of the form $G^4 |H|^4 \ln G |H|^2/\mu^2$, where G is a typical Yukawa coupling constant, and will be irrelevant to most of the subsequent discussion anyway.]

In the previous section we studied $V_t(\mu)$ and showed that the $\lambda_i(\mu)$ can be specified in such a way that $V_t(\mu)$ is positive semi-definite and that $\lambda_C(\mu) = \lambda_J(\mu) = 0$, resulting in the desired symmetry-breaking pattern in the tree approximation. If the correction term $V_r(\mu)$ by itself gave the same pattern, we would have the desired gauge hierarchy with no further adjustment. Unfortunately, as shown in appendix 1, this is not the case. It turns out that the field configuration which minimizes $V_r(\mu)$ can be approximately determined by finding that which minimizes the quantity

$$R(\phi, H) \equiv \frac{\sum_i \eta_i m_i^4}{\sum_i \eta_i m_i^2} \quad (3.2)$$

If we neglect the quintuplet H , the quantity (3.2) will be a minimum for a v.e.v. which is invariant under $SU(3) \otimes SU(2) \otimes U(1)$ or under $SU(4) \otimes U(1)$ depending on whether the coefficient of $\text{Tr} \phi^4$ in the numerator is positive or negative. In the absence of a quintuplet this coefficient is

$$\frac{15}{2} g^4 + 16 (\lambda_M - \lambda_C) \left(\frac{7}{5} \lambda_C + \frac{18}{5} \lambda_M \right)$$

which is necessarily positive when $\lambda_C = 0$, giving $SU(3) \otimes SU(2) \otimes U(1)$ at the minima of both V_t and V_r , with the latter determining the magnitude of the v.e.v. [In contrast, $\lambda_M = 0$ is not sufficient to ensure $SU(4) \otimes U(1)$ invariance at the minimum of V_r .] However, when a quintuplet is included, one finds in general that the minimum of V_r occurs for a configuration with $|\langle H \rangle| \sim |\langle \phi \rangle|$ and $\langle \phi \rangle \neq \langle \hat{\phi} \rangle_{H=0}$. In appendix A this is shown to be the case for the vector meson loop contribution to V_r which can be solved explicitly, and we argue that it is also true for the full V_r where V_t is specified by $\lambda_C(\mu) = \lambda_J(\mu) = 0$, $\alpha(\mu) = 1/45$, $\lambda_D(\mu) = 0.007$, and any positive values of $\lambda_M(\mu)$ and $\lambda_A(\mu)$.

What is really relevant, however, is the global minimum of $V_t + V_r$. The preceding discussion implies that in order to get the desired pattern of symmetry breaking it is necessary that some λ_i be $\gg g^4$ at m_X so that the symmetry at the global minimum will be that of V_t alone. Specifically, we have to prevent the (1,2) field (i.e. the complex doublet of the standard model) from acquiring either a mass or a vacuum expectation value larger than $O(e^{-C/\alpha})$. In addition, we want to check that the (1,1) field, massless at the tree level, acquires a mass via radiative corrections. In appendix B we solve explicitly a simple toy model to show how a hierarchy of v.e.v. can be maintained at the one-loop level. While we cannot solve explicitly the full $SU(5)$ case, we show in what follows that an analogous picture is possible.

We make the following definitions:

$$\phi = \phi_0 + \tilde{\phi} \equiv \frac{v}{\sqrt{30}} \begin{pmatrix} 2 & 2 & 0 \\ 0 & 2 & -3 \\ & -3 & -3 \end{pmatrix} + \tilde{\phi}, \quad \text{Tr } \phi \tilde{\phi} = 0$$

$$\delta^2 \equiv |\text{Tr } \tilde{\phi}|^2 + |H|^2$$

so that δ parametrizes the amount of "leakage" away from the desired field configuration at the minimum. We shall set $\lambda_C(\mu) = \lambda_J(\mu) = 0$ in V_t and assume that λ_M , λ_A , and λ_D are $O(g^2)$. [Actually $\lambda_D \approx 0.3 \alpha \approx 0.03 g^2 \approx 14 \alpha^2$, but this is not crucial to the argument; λ_M and λ_J can easily be $O(g^2)$.] Then V_t takes the form:

$$V_t = g^2 (A \delta^2 v^2 + B \delta^3 v + C \delta^4) \quad (3.3)$$

with A, B, C being O(1). V_r is a sum of terms of the form

$$V_i = \frac{1}{64\pi^2} m_i^4 \ln m_i^2 / \mu^2 \quad (3.4)$$

each having a minimum for $m_i^2 = \mu^2 e^{-1/2}$:

$$V_i(\min) = -\frac{\mu^4 e^{-1}}{128\pi^2} \quad (3.5)$$

Then the sum is bounded by the quantity (3.5) times the number of terms in the sum; for our SU(5) model there are 108 m_i (each vector counting for 3), so

$$V_r > -\frac{27}{32} \frac{e^{-1}}{\pi^2} \quad (3.6)$$

Suppose first that $\delta \geq v$. Then $V_t + V_r$ can be negative only if

$$g^2 \delta^4 < \frac{27}{32} \frac{e^{-1}}{\pi^2} \mu^4 \quad (3.7)$$

But the masses are of the form:

$$m_i^2 = g^2 (a_i v + b_i \delta)^2 + c_i \delta^2 \quad (3.8)$$

with a_i, b_i, c_i being O(1), and for V_i to approach its minimum value (3.5) we must have $m_i^2 \sim \mu^2 e^{-1/2} \sim g^2 \delta^2$. Then the condition (3.7) becomes

$$1 < \frac{27}{32} \frac{g^2}{\pi^2}$$

which is impossible. Therefore we necessarily have $\delta \ll v$ at the global minimum, so we may expand in δ/v those of the m_i^2 which are non-vanishing for $\delta = 0$. This expansion contains no term which is linear in δ/v because it must display the $SU(3) \otimes SU(2)$ symmetry of ϕ_0 , and one cannot construct an invariant which is linear in $\hat{\phi}$ or in H. Therefore, the expression will be of the form:

$$V_r(\delta) = \sum_i^{(1)} \left(V_i(0) + \alpha_i g^2 v^2 \frac{\partial V_i(0)}{\partial m_i^2} \frac{\delta^2}{v^2} + \dots \right) + \frac{1}{64\pi^2} \sum_i^{(2)} g^4 \beta_i^2 \delta^4 \ln \beta_i \delta^2 / \mu^2 \quad (3.9)$$

where α_i and β_i are $O(1)$. The first sum in (3.9) is over the i 's for which $m_i(\delta = 0) \neq 0$, and the second sum is over the remaining ones. In order for a configuration with $\delta \neq 0$ to give a lower potential than for $\delta = 0$ we must have

$$V_t + V_r - \sum_i^{(1)} V_i(b) < 0 \quad (3.10)$$

Since the coefficients of $(\delta^2/v^2)^n$ in (3.10) are $O(m_i^2/64\pi^2) = O(\alpha^2 v^4)$, the terms with $n = 2, 3, 4$ are negligible with respect to V_t for $A, B, C = O(1)$. Furthermore, since $\delta/v \ll 1$ and $\mu = O(gv)$,

$$\left(\frac{\delta}{v}\right)^{n>4} \ll \left|\delta^4 \ln \delta^2/v^2\right|$$

and the condition (3.10) becomes

$$V_t + \frac{1}{64\pi^2} \sum_i^{(1)} g^4 \beta_i \delta^4 \ln \delta^2/v^2 < 0 \quad (3.11)$$

which for $A = O(1)$ implies

$$\frac{1}{\alpha} < \left| \frac{\delta^2}{v^2} \ln \delta^2/v^2 \right| \quad (3.12)$$

However for $x < 1$ the function $f(x) = x \ln x$ is bounded by

$$|f(x)| < e^{-1}$$

so that (3.12) is again impossible. This means that no field configuration for which there is a quadratic term in V_t will acquire a vacuum expectation value.

Now V_t was constructed precisely to give $A = 0$ for the configuration

$$\tilde{\phi} = 0, \quad H = \begin{pmatrix} 0 \\ 0 \\ 0 \\ H_2 \end{pmatrix}, \quad |H_2|^2 \neq 0$$

In this case $B = 0$ as well; explicitly

$$V_t = \lambda_D |H_2|^4$$

Suppose first that $\ln |H_2|^2/v^2 \ll O(\lambda_D/\alpha^2)$ so that the last term in (3.9) can be neglected relative to V_t and the quadratic term in V_r . Then the potential takes the form

$$V(H) = \lambda_D |H_2|^4 + v^2 |H_2|^2 \quad (3.13)$$

where v^2 is the coefficient of δ^2 in (3.9). Explicit calculation gives

$$6 + \pi^2 v^2 = 9g^2 m_X^2 (2 \ln m_X^2/\mu^2 + 1) - 4\lambda_A m_{\phi_3}^2 (2 \ln m_{\phi_3}^2/\mu^2 + 1) \\ + 16 \lambda_A m_{\phi_8}^2 (2 \ln m_{\phi_8}^2/\mu^2 + 1) + 12 \lambda_D m_{H_3}^2 (2 \ln m_{H_3}^2/\mu^2 + 1) \quad (3.14)$$

where

$$m_X^2 = \frac{5}{12} g^2 v^2, \quad m_{\phi_3}^2 = \frac{8}{3} \lambda_M v^2 \\ m_{\phi_8}^2 = \frac{2}{3} \lambda_M v^2, \quad m_{H_3}^2 = \frac{5}{9} \lambda_A v^2 \quad (3.15)$$

are the squared masses of, respectively, the super heavy vectors, the SU(2) triplet and SU(3) octet scalars in ϕ , and the SU(3) triplet of scalars, as determined from the true potential with $\delta = 0$. The scale of v is determined by the requirement that $V_r(v, \delta = 0)$ be a minimum:

$$\frac{\partial V_r}{\partial v}(v, \delta = 0) = 0$$

giving the constraint

$$0 = 15g^2 m_X^2 [2 \ln m_X^2/\mu^2 + 1] + 8\lambda_M m_{\phi_3}^2 [2 \ln m_{\phi_3}^2/\mu^2 + 1] \\ + \frac{16}{3} \lambda_M m_{\phi_8}^2 [2 \ln m_{\phi_8}^2/\mu^2 + 1] + \frac{10}{3} \lambda_M m_{H_3}^2 [2 \ln m_{H_3}^2/\mu^2 + 1] \quad (3.16)$$

Solving eq. (3.16) for v gives:

$$-2 \ln v^2/\mu^2 = \left\{ \frac{25}{2} g^4 \left[\ln \frac{5}{12} g^2 + \frac{1}{2} \right] + \frac{64}{9} \lambda_M^2 \left[\lambda \ln \frac{2}{3} \lambda_M + \frac{7}{2} + 12 \ln 2 \right] \right. \\ \left. + \frac{50}{27} \lambda_A^2 [2 \ln \frac{5}{9} \lambda_A + 1] \right\} / \left(\frac{25}{4} g^4 + \frac{50}{27} \lambda_A^2 + \frac{224}{9} \lambda_M^2 \right) \quad (3.17)$$

The result (3.17) can now be used with eq. (3.14) to express v^2/v^2 in terms of g^2 and the λ_i . If $v^2 < 0$, H_2 acquires a v.e.v.:

$$\langle 0 | H_2 | 0 \rangle^2 = \frac{|v^2|}{2\lambda_D} = O\left(\frac{\alpha^2}{\lambda_D} v^2\right) = O(10^{-1} v^2)$$

and if $v^2 > 0$, H_2 acquires a mass:

$$m_{H_2}^2 = v^2 = O(\alpha^2 v^2) = O(10^{-4} v^2)$$

Either result is disastrous.

However, if $\lambda_J(\mu)$ is non-vanishing, the potential contains an additional quadratic term, $(5/9)\lambda_J v^2 |H_2|^2$, so by imposing

$$\lambda_J(\mu) = -\frac{9}{5} \frac{v^2}{v^2} \quad (3.18)$$

instead of $\lambda_J(\mu) = 0$, we retain the desired result. Given the complicated functional relation among couplings that it entails, this is admittedly not very palatable. As discussed above, as $\lambda_J = O(\alpha^2)$, we may absorb it into the definition of V_r and call this a different renormalization specification, but this is not really an aesthetic improvement. A natural prescription might be to define the $\lambda_i(\mu)$ as fourth derivatives of the full potential, but contrary to the case for a single spinorial Higgs representation (see, for example, Appendix B); this prescription is still somewhat arbitrary, it gives a complicated expression for $V_r(\mu)$, and does not seem to allow for a simpler formulation of the condition on λ_J .

Once we accept the constraint (3.18) (which presumably must be adjusted at each order in the loop expansion), the final symmetry breaking of $SU(3) \otimes SU(2) \otimes U(1)$ to $SU(3) \otimes (U(1))_{e.m.}$ goes through as discussed by Weinberg [9]. Since there can be no terms cubic in H_2 , the dominant contributions to $V(H_2)$ are now:

$$V(H_2) \approx |H_2|^4 \left[\lambda_D(\mu) + a \alpha^2 \ln(b |H_2|^2 / \mu^2) \right] \quad (3.19)$$

with $a, b = O(1)$. The potential (3.19) has a minimum for

$$\langle 0 | H_2 | 0 \rangle^2 = \frac{\mu^2}{b} \exp\left(-\frac{\lambda_D}{a \alpha^2}\right) \sim \mu^2 \exp\left(-\frac{14}{a}\right) \ll \mu^2$$

However eq. (3.19) is a poor approximation to the potential for H_2 if we wish to study its behaviour when $\left| \ln |H_2|^2 / \mu^2 \right| \gg 1$, so we must use the renormalization group equations to determine the potential near $\mu \approx \langle |H| \rangle$, i.e. near the value of μ for which $\lambda_D(\mu) = 0$.

As for the $SU(3) \otimes SU(2)$ singlet Higgs scalar which did not acquire a mass in the true approximation, a previous study [8] showed that vector meson loop contributions to V_r generate a mass

$$m_{\phi_0}^2 = \frac{25}{2} \alpha^2 v^2$$

There is no reason that other contributions should cancel this one unless they are constrained to do so, for example, by adding a term λ_C of order α^2 .

4. THE RENORMALIZATION GROUP EQUATIONS FOR THE HIGGS POTENTIAL

We have seen that the desired hierarchy of symmetry breaking: $SU(5) \rightarrow SU(3) \otimes SU(2) \otimes U(1)$ can be assured either at the tree level (section 2) or at the one-loop level (section 3). Also, Weinberg [9] has used the renormalization group [15] to show that a quartic Higgs coupling in the Weinberg-Salam model [1] which is zero at $\mu \approx 28$ GeV will evolve at larger scales to become ≈ 0.007 at a scale of order 2×10^{14} GeV. This is within an order of magnitude of the scale at which phenomenological estimates place grand unification. We will therefore set

$$\left. \begin{aligned} \lambda_D(\mu) &= 0.007 \\ \lambda_C(\mu) &= 0 \\ \lambda_S(\mu) &= 0 \end{aligned} \right\} \text{ at } \mu \approx 2 \times 10^{14} \text{ GeV} \quad (4.1)$$

and use the appropriate renormalization group equations to study the evolution of the coupling constants at larger scales. There are two points which we want to study.

- a) At what scale does one more of the quartic Higgs couplings λ_i become of order g^2 ? In view of a common theoretical prejudice that $\lambda_i \leq O(g^2)$, this will establish an approximate upper bound on the scale at which some new physics must intervene.
- b) Are the positivity inequalities of section 2 respected at larger scales? If one is violated, then some unwanted form of spontaneous symmetry breaking would take place in a way which would exclude the observed pattern, so that this condition gives another bound on the scale at which there would be more physics beyond grand unification.

Identifying any such new physics scale with the Planck mass m_P , we see that the considerations (a) and (b) above both give approximate upper bounds on the hierarchy m_P/m_X . We hope to find that this bound is much less than the approximate upper bound on the hierarchy m_X/m_W , thereby establishing the plausibility of a hierarchy of hierarchies.

The evolution of the various quartic scalar coupling constants is described in a well-known manner by the renormalization group [15] equations for these effective couplings. It should be noted that in a grand unified theory [2] heavy particles do not contribute to the renormalization group equations at scales much less than their masses [16]. Heavy particles may contribute to anomalous vertices at scales much smaller than their masses, but these do not appear in the renormalization group equations [15]. The effects of heavy particles may not disappear if their presence is required for renormalizability, but this is not the case in grand unified theories since the low-energy $SU(3) \otimes SU(2) \otimes U(1)$ theory is manifestly renormalizable [6] by itself. Since, in the example we discuss, one is interested in the presence of one adjoint and one fundamental representation, we shall be concerned with the five different couplings of eq. (2.26) in our quartic Lagrangian. For $SU(N)$, the renormalization group equations for the five couplings have been given in the one-loop approximation by Cheng, Eichten and Li [17].

Separate equations for the adjoint representation alone, or the fundamental one, had been given earlier by Gross and Wilczek [18]. Apart from typographical errors, they agree with those of Cheng et al., and with our own verification, for these special subcases. As we have written the basic Lagrangian in a form (2.26) different from that of Cheng et al. [17] in order to display the patterns of symmetry breaking, we here quote the renormalization group equations in terms of our notation:

$$8\pi^2 \frac{d\lambda_c}{dt} = \frac{52}{5} \lambda_c^2 + \frac{56}{15} \lambda_m^2 - \frac{4}{5} \lambda_c \lambda_m + \frac{604}{9} \lambda_A^2 - 102 \lambda_A \lambda_J + \frac{159}{4} \lambda_J^2 - 30 \lambda_c g^2 + 15 g^4 \quad (4.2a)$$

$$8\pi^2 \frac{d\lambda_m}{dt} = 9 \lambda_c^2 + \frac{22}{3} \lambda_m^2 - 3 \lambda_c \lambda_m + \frac{218}{3} \lambda_A^2 - 127 \lambda_A \lambda_J - \frac{593}{8} \lambda_J^2 - 30 \lambda_m g^2 + \frac{45}{2} g^4 \quad (4.2b)$$

$$8\pi^2 \frac{d\lambda_D}{dt} = 18 \lambda_D^2 + \frac{214}{3} \lambda_A^2 - 121 \lambda_A \lambda_J + \frac{489}{8} \lambda_J^2 - \frac{32}{5} \lambda_D g^2 + \frac{99}{50} g^4 \quad (4.2c)$$

$$8\pi^2 \frac{d\lambda_A}{dt} = \frac{10}{3} \lambda_m \lambda_A + \frac{25}{2} \lambda_c \lambda_A - \frac{45}{4} \lambda_c \lambda_J + 17 \lambda_D \lambda_A - \frac{45}{4} \lambda_D \lambda_J - \frac{889}{20} \lambda_A \lambda_J + \frac{161}{10} \lambda_A^2 + \frac{6561}{160} \lambda_J^2 - \frac{111}{5} \lambda_A g^2 + \frac{99}{80} g^4 \quad (4.2d)$$

$$8\pi^2 \frac{d\lambda_J}{dt} = \frac{10}{3} \lambda_m \lambda_J + \frac{20}{3} \lambda_c \lambda_A - \frac{15}{2} \lambda_c \lambda_J + \frac{20}{3} \lambda_D \lambda_A - 3 \lambda_D \lambda_J - \frac{111}{5} \lambda_A \lambda_J + \frac{302}{45} \lambda_A^2 + \frac{1039}{40} \lambda_J^2 - \frac{111}{5} \lambda_J g^2 + \frac{21}{20} g^4 \quad (4.2e)$$

Here t is defined as $t = \frac{1}{2} \log (Q^2/\Lambda_X^2)$, where $\Lambda_X = 2 \times 10^6$ GeV is a scale parameter chosen to ensure continuity of the gauge coupling constant at the superheavy threshold m_X .

As mentioned above, we use the initial conditions (4.1) for the renormalization group equations (4.2). The two unspecified couplings λ_M and λ_A should obey the inequality

$$\lambda_M \geq \frac{5}{3} \frac{\lambda_A^2}{\lambda_D} \quad (4.3)$$

derived in section 2. As a first example A, the solution of which is illustrated in fig. 2, we have chosen

$$\lambda_A \approx \lambda_D = 0.007 \quad ; \quad \lambda_M = 0.03 \quad (4.4)$$

thus satisfying (4.3). The gauge coupling at m_X was taken to be $g^2(m_X) = 0.3$, corresponding to $\alpha_{\text{GUM}} = 0.024$. It is therefore remarkable that whereas at m_X , λ_M is only one tenth of g^2 , at the Planck mass $Q = 2 \times 10^{19}$ GeV, $\lambda_M/g^2 \approx 0.79$ already. We should point out that our calculations are approximate and that a more refined solution will certainly yield values decreased by a few per cent^{*}). It is the qualitative aspect of the rises in the λ_M and λ_C couplings which is of interest, more than the exact numerical value. The adjoint representation couplings, over five decades of Q , may become of the order of the gauge coupling.

It would be dishonestly incomplete to present only this result, although we did say "may become". We have made a study of the (λ_M, λ_C) plane and plotted in fig. 3 the curves for $\lambda_{M-}(\lambda_C)$ and $\lambda_{C-}(\lambda_M)$, where $\lambda_{X-}(\lambda_Y)$ represents the smallest real root of the right-hand side of the equation for $d\lambda_X/dt$ as a function of λ_Y . Along these curves, λ_M or λ_C respectively is constant when x changes. The figure reveals a "funnel" in which both λ_M and λ_C lie close to λ_{M-} and λ_{C-} . To pass through such a funnel would take a large change in t as both couplings are quasi-stationary there.

^{*}) We can place analytical upper and lower bounds on $\lambda_M(m_P)/g^2(m_P)$ with the values 0.7984 and 0.7247, respectively, while our approximate numerical solution gave $\lambda_M/g^2 = 0.7934$.

Coming back to our result of rapid growth in λ_M when starting from $\lambda_M(m_X) \approx 0.03$, it is clear from the graph in fig. 3 that a $\lambda_M(m_X) \approx 0.3$ would stay stationary over the range $m_X \sim 2 \times 10^{14}$ GeV to $m_P \sim 2 \times 10^{19}$ GeV. An explicit integration of a sample solution B with this critical value has confirmed this qualitative feature. Using the same argument backwards: if one starts at m_P with $\lambda_M \approx 0.3$ we will go down to m_X with roughly the same value $\lambda_M \approx 0.3$. This shows the influence of the choice of initial conditions so that the conclusion of this discussion is that, while it is possible to have a $\lambda_M \leq 0(g^2)$ at $m_P \sim 2 \times 10^{19}$ GeV which decreases to a $\lambda_M(m_X \sim 2 \times 10^{14} \text{ GeV}) \approx 0.03 \sim \frac{1}{10} g^2(m_X)$, this is not necessarily so. If, however, we avoid solutions where $\lambda_M(m_P) \geq g^2(m_P)$, then the behaviour illustrated in our example A [$\lambda_M(m_X) \approx$ a few per cent] is the rule. Quite evidently the counter-example B mentioned above where $\lambda_M(m_P) \approx 0.3 \approx 1.6 g^2(m_P)$ does not belong to this class of solutions.

Further comments on the behaviour of the solutions in the range 10^{14} to 10^{19} GeV are in order. In this energy range, we must ensure that the solutions do not penetrate forbidden regions of the 5-dimensional λ_i space. These regions were specified by the positivity bounds obtained in section 2, eqs. (2.23) and (2.24). In the example A, $-\lambda_4/\lambda_5$ starts off with the value $\frac{3}{10}$ at $Q = m_X$, and then grows past $\frac{4}{5}$ up to infinity ($\lambda_5 \rightarrow -0$). [The condition (2.23a) is always satisfied when $\frac{4}{5} > -\lambda_4/\lambda_5 > \frac{3}{10}$.] This happens for $Q \leq 10^{16}$ GeV. Afterwards, $-\lambda_4/\lambda_5$ grows from $-\infty$ to reach -0.28 at $Q = m_P$. In this region, no condition need be satisfied except for $\lambda_C \geq 0$, which is always true. In example B, $-\lambda_4/\lambda_5$ always satisfies $\frac{3}{10} < -\lambda_4/\lambda_5 < \frac{4}{5}$ and the inequality (2.23a) always holds. There is no zero of λ_5 in the domain of integration of the renormalization group equations.

To conclude this section we make two observations:

I) As was said in the Introduction, if at a scale of $O(10^{19})$ GeV the quartic Higgs couplings are at most equal to g^2 ^{*)}, the adjoint couplings (λ_M, λ_C) decrease very fast so that they are zero (λ_C) or almost zero (λ_M) at $O(10^{14})$ GeV, yielding

*) From $0.5 g^2$ at m_P , λ_C decreases to zero at $m_X \approx 10^{-5} m_P$.

therefore symmetry breaking and the hierarchy (1.7). On the other hand, it is worth noting that in order to drive λ_D to zero at $O(10^{14})$ GeV it should not be much larger than $0.08 g^2$ at m_P . The rate of change of λ_D is much smaller than for λ_A and λ_J , reflecting the smaller non-Abelian "charge" of the quintuplet of Higgs fields. Below the grand unification mass, λ_D becomes the quartic Weinberg-Salam [1] Higgs coupling, which evolves much more slowly [17] even than λ_D because of the "dropping out" of heavy particles [16] and reduction of the symmetry group at scales less than the grand unification mass. These points are illustrated in Fig. 1 and more quantitatively in fig. 4. In fig. 4 the dashed lines are not exactly the evolution curves, they represent the mean slopes between m_X and m_P , for the same initial conditions for λ_M and λ_D .

II) The renormalization group [15] technique is of vital importance in this problem for analysing the evolution of the λ_i , in contrast with cases like QCD in the leading logarithmic approximation, where one can guess this evolution from an elementary resummation of the perturbation theory. Here the problem is much more complicated because of the interplay of several different coupling constants.

Table for Example A

Variation of $-\lambda_4/\lambda_5$ from m_X to m_P . In the interval $0.3 \leq -\lambda_4/\lambda_5 \leq 0.8$ one always has $\sqrt[3]{\lambda_M \lambda_D} \geq (3\lambda_J - \lambda_A)^2$

Q	$-\lambda_4/\lambda_5$	$\sqrt[3]{\lambda_M \lambda_D}$	$(3\lambda_J - \lambda_A)^2$
2×10^{14}	0.3	1.26×10^{-4}	4.9×10^{-5}
2×10^{15}	0.68	5.28×10^{-4}	3.92×10^{-7}
2×10^{16}	-12.32	No conditions	
2×10^{17}	-0.53		
2×10^{18}	-0.35		
2×10^{19}	-0.28		

Table for Example B

The variation of $-\lambda_4/\lambda_5$ and the quantities entering the condition $\sqrt[3]{5} \lambda_M \lambda_D \geq (3\lambda_J - \lambda_A)^2$

Q	$-\lambda_4/\lambda_5$	$\sqrt[3]{5} \lambda_M \lambda_D$	$(3\lambda_J - \lambda_A)^2$
2×10^{14}	0.3	1.26×10^{-3}	1.225×10^{-3}
2×10^{15}	0.3428	2.52×10^{-3}	5.76×10^{-4}
2×10^{16}	0.402	3.33×10^{-3}	2.5×10^{-4}
2×10^{17}	0.4772	3.81×10^{-3}	9.4×10^{-5}
2×10^{19}	0.7455	4.49×10^{-3}	8.1×10^{-7}

5. OUTLOOK

We have found that the desired hierarchy of hierarchies is plausible in grand unified theories [2], given the imposition [in the SU(5) model] of one unnatural condition [8], indispensable for the realization of the hierarchy $1 \gg m_W/m_X$ [9]. Given that the quartic Higgs couplings are $O(g^2)$ at the Planck mass, it is quite plausible that symmetry breaking by radiative corrections would yield $m_X/m_P = O(10^{-5})$. However, our studies are very incomplete:

- We have no understanding of the unnatural approximate coincidence of zeros in two Higgs coupling constants [λ_C and λ_J in our SU(5) example (2.26)].
- We have no understanding of the absence of mass terms for the Higgs fields above the grand unification mass.
- We have no real understanding of the magnitudes of the quartic Higgs couplings put in at the Planck mass.

It is a very aesthetic possibility that all symmetry breaking arises from radiative corrections, and this is certainly consistent with all our considerations, but we have no good understanding of why it should occur. However, if all symmetry breaking should arise from radiative corrections, then the renormalization group [15] provides us with some qualitative understanding of the gross

features of the ratios of interaction scales encountered below the Planck mass, and allows us to believe that this may be the only fundamental scale in Nature.

Acknowledgements

We would like to thank D.V. Nanopoulos and members of the CERN gauge theory workshop for comments and discussions on this work. One of us (J.E.) thanks Prof. P. Leal Ferreira and the Instituto de Física Teórica of São Paulo for kind hospitality while this work was in progress.

APPENDIX A

MINIMIZATION OF THE COLEMAN-WEINBERG [11] POTENTIAL

In this appendix we discuss the minimum of the Coleman-Weinberg potential. First we show that the vector-loop contribution has a global minimum for breaking to $SU(3) \otimes SU(2) \otimes U(1)$ in the absence of a quintuplet. This argument supports an ansatz for finding a good approximation to the global minimum in the general case, using the quantity defined in eq. (3.2). Using the latter procedure we show that, once the quintuplet is included, the vector-loop potential has a minimum in the direction $[SU(2)]^2 \otimes U(1)$ with $|\langle H \rangle| = O(|\langle \phi \rangle|)$. We then show that this solution lies extremely close to a true stationary point of the potential. Finally we use the above ansatz to investigate the full Coleman-Weinberg potential and conclude that its global minimum is not $SU(3) \otimes SU(2) \otimes U(1)$ invariant, and also has $|\langle H \rangle| = O(|\langle \phi \rangle|)$.

A.1 Minimization of vector loop potential with $H = 0$

For $H = 0$, the vector meson masses are given by (with a factor g^2 removed):

$$m_{i\alpha\beta}^2 = \frac{1}{2} (\varphi_\alpha - \varphi_\beta)^2 \quad (A.1)$$

where $m_{i\alpha\beta}$ is the mass of the $\alpha\beta$ element of the 5×5 matrix expression for the 24-plet of vector mesons. We have taken ϕ diagonal and the φ_α , $\alpha = 1, \dots, 5$, are its diagonal elements. There are at most 20 non-vanishing vector masses, and they satisfy

$$\sum_{i=1}^{20} m_i^2 = 5 \text{Tr } \phi^2 \quad ; \quad \sum_{i=1}^{20} m_i^4 = \frac{3}{2} (\text{Tr } \phi^2)^2 + \frac{5}{2} \text{Tr } \phi^4 \quad (A.2)$$

We shall minimize the function

$$\begin{aligned} F(\phi) = & \sum_{i=1}^{20} m_i^4 \ln m_i^2 / \mu^2 + \lambda_0 (\sum \varphi_\alpha) + \lambda_1 (\sum m_i^2 - 5 \sum \varphi_\alpha^2) \\ & + \lambda_2 \left(\sum m_i^4 - \frac{3}{2} (\sum \varphi_\alpha^2)^2 - \frac{5}{2} \sum \varphi_\alpha^4 \right) \end{aligned} \quad (A.3)$$

with respect to the m_i , λ_i , and ϕ_α . The λ_i are Lagrange multipliers corresponding to the necessary but not sufficient conditions on the masses (A.1). If $F(\phi)$ has a global minimum of the form (A.1), it will also be the global minimum when the full set of necessary and sufficient conditions is included. The conditions for an extremum of (A.3) are eqs. (A.2) and:

$$2m_i^2 \left(\ln m_i^2 / \mu^2 + \frac{1}{2} \right) + \lambda_1 + 2\lambda_2 m_i^2 = 0 \quad (\text{A.4})$$

$$\lambda_0 - 10 \rho_\alpha \lambda_1 - 4\lambda_2 \left(\frac{3}{2} \rho_\alpha \sum \phi_\alpha^2 + \frac{5}{2} \rho_\alpha^3 \right) = 0 \quad (\text{A.5})$$

and the traceless condition

$$\sum \rho_\alpha = 0 \quad (\text{A.6})$$

Equations (A.2) and (A.6) imply:

$$\frac{1}{12} \leq \frac{\sum m_i^4}{(\sum m_i^2)^2} \leq \frac{1}{8} \quad (\text{A.7})$$

where the lower and upper bounds correspond, respectively, to the configurations which are invariant under $SU(3) \otimes SU(2) \otimes U(1)$ and $SU(4) \otimes U(1)$. There are several solutions to eqs. (A.2) and (A.4) to (A.6).

A. $\lambda_i = 0$. Equation (A.4) has two solutions:

$$m_i = 0, \quad m_i^2 = \mu^2 e^{-\frac{1}{2}} \quad (\text{A.8})$$

If n of the m_i are non-vanishing we get

$$F = -\frac{n}{2} \mu^4 e^{-1} \quad (\text{A.9})$$

and

$$\frac{\sum m_i^4}{(\sum m_i^2)^2} = \frac{1}{n}$$

which implies with (A.7):

$$8 \leq n \leq 12$$

The lowest value of F is for $n = 12$, i.e. breaking down to $SU(3) \otimes SU(2) \otimes U(1)$.

B. $\lambda_1 \neq 0$. Then eqs. (A.5) and (A.6) imply:

$$0 = 5\lambda_1 \sum \rho_i^2 + [3(\sum \rho_i^2)^2 + 5 \sum \rho_i^4] \lambda_2 = \lambda_1 \sum m_i^2 + 2\lambda_2 \sum m_i^4 \quad (A.10)$$

Suppose m_1 and m_2 are two solutions of (A.4). These determine λ_1 and λ_2 ; in particular:

$$\lambda_2 = \lambda_2(m_1, m_2) \equiv \left[-m_1^2 \ln m_1^2/\mu^2 + m_2^2 \ln m_2^2/\mu^2 - \frac{1}{2}(m_1^2 - m_2^2) \right] / (m_1^2 - m_2^2)$$

If there is a third solution m_3 , we must have $\lambda_2(m_1, m_2) = \lambda_2(m_1, m_3)$, which implies:

$$0 = m_1^2 m_2^2 \ln m_1^2/m_2^2 + m_3^2 m_1^2 \ln m_3^2/m_1^2 + m_2^2 m_3^2 \ln m_2^2/m_3^2 \quad (A.11)$$

Taking $m_1 > m_2 > m_3$, the condition (A.11) can be expressed by

$$f(x, y) = 0 \quad : \quad x = m_1^2/m_2^2, \quad y = m_2^2/m_3^2$$

where

$$f(x, y) = \frac{\ln x}{x} - \frac{\ln xy}{xy} + \frac{\ln y}{x^2 y}$$

is a monotonically increasing function of x and y for $x, y > 1$, and

$$f(1, y) = f(x, 1) = 0$$

So there are only two non-degenerate solutions. Now using (A.10), eq. (A.4) implies:

$$\sum m_i^4 \ln m_i^2/\mu^2 = -\frac{1}{2} m^4 \quad (A.12)$$

Let

$$m_1^2 = a m^2, \quad m_2^2 = m^2, \quad a > 1$$

and suppose m_1 is n -fold degenerate; then m_2 is $(20 - n)$ -fold degenerate. The solution of (A.12) is

$$m^2 = \mu^2 e^{-1} \exp \left[-2na^2 \ln a / (na^2 + 20 - n) \right] \quad (A.13)$$

giving for the function (A.3):

$$F = -\frac{m^2}{2} [na^2 + 20 - n] \equiv -\frac{\mu^2}{2} e^{-1} V_0 \quad (A.14)$$

The minimum of F is the maximum of

$$V_0(n, a) = (na^2 + 20 - n) \exp \left[-2na^2 \ln a / (na^2 + 20 - n) \right]$$

We have

$$\frac{\partial V_0}{\partial a} = C \left[\frac{na^2}{na^2 + 20 - n} - 1 \right] \quad (A.15)$$

where

$$C = 4na \ln a \exp \left[-2na^2 \ln a / (na^2 + 20 - n) \right] > 0$$

Solutions with $n = 0$ or $n = 20$ are not allowed because they reduce to case A with $n = 20$, disallowed by eq. (A.7). Therefore for fixed n , $\partial V_0 / \partial a < 0$, and V_0 will be a maximum for the smallest value of a compatible with the constraint (A.7), i.e.

$$\left. \begin{aligned} \frac{1}{12} &\leq R(a) \leq \frac{1}{8} \\ R(a) &= \frac{a^2 n + 20 - n}{(an + 20 - n)^2} = \frac{\sum m^4}{(\sum m^2)^2} \end{aligned} \right\} \quad (A.16)$$

We have $\partial R(a) / \partial a > 0$ for $a > 1$, so the smallest value of a corresponds to the smallest allowed value of R . Solving the equation

$$R(a) = \frac{1}{12}$$

for $a > 1$ gives

$$a = \left[20 - n + \sqrt{96(20 - n)/n} \right] / (12 - n) \quad (A.17)$$

In addition

$$R(a) \leq R(\infty) = \frac{1}{n}$$

which with (A.7) implies $n \leq 12$. Therefore the solution will be of the form (A.14) with a given by (A.17) and $1 \leq n \leq 12$. Direct evaluation of (A.14) show that the lowest potential occurs for $n = 12$. But this implies $a = \infty$, i.e. $m_2 = 0$, which reduces to case A.

A.2 The full vector-loop potential

In the above example, the field configuration which minimized the potential turned out to be that which minimized the ratio (A.7). This can be understood by the fact that for fixed normalization, $\sum_{i=1}^N m_i^2 = \text{constant}$, both the functions $\sum m_i^4$ and $\sum m_i^4 \ln m_i^2/\mu^2$ are minimized by equipartition: $m_i = 1/N$. When the full constraints on the m_i are imposed: $m_i = m_i(\phi, H)$, this solution is not allowed, but both functions will be minimized by the configuration which is "closest" to equipartition. In general this will not be exactly the same for both functions, although it was in the preceding example. However, we expect the two minimal configurations to be sufficiently close that the minimum of $\sum m_i^4$ is sufficient to determine at least qualitatively the configuration which minimized the Coleman-Weinberg potential.

We normalize the set of Higgs fields by:

$$\sum m_i^2 = a \sum \phi^2 + \alpha |H|^2 \equiv 1 \quad (\text{A.18})$$

and make the following definitions:

$$\sum \phi^2 = y, \quad |H|^2 = x = (1-ay)/\alpha, \quad \rho = \sum \phi^4/y^2 \quad (\text{A.19})$$

Then we have in general

$$R = \frac{\sum m_i^4}{(\sum m_i^2)^2} = c y^2 + d \rho y + \beta (1-ay)^2/\alpha^2 + \gamma y(1-ay)/\alpha + \delta x y(1-ay)/\alpha \quad (\text{A.20})$$

where we chose x_M to be the minimum (maximum) of the ϕ_α^2/y if $\delta > 0$ ($\delta < 0$). The function (A.20) is identical in structure to the true potential with no mass parameters, and its minimization is similar to the procedure used for determining positivity conditions (section 2). If we consider only the vector-loop contribution we have:

$$\left. \begin{aligned} a = 5, \quad \alpha = \frac{24}{5}, \quad d = \frac{5}{2}, \quad \delta = 5, \quad c = \frac{3}{2}, \\ \beta = \frac{66}{25}, \quad \gamma = 1 \end{aligned} \right\} \quad (A.21)$$

Since $d, \delta > 0$, for fixed y we have to minimize the quantity:

$$f = dpy + \delta x_M (1 - ay)/\alpha \quad (A.22)$$

For fixed $x_M \equiv \phi_1^2/y$, $\phi_1^2 \leq \phi_{i>1}^2$, the extremum equations,

$$\left. \begin{aligned} \frac{\partial f}{\partial \phi_i} &= 0 \quad ; \quad i=2, \dots, 5 \\ \sum \phi_i^2 + \phi_1 &= y \quad ; \quad \sum \phi_i + \phi_1 = 0 \end{aligned} \right\} \quad (A.23)$$

have five solutions:

$$\left. \begin{aligned} \phi_1 &= (4, -1, -1, -1, -1)/\sqrt{20} \\ \phi_2^\pm &= (\phi_1, \phi_2, \phi_2, \phi_2, \phi_3): \begin{cases} \phi_2 = -\frac{\phi_1}{4} \pm \frac{\sqrt{4-5x_M}}{4\sqrt{3}} \\ \phi_3 = -\frac{\phi_1}{4} \mp \sqrt{3} \frac{\sqrt{4-5x_M}}{4} \end{cases} \\ \phi_3 &= (\phi_1, \phi_+, \phi_+, \phi_-, \phi_-): \quad \phi_\pm = -\frac{\phi_1}{4} \pm \frac{\sqrt{4-5\phi_1^2}}{4} \\ \phi_4 &= (\phi_1, -\phi_1, -\phi_1, \phi_2, \phi_1 - \phi_2): \quad \phi_2 = \frac{\phi_1}{2} + \frac{\sqrt{2-7\phi_1^2}}{2} \end{aligned} \right\} \quad (A.24)$$

The solutions ϕ_1 and ϕ_2^+ are excluded because $\phi_1^2 \not\leq \phi_{i>1}^2$; this condition requires $\phi_1^2 < 1/8$ for ϕ_2^- and ϕ_4 , and $\phi_1^2 < 2/15$ for ϕ_3 . Next we want to minimize

$$S_i \equiv \sum_{j>1} \phi_j^4 / y^2 \quad \text{for } \phi = \phi_i$$

for fixed ϕ_1 . By explicit evaluation using (A.24) and the condition $\phi_1^2 < 1/8$, we find $S_3 < S_2, S_4$. Substituting the expression for ϕ_3 in eq. (A.22) gives

$$f = \frac{1}{16} y d [4 - 4x_M + 15x_M^2] + \frac{\delta x_M}{\alpha} (1 - ay) \quad (A.25)$$

which is to be minimized as a function of x_M subject to the constraints

$$0 < y < \frac{1}{a}, \quad 0 < x_M < \frac{2}{15} \quad (\text{A.26})$$

The function (A.25) has a minimum for

$$x_M = \frac{2}{15} - \frac{8}{15} \frac{\delta}{\alpha d} \frac{(1-ay)}{y} \quad (\text{A.27})$$

which is $< \frac{2}{15}$ [the $SU(3) \otimes SU(2) \otimes U(1)$ value] unless $y = 1/a$, and is negative for

$$4\delta(1-ay) > \alpha dy$$

Using the values (A.21) we find that the minimum of (A.25) subject to (A.26) is given by (A.27) for $y > \frac{5}{28}$ and by $x_M = 0$ for $y \leq \frac{5}{28}$. For x_M in the range (A.26) and $\frac{7}{30} < \rho < \frac{13}{20}$, the slope of (A.20) is positive at $y = \frac{5}{28}$:

$$\left. \frac{\partial R(y)}{\partial y} \right|_{y=\frac{5}{28}} > 0$$

so the global minimum cannot have $y > \frac{5}{28}$. It will therefore have $x_M = 0$, giving $\rho = \frac{1}{4}$ for $\phi = \phi_3$. Using these values we finally minimize (A.20) with respect to y , and find a global minimum for

$$y = \frac{45}{379} \quad ; \quad \frac{y}{x} = \frac{\langle \text{Tr} \phi^2 \rangle}{\langle H^2 \rangle} = \frac{108}{77} \quad (\text{A.28})$$

which gives

$$V_{\min} = - 8.47 \mu^4 e^{-1} \quad (\text{A.29})$$

It is possible to show that there is a local minimum of the vector-loop potential of the form

$$\begin{aligned} \phi &= (\phi_1, \phi_2, \phi_3, \phi_4, \phi_5) \\ &= (\varphi, \varphi, -\varphi, -\varphi, 0) \end{aligned} \quad (\text{A.30})$$

and

$$H = (0, 0, 0, 0, h) \quad (A.31)$$

In such a configuration 3 of the 24 vectors remain massless, 1 has mass² equal to $\frac{4}{5} h^2 g^2$, 12 have mass² equal to $2\phi^2 g^2$ and 8 have mass² equal to $\frac{1}{2} g^2 (\phi^2 + h^2)$. The condition $dV/d\phi_5 = 0$ leads to

$$\sum_{i=1}^5 \phi_i [\phi_i^2 + h^2] \left[\ln \left(\frac{\phi_i^2 + h^2}{2\mu^2} \right) g^2 + \frac{1}{2} \right] = 0$$

which is automatically satisfied by ϕ of the form (A.30). The equations

$$\frac{\partial V}{\partial \phi_i} = 0 \quad (i=1,2,3,4) \quad \text{and} \quad \frac{\partial V}{\partial h} = 0$$

can be solved numerically and one finds

$$V_{\min} = -8.469 e^{-1} \mu^4 \quad (A.32)$$

This should be compared to $-6 e^{-1} \mu^4$ in the $SU(3) \otimes SU(2) \otimes U(1)$ case with $h = 0$ and to eq. (A.29) obtained by minimizing $\sum m_i^4 / (\sum m_i^2)^2$ as above. The value of $\sum \phi^2/h^2$ (i.e. y/x) at this minimum is 1.407, compared to the value 1.403 in eqs. (A.28).

It is simple to check that giving one of the first 4 components of H an infinitesimal (non-zero) value increases the potential; (A.32) corresponds to a local minimum with respect to changes in the values of the components of ϕ and H .

A.3 The full Coleman-Weinberg potential

We proceed once more by looking for the minimum of the ratio (A.7), which can again be expressed now in eqs. (A.18) to (A.20). Including a factor 3 for the vector-loop contribution, we have, for $\lambda_C = \lambda_J \equiv 0$,

$$d = \frac{15}{2} g^4 + 16 \left(\frac{18}{5} \lambda_M^2 + \frac{25}{18} \lambda_A^2 \right) \quad (A.33)$$

which is positive-definite, and

$$\delta = 15g^4 + \frac{80}{3} [-\lambda_D + 2\lambda_M + 3\lambda_A] \lambda_A \quad (\text{A.34})$$

which is positive for any λ_A if

$$\lambda_M > \frac{1}{2} \lambda_D - \frac{3\sqrt{3}}{4} g^2 \quad (\text{A.35})$$

Equation (A.35) is satisfied for our values

$$\lambda_D = 0.007, \quad \alpha = \frac{g^2}{4\pi} = 0.024 \quad (\text{A.36})$$

then the analysis proceeds as in (A.2), and we must minimize (A.25) subject to the constraints (A.26). The solution which is $SU(3) \otimes SU(2) \otimes U(1)$ invariant has $x_M = 2/15$, $\rho = 7/30$. From eq. (A.27) we see that for fixed $g \neq 1/a$ the minimum has $x_M < 2/15$ unless $\delta/\alpha d < 0$. This is ruled out since:

$$d = 8 \left[\frac{9}{5} g^2 + 3\lambda_D + 2\lambda_A \right] > 0 \quad (\text{A.37})$$

Therefore a necessary condition for the desired solution to be a global minimum is

$$\begin{aligned} \frac{\partial R}{\partial y} \Big|_{y=\frac{1}{a}} &= \frac{2c}{a} + \frac{2dp}{a} - \frac{\gamma}{\alpha} - \frac{\delta x_M}{\alpha} \\ x_M = \frac{2}{15} &= \frac{2c}{a} + \frac{14}{30} \frac{d}{a} - \frac{\gamma}{\alpha} - \frac{2}{15} \frac{\delta}{\alpha} > 0 \end{aligned} \quad (\text{A.38})$$

Using eqs. (A.33), (A.34), (A.37) and:

$$\left. \begin{aligned} a &= 5 \left[3g^2 + \frac{8}{3} \lambda_M + \frac{2}{3} \lambda_A \right] \\ \gamma &= 3g^4 + \frac{32}{3} \lambda_A \left[2\lambda_D + 5\lambda_M + \frac{17}{3} \lambda_A \right] \\ c &= \frac{9}{2} g^4 + 16 \left[\frac{161}{225} \lambda_M^2 - \frac{5}{24} \lambda_A^2 \right] \end{aligned} \right\} \quad (\text{A.39})$$

as well as the values (A.36) and the positivity constraints (section 2)

$$\lambda_M \geq 0 \qquad 0 \leq \lambda_A \leq \sqrt{\frac{3}{5} \lambda_M \lambda_D}$$

we find that the quantity (A.38) is positive-definite. This means that for $x_M = 2/15$, the lowest value of the potential has $y < 1/a$ and $x = |H|^2 \neq 0$, and the global minimum will have in addition $x_M < 2/15$.

A TOY MODEL

In this appendix we discuss a toy model which can be solved explicitly at the one-loop level, namely the Weinberg-Salam model [1] with two complex Higgs doublets ϕ_1, ϕ_2 for which we want to achieve a v.e.v. hierarchy

$$|\phi_1|^2 \gg |\phi_2|^2$$

the fields ϕ_1 and ϕ_2 are assumed to transform identically with respect to the local gauge transformations. In addition we shall impose invariance under a global and a discrete [peculiar to SU(2)] symmetry which leave the gauge couplings invariant:

$$\phi_1 \rightarrow e^{i\theta} \phi_1 \quad \text{or} \quad \phi_1 \rightarrow \tau_2 \phi_1^* \quad \text{with} \quad \phi_2 \rightarrow \phi_2 \quad (\text{B.1})$$

Then the most general Higgs coupling with no mass parameters is

$$V_L(\mu) = \lambda_1(\mu) |\phi_1|^4 + \lambda_2(\mu) |\phi_2|^4 + \lambda_3(\mu) |\phi_1|^2 |\phi_2|^2 \quad (\text{B.2})$$

The vector meson masses are of the form:

$$m_{V_i}^2 = a_i g^2 [|\phi_1|^2 + |\phi_2|^2]$$

with $a_i = O(1)$. To simplify the discussion we shall assume that a "non-vanishing" λ_i is $O(g^3)$ so that

$$\lambda_i \gg g^4 \gg \lambda_i^2 \quad (\text{B.3})$$

and we need retain only the vector-loop contribution to the Coleman-Weinberg potential:

$$V_r(\mu) = \left. \begin{aligned} & A (|\phi_1|^2 + |\phi_2|^2)^2 \ln \left(\frac{|\phi_1|^2 + |\phi_2|^2}{\mu^2} \right) \\ & + a (|\phi_1|^2 + |\phi_2|^2) + b |\phi_1|^2 |\phi_2|^2 \end{aligned} \right\} \quad (\text{B.4})$$

with $A = O(\alpha^2)$ and $a, b = O(1)$. Specification of a and b defines the renormalization prescription. We shall use the definitions

$$\lambda_i(\mu) \equiv \frac{1}{4!} \left. \frac{\partial^4 V}{\partial \phi_i^4} \right|_{\phi_i^2 = \mu^2} \quad (i=1,2) \quad (B.5a)$$

$$\lambda_3(\mu) \equiv \frac{1}{4} \left. \frac{\partial^4 V}{\partial \phi_1^2 \partial \phi_2^2} \right|_{\phi_1^2 = \phi_2^2 = \frac{\mu^2}{2}, \phi_{j \neq i} = 0} \quad (B.5b)$$

which give

$$a = -\frac{25}{6}, \quad b = -5$$

The potentials (B.2) and (B.4) contain a global $SU(2)_1 \otimes SU(2)_2$ invariance. If $\lambda_1(\mu) = 0$ there is a spontaneous breakdown of $SU(2)_1$, and if $\lambda_3(\mu) = 0$ the $SU(2)_2$ doublet ϕ_2 remains massless at the tree level. Including $V_r(\mu)$ the equations for a minimum become:

$$\frac{\partial V}{\partial \phi_1} = 2\lambda_3 \phi_1 \phi_2^2 + A \left[4\phi_1(\phi_1^2 + \phi_2^2) \ln \left(\frac{\phi_1^2 + \phi_2^2}{\mu^2} \right) - \frac{44}{3} \phi_1^3 - 8\phi_1 \phi_2^2 \right] = 0 \quad (B.6a)$$

$$\frac{\partial V}{\partial \phi_2} = 4\lambda_2 \phi_2^3 + 2\lambda_3 \phi_2 \phi_1^2 + A \left[4\phi_2(\phi_1^2 + \phi_2^2) \ln \left(\frac{\phi_1^2 + \phi_2^2}{\mu^2} \right) - \frac{44}{3} \phi_2^3 - 8\phi_2 \phi_1^2 \right] = 0 \quad (B.6b)$$

We assume that λ_3 is "non-vanishing", i.e.

$$\lambda_2 \gg \lambda_3, \quad A \quad (B.7)$$

which requires $|\phi_2| \ll |\phi_1|$ for eqs. (B.6) to be satisfied. An approximate solution to (B.6a) is obtained by setting $\phi_2 = 0$, giving:

$$\phi_1 = \mu \exp \frac{11}{6}$$

Expanding the logarithm in (B.6b) with respect to ϕ_2^2/ϕ_1^2 and using (B.7), we get the condition:

$$0 = 4\lambda_2 \phi_2^2 + 2\mu^2 \exp \frac{11}{3} \left(\lambda_3 + \frac{10}{3} A \right) + O(g^4 \phi_2^4 / \phi_1^2) \quad (B.8)$$

To avoid a minimum with $\phi_2^2/\phi_1^2 = O(A/\lambda_2) = O(g)$, we must impose:

$$\lambda_3 + \frac{10}{3} A > 0 \quad (B.9)$$

In addition we get

$$\left. \frac{\partial^2 V}{\partial \phi_1^2} \right|_{\substack{\phi_2=0 \\ \phi_1 = \mu \exp \frac{11}{6}}} = 8A\mu^2 \exp \frac{11}{3} \quad (\text{B.10})$$

so that the piece of ϕ_1 which is not eaten by a massive boson acquires a mass $O(\alpha\mu)$, and

$$\frac{\partial^2 V}{\partial \phi_2^2} = 2\mu^2 \exp \frac{11}{3} \left(\lambda_3 + \frac{10}{3} A \right) \quad (\text{B.11})$$

Then in order to avoid a similar mass scale for ϕ_2 we must set

$$\lambda_3 = -\frac{10}{3} A \quad (\text{B.12})$$

Equation (B.12) can equally well be regarded as a redefinition of b , i.e. a modification of the prescription (B.5b).

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Figure captions

- Fig. 1 : Qualitative illustration of the evolution of quartic Higgs couplings in a grand unified theory, with the SU(5) examples indicated in parentheses. The two arrows indicate the evolution of the (fundamental)⁴ coupling before and after the superheavy symmetry breaking.
- Fig. 2 : Example A of the evolution of the quartic Higgs couplings in the SU(5) model, starting with $\lambda_C = 0$ and $\lambda_M \ll g^2$ at 2×10^{14} GeV, and finding $\lambda_M, \lambda_C = O(g^2)$ at 2×10^{19} GeV.
- Fig. 3 : Evolution in the (λ_M, λ_C) plane. Shown are example A (qualitatively) and the limiting $\lambda_{C-}, \lambda_{M-}$ curves explained in the text.
- Fig. 4 : Comparative growth of λ_M and λ_D normalized at the same initial value. The mean slope for λ_C is also shown.

Fig. 1

Fig. 2

Fig. 3

Fig. 4