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SUPERSYMMETRIC POINT PARTICLES AND MONOPOLES WITH NO STRINGS

A.P. Balachandran ⁺⁾
Physics Department, Syracuse University

G. Marmo ^{*)}
Istituto di Fisica Teorica, Università di Napoli
and
Physics Department, Northeastern University, Boston

B.S. Skagerstam ^{")}
CERN -- Geneva

and
A. Stern ⁺⁾
Physics Department, Syracuse University

ABSTRACT

In previous work, an action principle for the charge monopole system was developed and it was canonically quantized. The action was single valued and free of string singularities. In this work, we supersymmetrize the preceding action and canonically quantize this new action. It describes a charged spin-half particle of gyromagnetic ratio 2 in a magnetic monopole field. It is shown that if a system admits only local Lagrangians for a configuration space Q , then under certain conditions, it admits a global Lagrangian when Q is enlarged to suitable $U(1)$ bundle over A . (The charge monopole system is an example of such a system.) Conditions under which a symplectic form is derivable from a Lagrangian are also found.

⁺⁾ Supported by the US Department of Energy.
Physics Department, Syracuse University, Syracuse,
NY 13210, USA.

^{*)} Supported by Istituto Nazionale di Fisica Nucleare, Italy.
Permanent address: Istituto di Fisica Teorica, Università
di Napoli, Mostra d'Oltremare, Pad. 19, 80125 Napoli,
Italy.

^{")} Permanent address: Institute of Theoretical Physics,
FACK, S-40220 Göteborg 5, Sweden.

1. INTRODUCTION

The formulation of the classical and quantum mechanics of a non-relativistic charged particle in the field of a magnetic monopole is of considerable theoretical interest. Previous treatments of this system have used either a Lagrangian singular along the Dirac string¹⁾ or a multiple valued action integral²⁾. Recently, a Lagrangian for this system has been constructed which is single valued and free of strings^{3),4)}. It has also been canonically quantized in a simple way^{3),4)}.

In this paper, we supersymmetrize the Lagrangian of Refs 3) and 4) and quantize it. We thus obtain the description of a non-relativistic charged spin half particle in a magnetic monopole field. The gyromagnetic ratio of the particle is 2.

Let $x = (x_1, x_2, x_3)$ denote the relative co-ordinate of the charge and the monopole and let

$$r = \left(\sum_{\alpha=1}^3 x_{\alpha}^2 \right)^{1/2} \quad (1.1)$$

$$\hat{x} = \frac{x}{r} \quad (1.2)$$

The variables in the Lagrangian of Refs 3) and 4) are not $r, \dot{r}, \hat{x}$ and $\dot{\hat{x}}$. (Here, the dot denotes time differentiation). They are instead $r, \dot{r}, g$ and $\dot{g}$, where g is a 2×2 unitary matrix of determinant 1 [thus, $g \in SU(2)$ where by $SU(2)$ is meant its conventional defining representation]. The relation of $\hat{x}$ to g is

$$\sigma_{\alpha} \hat{x}_{\alpha} = g \sigma_3 g^{\dagger} \quad (1.3)$$

where σ_{α} are the Pauli matrices.

In the supersymmetric charge-monopole system, $x, \hat{x}$ and r become the even elements $x_*, \hat{x}_*$ and r_* of a Grassmann algebra. Correspondingly, our Lagrangian for this system depends on r_*, g_* and their derivatives where g_* is a 2×2 unitary, unimodular matrix whose matrix elements are even Grassmann variables. Thus

$$g_* \in SU(2)_* \quad (1.4)$$

where $SU(2)_*$ is the Grassmannian generalization of $SU(2)$.

In Section 2, we write down the supersymmetric Lagrangian and derive the equations of motion. In Section 3, this Lagrangian is quantized using canonical methods.

Slightly more technical results have been gathered together in Section 4. The relation of our supersymmetrization procedure to a supersymmetric version of the Hopf fibration $U(1) \rightarrow SU(2) \rightarrow S^2$ ⁵⁾ is discussed in Section 4.1. [The latter is important in the absence of supersymmetry, cf. Refs. 3) and 4).] It is known that the charge-monopole interaction can be described by a system of local Lagrangians on Q (with no string singularities)²⁾. Here Q is the space of relative co-ordinates with the origin $[x = 0]$ removed. Thus if $\{U_\alpha\}$ is a covering of Q by co-ordinate neighbourhoods U_α , then we can find Lagrangians $\mathcal{L}_\alpha$ defined on U_α . In the intersection $U_\alpha \cap U_\beta$, $\mathcal{L}_\alpha - \mathcal{L}_\beta$ is the time derivative of a function. However, there exists no one Lagrangian defined on all of Q . In Refs 3) and 4), a global Lagrangian $\mathcal{L}$ was found by enlarging the degrees of freedom in $\mathcal{L}$ from those of Q to those of a $U(1)$ bundle on Q . In Section 4.2, we investigate when some such procedure will in general yield a global Lagrangian. It is proved that any system which admits local Lagrangians admits a global Lagrangian under certain conditions. (One of these conditions is a quantization condition at the classical level. All these conditions are, of course, fulfilled by the charge-monopole system^{*)}). If Q is the configuration space for the local Lagrangians, a certain $U(1)$ bundle on Q is the configuration space for the global Lagrangian. Thus it seems that nothing more involved than a $U(1)$ bundle on Q is normally necessary to construct a global Lagrangian for a system. In Section 4.3, we find the necessary and sufficient conditions on a symplectic form in order that it is derivable from a Lagrangian. Sections 4.1 - 4.3 exclusively deal with real Lagrangians and do not study supersymmetric Lagrangians.

2. THE SUPERSYMMETRIC LAGRANGIAN

2.1 The Lagrangian

The Lagrangian of Refs 3) and 4) is

$$\mathcal{L} = \frac{1}{2} m \frac{dx_\alpha}{dt} \frac{dx_\alpha}{dt} + i n \text{Tr} \sigma_3 \dot{g}^\dagger \dot{g} \quad (2.1)$$

*) For one charge and one monopole, the quantization condition [namely (4.14)] leads to no restriction [see discussion after (4.18)]. For several charges and monopoles, the need for the quantization condition has been shown before in Ref. 4). It implies that $\bar{e}\bar{g}/\bar{e}'\bar{g}'$ is rational, where $\bar{e}$ and $\bar{e}'$ are the electric charges of any two electric monopoles, and $\bar{g}$ and $\bar{g}'$ are the magnetic charges of any two magnetic monopoles.

Here m is the reduced mass, n is a constant (proportional to the product of the electric and magnetic charges [cf. (2.34)] and the rest of the variables were explained in Section 1. In the first term in (2.1), it is understood that x_α is to be regarded as a function of r and g . This is accomplished by using Eqs (1.1) and (1.2). Thus $\mathcal{L}$ is a function of r, g and their time derivatives.

The supersymmetric version of (2.1) is

$$\mathcal{L}_* = -i \frac{m}{2} \dot{x}_{*\alpha}(t, \theta) \Delta x_{*\alpha}(t, \theta) + n \text{Tr} g_*^+(t, \theta) \Delta g(t, \theta) \quad (2.2)$$

Here the parameter space has been enhanced by the addition of an odd Hermitean Grassmann variable θ . The operator Δ is defined as follows⁶⁾:

$$\Delta = \frac{\partial}{\partial \theta} + i \theta \frac{\partial}{\partial t} \quad (2.3)$$

The superfields $x_{*\alpha}$ are even and Hermitean, while g_* has been defined in Section 1. The relation between x_* and g_* is a generalization of (1.3):

$$r_* = \left(\sum_{\alpha=1}^3 x_*^2 \right)^{1/2} \quad (2.4)$$

$$\hat{x}_* = x_* / r_* \quad (2.5)$$

$$\sigma_\alpha \hat{x}_{*\alpha} = g_* \sigma_3 g_*^+ \quad (2.6)$$

We will now simplify the expression for the action

$$\mathcal{S}_* = \int dt d\theta \mathcal{L}_* \quad (2.7)$$

and derive the equations of motion. This is accomplished by the following substitutions:

1) We can write

$$x_{*\alpha}(t, \theta) = x_\alpha(t) + \frac{i\theta}{\sqrt{m}} f_\alpha(t) \quad (2.8)$$

where x_{α} and f_{α} are both Hermitean Grassmann variables, the first of which is even and the second is odd. It follows that

$$r_*(t, \theta) = r(t) + i\theta \frac{x(t) \cdot f(t)}{\sqrt{m} r(t)} \quad (2.9)$$

$$\hat{x}_*(t, \theta) = \hat{x}(t) + \frac{i\theta}{\sqrt{m} r(t)} (f(t) - \hat{x}(t) \cdot f(t) \hat{x}(t)) \quad (2.10)$$

where r and $\hat{x}$ are as in (1.1) and (1.2).

2) We can write

$$g_*(t, \theta) = \left(1 + \frac{\theta \Gamma(t)}{\sqrt{m}}\right) g(t) \quad (2.11)$$

$$\Gamma(t) = \sigma_{\alpha} \gamma_{\alpha}(t) \quad (2.12)$$

where $g(t) \in \text{SU}(2)$ and $\gamma_{\alpha}(t)$ is Hermitean and odd. (It is easy to verify that $g_{**}^{\dagger} g_{**} = 1$ for the form (2.11). Also since $\theta^2 = 0$, $\det[1 + \theta \Gamma/\sqrt{m}] = \exp\{\text{Tr} \log [1 + \theta \Gamma/\sqrt{m}]\} = 1$ and hence $\det g_* = \det g = 1$.)

3) The substitution of (2.11) in (2.6) gives

$$\hat{x}_* = \hat{x} + \frac{\theta}{\sqrt{m}} [\Gamma, \hat{x}] \quad (2.13)$$

where

$$\hat{x}_* = \sigma_{\alpha} x_{*\alpha} \quad (2.14)$$

$$\hat{x} = \sigma_{\alpha} x_{\alpha} = g \sigma_3 g^{\dagger} \quad (2.15)$$

Comparison of (2.10) and (2.13) gives

$$\frac{F}{r} = -i [\Gamma, \hat{x}] + \frac{\hat{x} \cdot f}{r} \hat{x} \quad (2.16)$$

where

$$F = f_\alpha \sigma_\alpha \quad (2.17)$$

We can solve (2.16) for Γ . It is determined up to a term which commutes with $\hat{X}$. We find

$$\Gamma = \frac{i}{4\pi} [F, \hat{X}] + \gamma \cdot \hat{x} \hat{X} \quad (2.18)$$

The following identity can be proved from (2.16) and (2.18):

$$\text{Tr}(\hat{X} \Gamma^2) = \frac{1}{4\pi^2} \text{Tr}(\hat{X} F^2) \quad (2.19)$$

To show this, compute $\text{Tr} \Gamma F$ from (2.16) and $\text{Tr} F \Gamma$ from (2.18) and use the fact that $\text{Tr} [\Gamma F + F \Gamma] = 0$.

We now substitute (2.8) and (2.11) in $\mathcal{L}_*$. Then after the θ integration is done, $\mathcal{S}_*$ takes the form

$$\mathcal{S}_* = \int dt \tilde{\mathcal{L}} \quad (2.20)$$

$$\tilde{\mathcal{L}} = \frac{m}{2} \dot{x}^2 + \frac{i}{2} f_\alpha \dot{f}_\alpha + i n \text{Tr} \sigma_3 \dot{g}^\dagger \dot{g} - \frac{n}{m} \text{Tr}(\hat{X} \Gamma^2) \quad (2.21)$$

Thus by (2.19),

$$\tilde{\mathcal{L}} = \frac{m}{2} \dot{x}^2 + \frac{i}{2} f_\alpha \dot{f}_\alpha + i n \text{Tr} \sigma_3 \dot{g}^\dagger \dot{g} - \frac{n i}{m 2\pi^2} \epsilon_{\alpha\beta\gamma} \hat{x}_\alpha f_\beta \dot{f}_\gamma \quad (2.22)$$

In this expression, x is to be regarded as a function of r and g [cf. Eq. (2.15)]. For $f_\alpha = 0$, (2.22) becomes the Lagrangian of Refs 3) and 4).

2.2. The Equations of Motion

The variation of r in (2.22) gives

$$m \ddot{r} = m r \dot{x}^2 + \frac{n}{m} \frac{i}{\pi^2} \epsilon_{\alpha\beta\gamma} x_\alpha f_\beta \dot{f}_\gamma \quad (2.23)$$

The variation of f_α gives

$$\dot{f}_\alpha = - \frac{n}{m} \frac{1}{\pi^2} \epsilon_{\alpha\beta\gamma} x_\beta \dot{f}_\gamma \quad (2.24)$$

The most general variation of g is induced by an element of its Lie algebra:

$$\delta g = i \epsilon_\alpha \sigma_\alpha g \quad (2.25)$$

Such a variation causes an infinitesimal rotation of $\hat{x}$. Let us couple it with an infinitesimal rotation of f as well:

$$\delta f_\alpha = 2 \epsilon_{\alpha\beta\gamma} f_\beta \epsilon_\gamma \quad (2.26)$$

The resultant equation of motion expresses angular momentum conservation:

$$\frac{d}{dt} J_\alpha = 0 \quad (2.27)$$

$$J_\alpha = \epsilon_{\alpha\beta\gamma} x_\beta m \dot{x}_\gamma + n \dot{x}_\alpha - \frac{i}{2} \epsilon_{\alpha\beta\gamma} f_\beta f_\gamma \quad (2.28)$$

If g alone is varied, the resultant equation is clearly a combination of (2.27) and (2.24). Thus such a variation need not be considered separately.

Note that in view of (2.24) the component

$$S_\alpha = -\frac{i}{2} \epsilon_{\alpha\beta\gamma} f_\beta f_\gamma \quad (2.29)$$

of the spin vector $\vec{S}$ obeys the equation

$$\dot{S}_\alpha = \frac{n}{m r^3} \epsilon_{\alpha\beta\gamma} S_\beta x_\gamma \quad (2.30)$$

2.3. Interpretation

Consider the motion of a particle of electric charge $\bar{e}$ and spin $\vec{S}$ in the magnetic monopole field

$$\vec{B} = \frac{\bar{g}}{4\pi} \frac{\vec{x}}{r^3} \quad (2.31)$$

Let the Hamiltonian have the conventional form

$$H = \frac{\vec{p}^2}{2m} + \frac{\bar{e}}{m} \vec{S} \cdot \vec{B} \quad (2.32)$$

As is known⁷⁾, to take the effect of $\vec{B}$ into account, in conjunction with (2.32) we should use the Poisson brackets (PB's)

$$\{x_i, p_j\} = \delta_{ij} \quad (2.33)$$

$$\{p_i, p_j\} = -\eta \epsilon_{ijk} \frac{x_k}{r^3}, \quad \eta = \frac{e\bar{g}}{4\pi} \quad (2.34)$$

$$\{S_\alpha, S_\beta\} = \epsilon_{\alpha\beta\gamma} S_\gamma \quad (2.35)$$

The remaining PB's are zero.

It is easy to show that with the identification (2.29), the equations of motion for this system are precisely (2.23), (2.27) and (2.30). Thus the gyro-magnetic ratio of the particle is 2.

There was an additional equation of motion (2.24) in Section 2.2. The equation for spin precession was a result of this equation and the definition (2.29). When we quantize the system, we will see that as a consequence of (2.29), the spin of the particle is $\frac{1}{2}$.

Thus the action $\oint_{\star}$ describes the non-relativistic limit of a Dirac particle in the presence of the magnetic monopole field $\vec{B}$.

3. THE CANONICAL FORMALISM

In this section, we will quantize the Lagrangian (2.22) and show how it leads to the system described in Section 2.3.

3.1 The classical canonical formalism

We recall that in (2.22), x is to be regarded as a function of r and g . Let $\xi = (\xi_1, \xi_2, \xi_3)$ be some local co-ordinates for g so that $g = g(\xi)$. The specific details of this parametrization of g are not important for us. Thus $\mathcal{L}$ is a function of $r, \xi, \dot{r}, \dot{\xi}$ and their derivatives.

For deriving the canonical formalism, we follow the methods of Refs. 8), 9) and 3) to deal with group elements and the methods of Ref. 10) to deal with the anti-commuting variables.

Let p_r, π_ρ and χ_α denote the momenta conjugate to r, ξ_ρ and f_α . The PB between a pair of anticommuting variables C and D is defined as follows:

$$\{C, D\} = -\left(\frac{\partial C}{\partial f_\alpha} \frac{\partial D}{\partial \chi_\alpha} + \frac{\partial C}{\partial \chi_\alpha} \frac{\partial D}{\partial f_\alpha}\right) \quad (3.1)$$

Hence

$$\{f_\alpha, f_\beta\} = \{\chi_\alpha, \chi_\beta\} = 0 \quad (3.2)$$

$$\{f_\alpha, \chi_\beta\} = -\delta_{\alpha\beta} \quad (3.3)$$

The remaining PB's are defined in the normal way.

It is convenient to replace π_ρ by another set of variables

$$L_\alpha = -\frac{1}{2} \pi_\rho N_{\rho\alpha}(\xi) \quad (3.4)$$

where the non-singular matrix N was defined elsewhere^{9), 8)}. It fulfils the identity

$$\frac{\partial g(\xi)}{\partial \xi_\rho} N_{\rho\alpha}(\xi) = i\sigma_\alpha g(\xi) \quad (3.5)$$

As a consequence,

$$\{L_\alpha, g\}(\xi) = \frac{1}{2} \frac{\partial g(\xi)}{\partial \xi_\rho} N_{\rho\alpha}(\xi) = i\frac{\sigma_\alpha}{2} g(\xi) \quad (3.6)$$

Similarly, since $gg^{-1} = 1$,

$$\{L_\alpha, g^{-1}\}(\xi) = -ig^{-1}(\xi) \frac{\sigma_\alpha}{2} \quad (3.7)$$

It is also easy to show from (3.6) or (3.7) that^{8), 9)}

$$\{L_\alpha, L_\beta\} = \epsilon_{\alpha\beta\gamma} L_\gamma \quad (3.8)$$

The PB's (3.6) to (3.8) do not depend on the parametrization of g and are simple. Hence it is more convenient to work with L_α instead of π_ρ . In view

of (2.15), (3.6) and (3.7), L generates spatial rotations on $\hat{x}$ and is thus the orbital part of the total angular momentum. For the Lagrangian (2.1), it is thus identical in form to the conserved angular momentum of the charge monopole system when no intrinsic spin is present [cf. Ref. 3) and Eq. (3.10)].

For the Lagrangian $\tilde{L}$, the expressions for the momenta are as follows:

$$\vec{p}_r = m\dot{\vec{r}} \quad (3.9)$$

$$L_\alpha = -\frac{1}{2} \frac{\partial \tilde{L}}{\partial \dot{\vec{f}}_\alpha} \dot{\vec{f}}_\alpha = \epsilon_{\alpha\beta\gamma} \vec{x}_\beta \frac{d}{dt} m \vec{x}_\gamma + n \vec{x}_\alpha \quad (3.10)$$

$$\chi_\alpha = \frac{\partial \tilde{L}}{\partial \dot{f}_\alpha} \quad (3.11)$$

$$= -\frac{i}{2} f_\alpha \quad (3.12)$$

[Equation (3.5) is used in the derivation of Eq. (3.10), cf. Ref. 3).] Thus we have the primary constraints

$$\varphi = \vec{x}_\alpha L_\alpha - n \quad (3.13)$$

$$\psi_\alpha = \chi_\alpha + \frac{i}{2} f_\alpha \quad (3.14)$$

which vanish weakly.

The Hamiltonian is

$$H = \dot{\vec{r}} \cdot \vec{p}_r + \dot{\vec{f}} \cdot \vec{\pi}_f + \dot{f}_\alpha \chi_\alpha - \tilde{L} + v\varphi + v_\alpha \psi_\alpha \quad (3.15)$$

$$= \frac{\vec{p}_r^2}{2m} + \frac{1}{2mr^2} (L_\alpha L_\alpha - n^2) + \frac{n}{m} \frac{i}{2r^2} \epsilon_{\alpha\beta\gamma} \vec{x}_\alpha \dot{\vec{f}}_\beta \dot{\vec{f}}_\gamma + v\varphi + v_\alpha \psi_\alpha \quad (3.16)$$

where v and v_α are Lagrange multipliers.

The requirement $\{\phi, H\} \approx 0$ leads to no new equation while the requirement $\{\psi_\alpha, H\} \approx 0$ only leads to a determination of v_α . Thus there are no secondary constraints.

The constraints fulfil the PB's

$$\{\phi, \psi_\alpha\} = 0 \quad (3.17)$$

$$\{\psi_\alpha, \psi_\beta\} = -i\delta_{\alpha\beta} \quad (3.18)$$

Hence ϕ is first class and ψ_α 's are second class.

The second class constraints can be eliminated by introducing the Dirac brackets (DB's) ¹⁰.

$$\{f_\alpha, f_\beta\}^* = -i\delta_{\alpha\beta} \quad (3.19)$$

$$\{f_\alpha, \chi_\beta\}^* = -\frac{1}{2}\delta_{\alpha\beta} \quad (3.20)$$

$$\{\chi_\alpha, \chi_\beta\}^* = -\frac{i}{4}\delta_{\alpha\beta} \quad (3.21)$$

The DB's which involve r, p_r, g or L_α are equal to their corresponding PB's. The constraints ψ_α have zero DB's with all the variables. Thus we can replace PB's by DB's and then eliminate χ_α by setting $\chi_\alpha = -i/2 f_\alpha$. They are subject to the constraint $\phi \approx 0$.

The Lagrange multiplier v in (3.16) is arbitrary. Thus only those variables with (weakly) zero DB's with ϕ have a well defined time evolution. They are the first class variables or observables of the theory. Only such variables are of physical interest. The discussion of Ref. 3) shows that a complete set of such observables is $r, p_r, \hat{x}_\alpha, L_\alpha$ and f_α . They are subject to the one constraint $\phi \approx 0$.

3.2. Quantum Mechanics

The algebra of observables in quantum theory is obtained by replacing the DB's in (3.19) - (3.21) by $(-i)$ times the anticommutator brackets and the remaining PB's by $(-i)$ times the commutator brackets. In particular ($\hbar = 1$)

$$[f_\alpha, f_\beta]_+ = \delta_{\alpha\beta} \quad (3.22)$$

It is known as a consequence¹⁰⁾ that an irreducible representation of f_α 's is obtained in quantum theory by the identification

$$f_\alpha = \frac{1}{\sqrt{2}} \sigma_\alpha \quad (3.23)$$

Thus the spin variables (2.29) are represented by the operators

$$S_\alpha = \frac{1}{2} \sigma_\alpha \quad (3.24)$$

Hence the spin of the particle in quantum mechanics is $\frac{1}{2}$.

The quantization of the remaining variables has been discussed elsewhere^{3),4)}. In particular, it has been shown that n must fulfil the Dirac quantization condition

$$2n = \text{integer} \quad (3.25)$$

Finally, we note the following. The equations of motion for (2.32) show that $p_i = m\dot{x}_i$. It is simple to manipulate the Hamiltonian (2.32) to the form (3.16) (with $v = v_\alpha = 0$) if this and the identity $\varepsilon_{\alpha\beta\gamma} x_\beta p_\gamma = L_\alpha - n\hat{x}_\alpha$ are taken into account. (Of course, when computing the time evolution of observables, it is permissible to set $v = v_\alpha = 0$.) The bracket relations (2.33) - (2.35) are also implied by the algebra of observables appropriate to (3.16).

4. FINAL REMARKS

4.1. The supersymmetric Hopf Fibration

It is well known that the description of a spinless charged particle in a magnetic monopole field involves the use of non-trivial $U(1)$ bundles L_M over the two-sphere S^2 . They may all be constructed from the $U(1)$ Hopf bundle over S^2 ⁵⁾. The latter is topologically S^3 which we may regard as $SU(2)$. The group $U(1)$ acts on $SU(2)$ as follows:

$$g \longrightarrow g \exp(i\sigma_3 \alpha) \quad (4.1)$$

Hence the projection map to S^2 is

$$g \longrightarrow g \sigma_3 g^\dagger = \sigma_\alpha \hat{x}_\alpha \quad (4.2)$$

Let $(M \geq 1)$ and

$$\mathbb{Z}_M = \left\{ \mathbb{Z}_k = \exp(i\sigma_3 \frac{2\pi k}{M}) \mid k=0,1,2,\dots,M-1 \right\} \quad (4.3)$$

It is a cyclic group of order M . It has an action

$$g \longrightarrow g \mathbb{Z}_k \quad (4.4)$$

on $SU(2)$. The space L_M is the quotient of $SU(2)$ by this action¹¹⁾:

$$L_M = SU(2)/\mathbb{Z}_M \quad (4.5)$$

In this paper, we have worked with the supersymmetric versions of these fibrations. We now briefly summarize their structure.

The supersymmetric version $SU(2)_*$ of $SU(2)$ was defined in Section 1.

Let

$$U(1)_* = \left\{ \exp(i\sigma_3 \alpha) \right\} \quad (4.6)$$

where α is an even Grassmann variable. It has an action

$$g_* \longrightarrow g_* \exp(i\sigma_3 \alpha) \quad (4.7)$$

on $SU(2)_*$. Hence the projection map is

$$g_* \longrightarrow g_* \sigma_3 g_*^\dagger = \sigma_\alpha \hat{x}_* \alpha \quad (4.8)$$

The image under this map is a supersymmetric version S^2_* of the two-sphere S^2 since

$$\hat{x}_* \alpha \hat{x}_* \alpha = 1 \quad (4.9)$$

The supersymmetric version L_{M*} of L_M can be constructed as before:

$$L_{M*} = SU(2)_* / \mathbb{Z}_M \quad (4.10)$$

It is of interest to examine the variation of Γ in (2.11) under a gauge transformation of the form (4.7). Writing

$$\exp(i\sigma_3\alpha(t,\theta)) = (1 + \theta\sigma_3\rho)\exp(i\sigma_3\beta(t)) \quad (4.11)$$

we find

$$g(t,\theta)\exp(i\sigma_3\alpha(t,\theta)) = (1 + \theta\frac{\Gamma(t)}{\sqrt{m}} + \theta\rho\hat{I})g(t)\exp(i\sigma_3\beta(t)) \quad (4.12)$$

Thus under a gauge transformation (4.7), Γ changes by a term parallel to $\hat{x}$. Since $\hat{x}_*$ is gauge invariant, it determines Γ only up to such a term. Equation (2.18) is consistent with this observation.

4.2. Local and global Lagrangians

The problem considered in this Section is best explained by recalling some unusual features of the charge-monopole system and its Hamiltonian and Lagrangian descriptions.

Consider the latter and assume that there is no intrinsic spin ($\vec{S} = 0$). It has a perfectly well defined canonical description with the Hamiltonian (2.32) and PB's (2.33) and (2.34). Let

$$Q = \{x \in \mathbb{R}^3 \mid x \neq 0\} \quad (4.13)$$

denote the configuration space for this system and let TQ be the associated tangent bundle (that is, the space of co-ordinates and velocities). Then it is well-known that despite the existence of a global canonical description, the system admits no globally defined Lagrangian on TQ . But it does admit a local Lagrangian description on TQ . That is, we can cover Q by coordinate patches U_α and find a Lagrangian $\mathcal{L}^{(\alpha)}$ on each tangent bundle TU_α over U_α . However, it is impossible to choose these Lagrangians such that $\mathcal{L}^{(\alpha)} = \mathcal{L}^{(\beta)}$ on all the intersections $TU_\alpha \cap TU_\beta$.

In previous work^{3),4)} and in this paper, a global Lagrangian description of the charge-monopole system has been discussed. This Lagrangian, however, is not defined on TQ , but on TE where E is a $U(1)$ bundle on Q ^{*}).

The problem we wish to consider in this Section can now be explained. Consider a system with a configuration space Q which admits a local Lagrangian description. [Thus if $\{U_\alpha\}$ is a suitable covering of Q by co-ordinate neighbourhoods, then there are Lagrangians $\mathcal{L}^{(\alpha)}$ defined on TU_α]. Let the system also admit a global canonical description. We want to find the conditions under which we can find a global Lagrangian for this system by a suitable enlargement of TQ (as in the charge-monopole case). Here, Q is of course not restricted to be of the form (4.13).

In the statement of our result, we regard the Hamiltonian or the energy and the symplectic form or essentially equivalently, the Poisson brackets as defined in terms of co-ordinates and velocities (and not in terms of co-ordinates and momenta).

We shall prove the following: suppose we are given a dynamical system with local Lagrangians $\mathcal{L}^{(\alpha)}$ defined on TU_α . We assume that the associated Hamiltonian description exists globally, that is, that the associated symplectic form ω and the energy are globally defined on TQ . Suppose now that the integral of ω over any closed two dimensional surface M in Q is an integral multiple of $2\pi\lambda$ where λ is an over-all constant:

$$\int_M \omega = 2\pi v \lambda, v = 0, \pm 1, \pm 2, \dots \quad (4.14)$$

(Here λ is the same for all M and v is characteristic of M). Then there exists a $U(1)$ bundle E over Q and a global Lagrangian on TE for this system ^{**}).

The expression for ω in local co-ordinates on TU_α is

$$\omega = d\left(\frac{\partial \mathcal{L}^{(\alpha)}}{\partial \dot{x}_i} dx_i\right) \quad (4.15)$$

*) Similarly, consider a Yang-Mills particle with co-ordinate x and internal vector $I = (I_1, I_2, \dots, I_N)$. Its configuration space Q is thus the set $\{(x, I)\}$. In general, this system also admits no global Lagrangian on TQ . A global Lagrangian however can be found on TE , where E , the group manifold of the internal symmetry group, can be regarded as a fibre bundle over Q [see Ref. 9)]. Elsewhere, it will be shown that a similar situation prevails for many spinning systems, including relativistic spinning point particles.

***) See the footnote on page 2.

$$= \frac{1}{2} \left(\frac{\partial}{\partial x_j} \left(\frac{\partial \mathcal{L}^{(\alpha)}}{\partial \dot{x}_i} \right) - \frac{\partial}{\partial x_i} \left(\frac{\partial \mathcal{L}^{(\alpha)}}{\partial \dot{x}_j} \right) \right) dx_j \wedge dx_i + \frac{\partial^2 \mathcal{L}^{(\alpha)}}{\partial \dot{x}_j \partial \dot{x}_i} dx_j \wedge dx_i \quad (4.16)$$

where the x 's are co-ordinates for U_α . The global existence of ω means

$$d \left(\frac{\partial \mathcal{L}^{(\alpha)}}{\partial \dot{x}_i} dx_i \right) = d \left(\frac{\partial \mathcal{L}^{(\beta)}}{\partial \dot{x}_j} dx_j \right) \text{ on } TU_\alpha \cap TU_\beta \quad (4.17)$$

Similarly, the global existence of energy means

$$\frac{\partial \mathcal{L}^{(\alpha)}}{\partial \dot{x}_j} \dot{x}_i - \mathcal{L}^{(\alpha)} = \frac{\partial \mathcal{L}^{(\beta)}}{\partial \dot{x}_j} \dot{x}_i - \mathcal{L}^{(\beta)} \text{ on } TU_\alpha \cap TU_\beta \quad (4.18)$$

We note that the condition (4.14) is empty for the charge-monopole system (with one charge and one monopole) for the following reason: there is only one class of homotopically non-trivial closed surfaces M for this system. A representative of this class is the two-sphere S^2 in Q which surrounds the origin. Thus, since the integral in (4.14) is zero by Stokes' theorem if M is homotopically trivial, (4.14) only defines λ . (Of course, quantum theory restricts λ by the Dirac quantization condition.)

The points of TQ are invariant under the action of $U(1)$. Since the equations of motion are on TQ , it follows that a gauge transformation by a time dependent element $u \in U(1)$ can only change the global Lagrangian $\mathcal{L}$ to an equivalent Lagrangian $\mathcal{L}'$ ⁽¹²⁾. Here $\mathcal{L}'$ is equivalent to $\mathcal{L}$ in the sense that both give the same equations of motion. Thus in this enlarged description, the system exhibits a $U(1)$ gauge invariance. It is a sort of a generalized charge-monopole system. It is interesting that nothing more involved than such generalized charge-monopole systems and $U(1)$ bundles can arise under our assumptions.

As a prelude to the proof, we first state and prove a lemma:

Let Ω be a closed two form on Q :

$$\Omega = \Omega_{ij} dx_i \wedge dx_j, \quad d\Omega = 0 \quad (4.19)$$

or

$$\partial_i \Omega_{jk} + \partial_j \Omega_{ki} + \partial_k \Omega_{ij} = 0 \quad (4.20)$$

Further, let it fulfil the analogue of (4.14) for every closed two-surface M in Q :

$$\int_M \Omega = 2\pi \nu \lambda, \nu = 0, \pm 1, \pm 2, \dots \quad (4.21)$$

Then, there exists a $U(1)$ bundle E on Q and a form $\tilde{\Omega}$ on E with the following properties:

1) $\tilde{\Omega}$ is exact:

$$\tilde{\Omega} = d\Lambda \quad (4.22)$$

Here Λ is a one form globally defined on E .

2) $\tilde{\Omega}$ is "gauge invariant" and hence projects down to a form on Q . The latter is precisely Ω . (Here by gauge invariance we mean the following: let ϕ and ϕ' be two sections from a co-ordinate neighbourhood in Q to E . Then the pull backs $\phi^* \tilde{\Omega}$ and $\phi'^* \tilde{\Omega}$ are equal¹³⁾). This result can also be stated in another way. Let $\pi : E \rightarrow Q$ be the projection map from the bundle E to the base Q . Then $\tilde{\Omega} = \pi^* \Omega$.

We assume that Q is paracompact. Then Q has a covering $\{U_\alpha\}$ such that both U_α 's and their intersections are contractible. The proof of the lemma proceeds by observing that by (4.20) and the converse of the Poincaré lemma¹⁴⁾,

$$\Omega|_{U_\alpha} = d\theta_\alpha \quad (4.23)$$

Thus

$$\theta_\alpha - \theta_\beta = df_{\alpha\beta} \text{ on } U_\alpha \cap U_\beta \quad (4.24)$$

where

$$d(f_{\alpha\beta} + f_{\beta\gamma} + f_{\gamma\alpha}) = 0 \text{ on } U_\alpha \cap U_\beta \cap U_\gamma \quad (4.25)$$

Suppose now that

$$f_{\alpha\beta} + f_{\beta\gamma} + f_{\gamma\alpha} = 2\pi n_{\alpha\beta\gamma} \lambda \quad (4.26)$$

where $n_{\alpha\beta\gamma}$ has integer values. Then we can define the following map F :

$$F(f_{\alpha\beta}) \equiv g_{\alpha\beta} = \exp\left(\frac{i}{2} f_{\alpha\beta}\right) \quad (4.27)$$

The functions $g_{\alpha\beta}$ are defined on $U_\alpha \cap U_\beta$ and have values in $U(1)$. Further, by (4.26), they fulfil the co-cycle property

$$g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = 1 \quad (4.28)$$

Hence they define a $U(1)$ bundle E on Q ¹⁵⁾.

It may be shown ¹⁶⁾ that (4.21) is equivalent to (4.26). Thus with (4.21), we have a $U(1)$ bundle on Q . Let x and x' be the co-ordinates of the same point p in $U_\alpha \cap U_\beta$ for the co-ordinate systems appropriate to U_α and U_β . Then the co-ordinates $[x, h^{(\alpha)}]$ and $[x', h^{(\alpha)} g_{\alpha\beta}(p)]$ describe the same point, in the fibre over p . Here $h^{(\alpha)} \in U(1)$.

Let

$$m_\alpha = -i\lambda h^{(\alpha)-1} dh^{(\alpha)} \quad (4.29)$$

The form m_α is defined on the fibres over U_α in the co-ordinate system appropriate to U_α . We have

$$dm_\alpha = 0 \quad (4.30)$$

Also

$$m_\alpha - m_\beta = i\lambda g_{\alpha\beta}^{-1} dg_{\alpha\beta} = -df_{\alpha\beta} \quad (4.31)$$

at a common point in the fibres over $U_\alpha \cap U_\beta$. Comparison of (4.24) and (4.31) shows that

$$\theta_\alpha + m_\alpha = \theta_\beta + m_\beta \quad (4.32)$$

Thus the one form

$$\theta = \theta_\alpha + m_\alpha \quad (4.33)$$

is globally defined on E . Further, since

$$dm_\alpha = 0 \quad (4.34)$$

we can write $\Omega = d\theta$ if we regard Ω as a form on E (in local co-ordinates, this form has the same expression as the two form we started with on Q . It is the pull back of the form on Q by the projection map $\pi : E \rightarrow Q$). The lemma is thus proved.

Let us now prove the main result. We can write

$$\omega|_{TU_\alpha} = d\psi_\alpha \quad (4.35)$$

where

$$\psi_\alpha = \frac{\partial L}{\partial x_i} dx_i \quad (4.36)$$

The identity $\omega|_{TU_\alpha} = \omega|_{TU_\beta}$ shows that $d(\psi_\alpha - \psi_\beta)$ is zero. By (4.16), this also implies that $\psi_\alpha - \psi_\beta$ is independent of $\dot{x}$ and defines a form on $U_\alpha \cap U_\beta$. Thus

$$\psi_\alpha - \psi_\beta = df_{\alpha\beta} \text{ on } U_\alpha \cap U_\beta \quad (4.37)$$

where $f_{\alpha\beta}$ fulfils (4.26) by (4.14). As before, we can construct a $U(1)$ bundle E on Q and define forms m_α with the property and (4.31). Hence the form χ defined by

$$\chi = \psi_\alpha + m_\alpha \quad (4.38)$$

exists globally and

$$d\chi = \omega \quad (4.39)$$

[Here χ is a form on TE . So m_α , ψ_α and ω are to be regarded as forms on TE in (4.38) and (4.39). That is, they are suitable pull backs of the forms we started with. For simplicity, as before, we do not introduce new notations for them].

Now by (4.18) and (4.37)*)

$$\mathcal{L}^{(\alpha)} - \mathcal{L}^{(\beta)} = \frac{df_{\alpha\beta}}{dx_i} x_i \quad (4.40)$$

Thus the Lagrangian $\tilde{\mathcal{L}}$ defined on the fibres over TU_α as

$$\tilde{\mathcal{L}} = \mathcal{L}^{(\alpha)} - i\lambda h^{(\alpha)-1} \frac{dh^{(\alpha)}}{dt} \quad (4.41)$$

is in fact globally defined on TE. Since the last term is (locally) the time derivative of a function, $\mathcal{L}^{(\alpha)}$ and $\tilde{\mathcal{L}}$ also give the same equations of motion. The result is thus proved.

We may also note here the transformation property of $\tilde{\mathcal{L}}$ under a $U(1)$ gauge transformation. Under the latter,

$$h^{(\alpha)}(t) \rightarrow h^{(\alpha)}(t) U(t); \quad U(t) \in U(1) \quad (4.42)$$

and hence

$$\tilde{\mathcal{L}} \rightarrow \tilde{\mathcal{L}} - i\lambda U(t)^{-1} \frac{dU(t)}{dt} \quad (4.43)$$

That is, $\tilde{\mathcal{L}}$ changes by the time derivative of a function. Thus the enlarged description is a gauge theory with a $U(1)$ gauge group.

4.3. When does a symplectic form admit a Lagrangian?

Suppose we are given a dynamical system specified by PB's and a Hamiltonian. It is not evident that such a system admits either a local or global Lagrangian description on TQ. In this section, we derive the necessary and sufficient conditions in order that a symplectic form ω on TQ is derivable from a Lagrangian, so that it is of the form

*) At this point, it is possible to dispense with the global energy condition (4.18) in the following way. Without this condition, in view of (4.17), (4.40) is modified to $\mathcal{L}^{(\alpha)} - \mathcal{L}^{(\beta)} = \partial f_{\alpha\beta} / \partial x_i \dot{x}_i + P_{\alpha\beta}$. Here a) $P_{\alpha\beta} = -P_{\beta\alpha}$ and $P_{\alpha\beta} + P_{\beta\gamma} + P_{\gamma\alpha} = 0$, and b) since $\mathcal{L}^{(\alpha)}$ and $\mathcal{L}^{(\beta)}$ give the same equations of motion, $P_{\alpha\beta}$ are constants and, hence, globally defined. Let $\{\psi_\alpha\}$ be a partition of unity for the covering $\{U_\alpha\}$ [$\text{supp } \psi_\alpha = U_\alpha$, $\psi_\alpha \geq 0$ and $\sum_\alpha \psi_\alpha = 1$]. Then the globally defined functions $k^{(\alpha)} = \sum_\lambda P_{\alpha\lambda} \psi_\lambda$ have the property $k^{(\alpha)} - k^{(\beta)} = P_{\alpha\beta}$ in view of a). Thus $\hat{\mathcal{L}}^{(\alpha)} = \mathcal{L}^{(\alpha)} - k^{(\alpha)}$ fulfil an equation of the form (4.40) and $\hat{\mathcal{L}} = \mathcal{L}^{(\alpha)} - i\lambda h^{(\alpha)-1} dh^{(\alpha)}/dt$ is globally defined on TE. Further, since $k^{(\alpha)}$ is independent of $\dot{x}$, ω has the usual relation to $\hat{\mathcal{L}}$. However, since $k^{(\alpha)}$ can depend on x , $\hat{\mathcal{L}}$ may not give the original equations of motion.

$$\omega = d\left(\frac{\partial L}{\partial \dot{x}_i} dx_i\right) \quad (4.44)$$

The considerations are local. Hence if the conditions are fulfilled, the system may admit only local Lagrangians and no global Lagrangian.

The conditions are the following:

$$a) \quad d\omega = 0 \quad (4.45)$$

b) The form ω is identically zero when evaluated on $\partial/\partial \dot{x}_i$ $\partial/\partial \dot{x}_j$. Thus ω contains no piece of the form $a_{ij} dx_i \wedge dx_j$ and the fibres of TQ are Lagrangian submanifolds for ω ¹⁷⁾.

c) There exists a vector field Δ of the form

$$\Delta(x, \dot{x}) = \dot{x}_i \frac{\partial}{\partial x_i} + V_i(x, \dot{x}) \frac{\partial}{\partial \dot{x}_i} \quad (4.46)$$

(such that in each co-ordinate neighbourhood)

$$L_{\Delta} \omega = 0 \quad (4.47)$$

Here L_{Δ} is the Lie derivative associated with the vector field Δ . Note that Δ is just a general second order vector field [cf. Marmo and Saletan Ref. 12)].

The necessity of the conditions a) and b) is evident from (4.44). The necessity of c) can be seen as follows. If there exists a Lagrangian for ω , its equations of motion specify a vector field Δ on TR. The time evolution due to Δ is given by

$$\frac{dF}{dt} = \Delta F \quad (4.48)$$

for any function F on TQ. Since Δ will be such that (4.48) gives $dx_i/dt = \dot{x}_i$, Δ is of second order. Further, time evolution due to Δ will be a canonical transformation for ω . The necessity of c) is thus shown.

Let us now prove the sufficiency. We work in a co-ordinate neighbourhood N with local co-ordinates $(x, \dot{x})$. Since N is star shaped, by a), we can write

$$\omega = d(A_i(x, \dot{x}) dx_i + B_i(x, \dot{x}) d\dot{x}_i) \quad (4.49)$$

By b),

$$B_i = \frac{\partial B}{\partial \dot{x}_i} \quad (4.50)$$

Hence

$$\omega = d(A_i dx_i - \frac{\partial B}{\partial \dot{x}_i} dx_i + dB) = d(C_i dx_i) \quad (4.51)$$

$$C_i = A_i - \frac{\partial B}{\partial \dot{x}_i} \quad (4.52)$$

By c),

$$L_\Delta \omega = d(L_\Delta (C_i dx_i)) = d((\Delta C_i) dx_i + C_i d\dot{x}_i) = 0 \quad (4.53)$$

By examining terms of the form $\alpha_{ij} dx_i \wedge dx_j$, we see that there exists a function $\mathcal{L}$ on N with the property

$$C_i = \frac{\partial \mathcal{L}}{\partial \dot{x}_i} \quad (4.54)$$

In view of (4.51), this proves the sufficiency of the conditions.

We have yet to discuss the remaining terms in (4.53). Suppose that the Hessian matrix H defined by

$$H_{ij} = \frac{\partial^2 \mathcal{L}}{\partial \dot{x}_i \partial \dot{x}_j} \quad (4.55)$$

is non-singular and that the force function V_i has the conventional form

$$V_i = H_{ij}^{-1} \left(\frac{\partial \mathcal{L}}{\partial x_j} - \frac{\partial^2 \mathcal{L}}{\partial \dot{x}_j \partial \dot{x}_k} \dot{x}_k \right) \quad (4.56)$$

Then it is simple to verify that

$$\Delta C_i dx_i + C_i d\dot{x}_i = d\mathcal{L} \quad (4.57)$$

So $L_{\Delta}\omega = 0$. Thus we can take Δ to be the vector field which determines the time evolution for the "Lagrangian" $\mathcal{L}$ if $\det H \neq 0$. The requirement (4.53) is then fulfilled.

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