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STRONG INTERACTIONS -- A TUNNELLING PHENOMENON ?

I. THEORY AND APPLICATION TO HADRONIC DIFFRACTION

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A B S T R A C T

A new approach to strong interactions at high energies is proposed. Hadronic scattering in a given channel emerges (for $\Theta \neq 0$) as a tunnelling phenomenon in the scattering sector due to the presence of many other competing channels. This hadronic tunnelling takes place at the interface between highly excited hadronic matter - being built up during the interaction - and the surrounding vacuum. The tunnelling amplitude is non-perturbative : it depends only on the geometry of this interface and not on the strength of hadronic interactions. This geometry turns out to be identical to the one arising in colour confinement schemes : the geometry of the surface of a long intermediate "colour electric flux tube" or "fat string", which separates the (inside) coloured quark-gluon phase from the (outside) colour singlet phase. We study such tunnelling mechanisms rigorously in a quantum mechanical laboratory using a semi-classical approximation to functional integrals extended to include complex classical paths.

The tunnelling amplitude for elastic scattering is obtained in closed analytic form for all angles ; it exhibits a strong diffraction peak, followed by an Orear-type behaviour $\exp(-A p_T)$ which automatically turns into a power law behaviour $(p_T^2)^{-N_f(\Theta)}$ at large p_T . This power law does not arise from an interaction of quasi-free quarks but from the (fat) string-like shape of their "confining walls". An extension of the tunnelling picture to multiparticle channels is indicated ; it will be the subject of a forthcoming paper. A possible new interpretation of hadronic scattering off nuclei at intermediate and high energies is discussed. An interesting relation to dual models is pointed out.

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1. INTRODUCTION AND OUTLINE OF THE IDEAS

There is much hope that a gauge theory in terms of elementary quark and gluon fields, like quantum chromodynamics (QCD), might be the key to understanding strong interactions. Unfortunately, though much effort has been spent, confinement of these constituent fields and -- related to this -- the characteristic spatial extension of hadrons and the finite range of their interactions have not yet emerged from such a framework.

In this paper we propose a new mechanism for strong interactions, which puts forward a rather complementary view. The characteristic and universal spatial extension of interacting hadronic matter plays the central role rather than its constituent aspects. Nevertheless this approach will provide a uniform picture of *small and large p_T phenomena*.

Hadronic scattering at high energies will be interpreted (for $\theta \neq 0$) as a *tunnelling phenomenon**) taking place near the interface of excited hadronic matter -- built up during the interaction -- and the surrounding vacuum. (The dynamical importance of this interface will be somewhat reminiscent of the significance of the surface of a single hadron bag¹⁻⁵), which is the interface between the inside medium of confined quarks and gluons and the outside vacuum.)

An important result of this approach concerning constituent pictures will be: out to the largest p_T presently available the "confining walls" (= interface) rather than point-like constituents inside it are probed. In our approach quarks and gluons never enter as individual (spatial) degrees of freedom; it is rather the particular shape of the "confining walls", which will give rise to a power law behaviour at large p_T : the interaction region comes out to be a longitudinal "fat string" stretching with increasing energy [very similar to the long intermediate colour electric flux tube in models of bag-bag scattering⁶⁻⁹) and/or the "fire-sausage" of Ref. 4]. For large p_T the (essentially) point-like tip of this long "fat string" is probed, thus mimicking a point-like interaction.

First preliminary versions of our ideas may be found in Ref. 10. Since a number of sections will have to be spent before a final formulation of our approach is reached, let us give some idea about the tunnelling mechanism already in this introduction.

As a mathematical laboratory to explore tunnelling phenomena in the scattering sector, we choose the certainly oversimplified framework of quantum mechanical scattering with an effective (energy dependent) potential (and relativistic kinematics). The main advantage is that it allows to investigate explicitly and rigorously such tunnelling mechanisms, which may well turn out to be equally

*) We shall use the word "tunnelling" in a generalized sense for a quantal penetration of intensity into a *classically forbidden* region (arising from *complex* classical action contributions).

important in field theory¹¹⁾. In order to facilitate such an extension we use the so-called "*complex* semiclassical approximation" method. This technique may be viewed as an extended semiclassical approximation of functional integrals involving *complex* classical paths (see Ref. 12) or as a semiclassical approximation of the formulation of quantum mechanics by Balian and Bloch¹³⁾.

We start from some short range (effective) potential which is distinguished by a strong variation across a (real or complex) surface close to its periphery as, for example, idealized by a sharp cut-off or more realistically effected by a complex singularity. This variation gives a strong dynamical significance to the peripheral region of the potential, which we naturally associate with the interface between the intermediate, excited hadronic matter and the vacuum. It is the presence of such a "geometrical" discontinuity or singularity "surrounding" the interaction region which gives rise to a *tunnelling phenomenon, being nonperturbative* in the sense that it only depends on the geometry of the interaction region and not on the strength of the potential. We shall argue that it is the important degree of freedom of copious particle production which promotes the tunnelling component to become (for $\theta \neq 0$) the dominant reaction mechanism in a given channel. Its origin makes it most insensitive to the influence of competing channels. This in turn justifies its calculation within a potential framework. In our tunnelling approach the geometry of the interface ("confining walls") enters as a general "variable" to be determined from a few key features of the data. Its dynamical origin is beyond the scope of the present framework and remains to be investigated (presumably within a microscopic theory of confined quarks and gluons).

In this paper we shall mainly concentrate on elastic hadron scattering. An extension of our tunnelling approach to inelastic and in particular to inclusive reactions will only be sketched in the last subsections (7.1 and 7.3) and presented in detail in a forthcoming paper¹⁴⁾. We shall reach a uniform and semi-quantitative understanding of inclusive reactions over the whole kinematical region (from fragmentation to central production, from small to large p_T).

This paper is organized as follows. In the next section we introduce our hypothesis of "maximal importance" of particle production and set up the framework for our investigations. In Section 3 we first summarize and illustrate the complex semiclassical approximation method. Using this technique we then investigate what may be viewed as a prototype of the tunnelling mechanism which we have in mind for strong interactions. In Section 4 the geometry of the hadronic interaction region is determined and possible relations of our approach to constituent pictures (for example colour flux tubes etc.) are discussed. In Section 5 we are ready to present a realistic tunnelling amplitude for hadronic scattering, which we discuss at length and compare to high-energy $pp \rightarrow pp$ data in Section 6. The same type of approach surprisingly turns out to be also

applicable to nuclear physics already at intermediate energies (Section 6.1). This is at the same time an almost parameter-free support of our ideas and allows us to work out the important differences between nuclear and hadronic scattering. Section 7 contains an extension of our approach to inelastic processes. Moreover an interesting relation to dual models is established.

2. SETTING UP THE FRAMEWORK

2.1 Why hadronic tunnelling?

This subsection contains an essential ingredient of our picture, which may be summarized by "maximal importance" of multiparticle production. In fact, it is the presence of many open competing channels, which will leave us with tunnelling as the dominant reaction mechanism in a given channel for $\theta \neq 0$. Though the following (idealized) hypothesis about the effect of absorption into open channels is very intuitive, its final justification will come *a posteriori* from its success.

Let us consider in the following *elastic* scattering^{*)} of two (extended) hadrons in their centre-of-mass system (c.m.s.) with high cm momentum $k \approx \sqrt{s}/2$. The two hadrons approach each other with impact parameter b . According to the short range nature of strong interactions, the reaction will essentially take place within a finite volume in space (-time), the interaction region^{**)}.

Let us now envisage the extreme case of "*maximal importance*" of *particle production*. We assume that whenever the two hadrons interpenetrate each other, particle production takes place and the scattering intensity is lost for the elastic channel. Classically interpenetration happens for impact parameters $b < R_T$, where $2R_T$ is the maximal transverse range of the interaction region. For the elastic channel we are then left only with the contributions^{***)} from $b \geq R_T$, which -- for $b > R_T$ -- are (essentially) *quantum effects*. Since these contributions from $b \geq R_T$ are intuitively insensitive to the details of particle production, it seems rather safe to investigate *them* within a quantum mechanical framework. There we shall study the conditions under which these quantum effects become an important (nonperturbative) tunnelling phenomenon (shadow scattering in the literal sense of the word).

*) An extension of such ideas to inelastic channels is similarly intuitive, though not quite straightforward; see Sections 7.1 and 7.3 and a forthcoming paper¹⁴⁾.

**) Which neither has to be nor will turn out to be merely the geometrical overlap of two Lorentz-contracted pancakes (see Sections 4.2 and 4.3).

***) In order to avoid confusion, let us strongly emphasize that this absorption of "*semiclassical contributions*" from $b < R_T$ is very different from absorbing *small b contributions* in the usual impact parameter amplitude. This will be worked out very clearly in Sections 3.2 and 5.3. There we shall also give a more precise meaning to the sloppy conditions " $b < R_T$ " and " $b \geq R_T$ ".

2.2 The quantum mechanical laboratory

As is obvious from the preceding subsection, we certainly do *not* attempt a full potential description of high-energy hadron scattering. It is only the scattering component for $b \gtrsim R_T$, being insensitive to the presence of multi-particle channels, which we shall investigate in a potential framework.

In this spirit let us consider our quantum mechanical laboratory: two incoming hadrons move within their mutual effective (energy dependent) potential $V(\vec{x}_1 - \vec{x}_2, k)$, the finite range of which is naturally identified with the spatial extensions of the interaction region. A standard reduction (splitting off of the cm motion) leads us to treat the stationary problem of a plane wave with wave number k (moving in z -direction, say: e^{ikz}), which scatters*) off the static, effective potential $U(\vec{x}, k) [\equiv 2m/\hbar^2 V(\vec{x}, k)]$

$$[\Delta + k^2 - U(\vec{x}, k)] \psi(\vec{x}, k) = 0, \quad (1)$$

where the wave function obeys the standard radiation condition at infinity

$$\psi(\vec{x}, k) \underset{|\vec{x}| \rightarrow \infty}{\sim} e^{ikz} + F(k, \theta) \frac{e^{ik|\vec{x}|}}{k|\vec{x}|} \quad (2)$$

defining the scattering amplitude $F(k, \theta)$.

We shall be interested in high-energy scattering, where the product of k times the characteristic extension of the potential is large as compared to one. At these energies the semiclassical ray language becomes appropriate.

The hypothesis of maximal importance of particle production -- translated into this analogue picture -- reads: all semiclassical contributions due to incoming rays with impact parameters $b < R_T$ ("impinging" on the scatterer) are absorbed**) into inelastic channels, where R_T is now the maximal transverse range of the potential. For the elastic channel we are then left with the contributions from rays with $b \gtrsim R_T$, which barely touch the interaction region (see Fig. 1).

*) Clearly neither the plane wave nor the scatterer are to be identified with either of the two incoming hadrons.

) See footnote (*) on page 3.

One might think that these contributions, in particular the quantum effects for $b > R_T$, are vanishingly small. We shall see, however, in Sections 3.2 and 5.2 how a strong variation of the potential at its periphery (idealized by a sharp cut-off or more realistically effected by a complex pole) strongly enhances such quantum effects: it induces "tunnelling rays" with high concentration near a (real or complex) caustic^{*}) surface surrounding the periphery. Quantum effects with $b \gtrsim R_T$ of this kind turn out to give sizeable (nonperturbative) contributions. They still lead to a tremendously fast dropping behaviour in the angular distribution. But then we know that elastic $pp \rightarrow pp$ data at ISR energies show an angular decrease of 9 to 10 orders of magnitude¹⁵⁾ within the first few angular degrees. In fact we shall see that the tunnelling contributions from $b \gtrsim R_T$ easily accommodate -- with two to three free parameters characterizing the geometry of the caustic -- the correct normalization and decrease of the $pp \rightarrow pp$ data out to the largest p_T available.

For a good part of this paper we shall study the tunnelling component due to a cut-off potential (i.e. an interaction region with sharp boundary). Although it is probably *not* the realistic case, it may, however, be treated most rigorously. In Section 5.2 we shall extend the discussion to tunnelling along complex caustics as, for example, due to a complex pole of the potential close to its periphery which allows for a "blurred" boundary of the potential. As we shall see, the mathematical treatment of this extension is much more involved, some details of the resulting tunnelling amplitude change, but the main features and in particular the intuition about its origin remain very similar to the sharp boundary case. This fact justifies our extensive discussion of cut-off potentials.

For the discussion of cut-off potentials, it is convenient to use the optical language in terms of a (complex) refraction index $N(k)$, characterizing the (absorbing) medium inside the scatterer:

$$N(k) = \sqrt{1 - \frac{U(k)}{k^2}} \quad \text{inside the scatterer} \quad (3)$$

leading to the free wave equation

$$[\Delta + k^2] \psi(\vec{x}, k) = 0 \quad \text{outside the scatterer} \quad (4)$$

and to

$$[\Delta + N^2 k^2] \psi(\vec{x}, k) = 0 \quad \text{inside the scatterer} \quad (5)$$

with the additional boundary condition that the wave function

$$\psi \text{ (or its normal derivative or any linear combination of both) be continuous on the sharp boundary.} \quad (6)$$

*) A caustic is an envelope of a family of (real or complex) rays. It is the three-dimensional equivalent of a turning point. Along a caustic surface *the ray concentration is high* (semiclassically infinite).

The next section is devoted to the "complex semiclassical approximation" method and its application to a prototype for hadronic tunnelling.

3. SEMICLASSICAL APPROXIMATION INCLUDING COMPLEX CLASSICAL PATHS

In the formulation of quantum mechanics as given by Balian and Bloch¹³⁾ the *exact* solution of the Schrödinger equation is expressed in terms of a multiple scattering expansion involving values of the *classical action* along all possible *real and complex classical paths*. This formulation is capable of providing a semiclassical approximation scheme ($\hbar \rightarrow 0$ or $k \rightarrow \infty$), much superior to the usual semiclassical approximation of functional integrals involving only real classical paths (WKB). Important quantum effects, in particular tunnelling into shadow regions in the presence of real or complex caustics (= three-dimensional equivalents of turning points), are automatically included.

In practice we shall make use of the extended semiclassical approximation scheme as given by Knoll and Schaeffer¹²⁾ which approximates and simplifies the formulation of Balian and Bloch for $\hbar \rightarrow 0$, viz. $k \rightarrow \infty$ [and has met with much success already in quantitative applications to heavy ion scattering¹⁶⁾].

In the one-dimensional case, say, all turning points ($k^2 - U(x) = 0$) and singularities of the potential in the *complex* x -plane are first determined. The wave function (or Green's function) is then given i) by the usual WKB approximant (with real classical action) and ii) in addition by WKB-type contributions with complex classical actions S arising from multiple reflections¹²⁾ between those turning points (and singularities) along complex classical paths. Each single contribution in the series of infinitely many complex reflections describes a small quantum effect $[O(\exp(-\text{Im } S/\hbar))]$. However, their sum can give rise to important tunnelling type phenomena of *nonperturbative nature* (with respect to the strength of the potential).

Quantum mechanical scattering in the presence of several turning points (caustics) and singularities of the potential is generally a complicated mathematical problem. We approach this problem by the method of complex semiclassical approximation because:

- i) it provides an intuitive insight into the origin of the tunnelling effects under consideration;
- ii) it seems flexible enough to apply to more general cases, where the more orthodox method of uniform approximation^{17,18)} fails;
- iii) it seems most suitable for an extension to field theory. [The possible importance of complex trajectories in field theory has been recently pointed out by Balian, Parisi and Voros¹¹⁾.]

In the subsequent applications we shall give considerable preference to intuition over mathematical rigour. Whenever necessary, mathematically rigorous treatments have been banned into the appendices. For a detailed derivation and discussion of the complex multiple reflection theory, we have to refer to Refs. 12 and 13.

3.1 Complex multiple reflection series in one dimension

As an illustration let us discuss the multiple reflection expansion given by Knoll and Schaeffer¹²⁾ for the simplest one-dimensional case

$$\left[\frac{d^2}{dx^2} + k^2 - U(x) \right] \psi(x) = 0, \quad (7)$$

assuming that there are just two *complex* turning points x_1, x_2

$$k^2 - U(x_{1,2}) = 0 \quad (8)$$

and that the potential $U(x)$ is analytic.

The classical paths contributing in a complex semiclassical approximation, joining two prescribed *real* points x' and x'' , are those with zero, one, two, ... reflections between the complex turning points x_1 and x_2 . However, the specific form of the resulting multiple reflection series for $\psi(x'', x')$ depends on the situation of the regions of analyticity of the WKB approximants $\propto \exp(iS_{1,2})$ with classical action

$$S_{1,2}(x_{1,2}, x) = - \int_{x_{1,2}}^x dx' \sqrt{k^2 - U(x')} \quad (9)$$

in the complex x plane relative to each other as well as on the positions of the starting and end points, x' and x'' . Near the turning points $x_{1,2}$ the action integrals $S_{1,2}$ have a branch point each of the kind $S_{1,2} \propto (x - x_{1,2})^{3/2}$; correspondingly there are three branch cuts, the so-called Stokes lines given by

$$\text{Re } S_{1,2}(x_{1,2}, x) \equiv 0 \quad (10)$$

emanating from $x_{1,2}$ and separating the regions of analyticity of the WKB approximants. In general, depending on the topology of these Stokes lines and the relative position of the boundary points x' and x'' , certain turning points cause reflections, others do not. For the detailed rules see Ref. 12.

Let us quote the resulting multiple reflection series which is a prototype for later situations: both, $x' = \infty$ and x'' , to the right of x_1 and x_2 , with Stokes lines as depicted in Fig. 2

FIG.2

The multiple reflection expansion reads¹²⁾ for this case

$$\psi(x'', \infty) \propto \frac{1}{(k^2 - U(x''))^{1/4}} \left[e^{-iS(x'', \infty)} \right] \quad (11)$$

with

$$-i e^{-i(S(x_1, \infty) - S(x'', x_1))} \sum_{\mu=0}^{\infty} (-i e^{iS(x_2, x_1)})^{2\mu}$$

$$\sum_{\mu=0}^{\infty} = \frac{1}{1 + e^{2iS(x_2, x_1)}} \quad (12)$$

where

$$\begin{aligned} S(x'', \infty) &= \lim_{x' \rightarrow \infty} \left[\int_{x'}^{x''} dx \sqrt{k^2 - U(x)} + kx' \right] \\ &= \int_{\infty}^{x''} dx (\sqrt{k^2 - U(x)} - k) + kx'' \end{aligned} \quad (13)$$

with

$$\text{Re} \sqrt{k^2 - U(x)} > 0.$$

The interpretation is intuitive: the classical trajectories building up the quantum mechanical wave are: the direct (real) trajectory from $x' = \infty$ to x'' , the

trajectory from ∞ returning to x'' after "complex" reflection at x_1 , the trajectories returning to x'' after a "complex" reflection at x_1 followed by $\mu = 1, 2, 3, \dots$ "complex" reflections between x_1 and x_2 leading to the geometrical series (11, 12). Obviously the usual semiclassical approximation with real classical paths just amounts to the direct term $\exp(-iS(x'', \infty))$ in Eq. (11).

The reflection coefficient R ($\hat{=}$ scattering amplitude in this one-dimensional problem) is given by

$$\psi(x'', \infty) \underset{x'' \rightarrow \infty}{\sim} e^{-ikx''} + R e^{ikx''} \quad (14)$$

such that from Eqs. (11) and (12)

$$R = \frac{-ie^{2iS(\infty, x_1)}}{1 + e^{2iS(x_2, x_1)}} \quad (15)$$

Separable three-dimensional problems (see next subsection) reduce to very similar one-dimensional expansions in terms of the separated variables. However, the positions of one or more turning points then depend on constants of motion such as angular momentum. Hence for certain complex values of these the multiple reflection term (15) may exhibit poles if

$$S(x_2, x_1) = (m + 1/2) \pi \quad \text{with } m \text{ integer.} \quad (16)$$

Such poles may add up to give important nonperturbative contributions, as we shall see below.

3.2 Tunnelling along a spheroidal boundary: surface creep waves

In this subsection we shall treat the three-dimensional wave mechanical problem of scattering of a plane wave off a *spheroid with sharp boundary*¹⁹⁾ (i.e. a spheroidal potential well). This contains as a limiting case the scattering off a sphere, which has been treated rigorously in Refs. 20 and 21, where (as late as 1969!) full asymptotic expansions for the solution have been derived. We discuss here the example of the more general shape of a spheroid (see Ref. 19) because it will be relevant for our applications to particle physics. We present a full treatment of the scattering problem in Appendices A, B, and C using the method of uniform approximation^{17,18)}. In this section we shall "translate" the results into the much more illustrative expressions as obtained from the complex semiclassical method¹²⁾ (being very sloppy about quoting domains of validity of the results).

In this scattering problem the sharp edge of the potential acts as a geometrical caustic surface for a nonperturbative tunnelling component of the kind we are interested in: the so-called surface creep waves^{22,23,19,20)}.

Let us first treat the simplest case of an infinitely deep potential well $[N \rightarrow i\infty$ in Eq. (3)], i.e. of a "totally reflecting" scatterer in optical terminology. This leads to the solution $\psi(\vec{x}, k) \equiv 0$ inside the spheroid; the remaining free wave equation (4) outside the spheroidal scatterer is separable in spheroidal coordinates ξ, θ, ϕ .

Let us introduce coordinates of a prolate spheroid^{*)} with the z axis being the axis of rotation (see Fig. 3)

$$\begin{aligned} x &= \Delta R \sinh \xi \sin \theta \cos \phi \\ y &= \Delta R \sinh \xi \sin \theta \sin \phi \\ z &= \Delta R \cosh \xi \cos \theta \end{aligned} \quad (17)$$

with

$$2\Delta R = 2\sqrt{R_L^2 - R_T^2} = \text{focal distance}$$

FIG.3

A spheroidal surface corresponds to $\xi = \text{const.}$ $\xi = \xi_B$ with $\sinh \xi_B = R_T/\Delta R$ is the equation of the sharp boundary of the scatterer. In these coordinates the wave equation (4) for $\xi \geq \xi_B$ separates into a "radial" and an angular equation with separation parameter $\lambda + 1/2$:

$$\left[\frac{d^2}{d\xi^2} + (k\Delta R \sinh \xi)^2 - (\lambda + 1/2)^2 + \frac{1}{4 \sinh^2 \xi} \right] \times \sqrt{\sinh \xi} R(k\Delta R, \cosh \xi) = 0 \quad (18)$$

$$\left[\frac{d^2}{d\theta^2} + (k\Delta R \sin \theta)^2 + (\lambda + 1/2)^2 + \frac{1}{4 \sin^2 \theta} \right] \times \sqrt{\sin \theta} S(k\Delta R, \cos \theta) = 0. \quad (19)$$

^{*)} A body obtained by rotating an ellipse around its major axis.

Due to the axial incidence of the plane wave the dependence on the azimuthal angle ϕ is trivial and has been omitted; (see, however, Appendix A).

An extensive discussion of general properties of the radial and angular spheroidal wave functions $R(k\Delta R, \cosh \xi)$ and $S(k\Delta R, \cos \theta)$ may be found in Refs. 24 and 25. Suitable approximations for the case we shall be interested in later -- $k \rightarrow \infty$, $\Delta R \rightarrow \infty$ and complex λ -- are derived in Appendices B and C.

The exact solution for the scattering amplitude

$$f(k, \theta) = \frac{2}{(k R_T)^2} F(k, \theta) \quad (20)$$

with normalization

$$\sigma_{\text{total}} = 2\pi R_T^2 \operatorname{Im} f(k, 0) \quad (21)$$

is easily derived in terms of the spheroidal partial wave expansion (see Appendix A)

$$f(k, \theta) = \frac{2}{(k R_T)^2} \sum_{n=0}^{\infty} (\lambda(n) + 1/2) \frac{d(\lambda(n) + 1/2)}{dn} f_{\lambda(n)}(k R_T, k R_L)^* \times S_{on}^{(1)}(k \Delta R, \cos \theta) \quad (22)$$

with partial waves

$$f_{\lambda(n)}(k R_T, k R_L) = 2i \frac{R_{on}^{(1)}(k \Delta R, R_L/\Delta R)}{R_{on}^{(3)}(k \Delta R, R_L/\Delta R)} \quad (23)$$

The quantum number n in Eq. (22) is a constant of motion representing a so-called action variable. In fact semiclassically, i.e. for large values of n , one finds (see Appendix C)

$$2n + 1 \approx \frac{2}{\pi} \int_0^\pi d\theta' \sqrt{(\lambda(n) + 1/2)^2 + (k \Delta R \sin \theta')^2} \equiv S_\theta(2\pi, 0), \quad (24)$$

where the right hand side of Eq. (24) is just the angular part of the classical action taken along a closed (elliptical) orbit. Moreover, in the spherical limit, $\Delta R \rightarrow 0$, we have from Eqs. (19) and (24)

$$n \xrightarrow{\Delta R \rightarrow 0} \lambda \xrightarrow{\Delta R \rightarrow 0} J = \text{angular momentum} \quad (25)$$

and

$$S_{on}^{(1)}(k \Delta R, \cos \theta) \xrightarrow{\Delta R \rightarrow 0} P_J(\cos \theta). \quad (26)$$

For

$$k \sin \theta \equiv p_T \ll \frac{|\lambda + 1/2|}{\Delta R} \quad (27)$$

we see from Eq. (19) that

$$S_{on}^{(1)}(k\Delta R, \cos \theta) \approx P_{\lambda(n)}(\cos \theta) \quad (28)$$

and thus, for sufficiently small angles θ , $\lambda(n)$ approximately equals the angular momentum.

For large values of k the representation (22) becomes practically useless due to its very bad convergence: the number of contributing partial waves increases proportional to k . The standard "trick" consists in continuing the partial wave expansion into the complex λ plane (corresponding to the complex angular momentum plane in the spherical case) by means of a Sommerfeld-Watson transformation. This transformation will finally serve^{20,21)} to distinguish the contributions from rays with impact parameters $b < R_T$ from those with $b \geq R_T$.

We first need approximations of the partial wave amplitudes valid for complex λ . Let us translate the expressions derived in Appendix B into the illustrative language of the complex multiple reflection approach¹²⁾ applied to the one-dimensional "radial" equation (18).

The discontinuity of the potential at $\xi = \xi_B$ introduces (roughly speaking) a fixed turning point at $\xi = \xi_B$. For convenience let us introduce the "transverse radial" variable

$$\rho = \Delta R \sinh \xi \quad (29)$$

In this variable the fixed turning point appears at $\rho = R_T$. In addition the wave equation (18) has a turning point at $\rho = (\lambda + 1/2)/k$. Let us introduce the impact parameter variable

$$\beta = \frac{\lambda + 1/2}{k} \quad (30)$$

The two turning points at $\rho = R_T$ and $\rho = \beta$ will give rise to a multiple reflection series for values of β in a certain region of the complex β plane (specified in Appendix B) in the quadrant $\text{Re } \beta \geq R_T$ and $\text{Im } \beta \geq 0$. The partial wave amplitudes $f_{\lambda(n)}(kR_T, kR_L)$ are built up from contributions of complex classical paths starting at $\rho = \infty$ and ending at $\rho = \infty$. For $\beta < R_T$ there is only one real path containing a single reflection at the turning point $\rho = R_T$, which results in the usual WKB approximation

$$f_{\lambda(n)}(kR_T, kR_L) \approx i - e^{2iS_\rho(\infty, R_T)} \quad \text{for } \beta < R_T \quad (31)$$

with the "radial" part, S_ρ , of the classical action given by

$$\begin{aligned}
 S_g(\infty, R_T) &= (n+1/2) \frac{\pi}{2} - k R_L + k \int_{R_T}^{\infty} d\rho \frac{(\sqrt{\rho^2 - \beta^2} - \rho)}{\sqrt{\rho^2 + \Delta R^2}} \\
 &= -k \int_{\beta}^{R_T} d\rho \frac{\sqrt{\rho^2 - \beta^2}}{\sqrt{\rho^2 + \Delta R^2}} = -S_g(R_T, \beta) = S_g(\beta, R_T).
 \end{aligned} \tag{32}$$

For $\text{Re } \beta \geq R_T$ and $\text{Im } \beta \geq 0$ a single reflection at $\rho = R_T$ is followed by complex multiple reflections between the two turning points $\rho = R_T$ and $\rho = \beta$ resulting in the multiple reflection series:

$$\begin{aligned}
 f_{\lambda(n)}(k R_T, k R_L) &\approx -e^{2i S_g(\infty, R_T)} \sum_{\mu=0}^{\infty} (-i e^{2i S_g(\beta, R_T)})^{\mu} \\
 &= \frac{-e^{2i S_g(\infty, R_T)}}{1 + i e^{2i S_g(\beta, R_T)}} \quad \text{for } \begin{cases} \text{Re } \beta \geq R_T \\ \text{Im } \beta \geq 0 \end{cases}
 \end{aligned} \tag{33}$$

In comparison with the multiple reflection series (15) obtained in the previous one-dimensional example (with $x', x'' = \infty$) there is an additional phase shift of $\pi/2$ for each reflection at $\rho = R_T$. This is due to the turning point at $\rho = R_T$ being introduced by a discontinuity of the potential [whereas Eq. (15) was derived for an analytic potential].

In Eq. (33) the action integrals $S_g(\infty, R_T)$ and $S_g(\beta, R_T)$ are complex (with analytic continuation $\sqrt{\rho^2 - \beta^2} = i \sqrt{\beta^2 - \rho^2}$). This leads to an exponentially damped partial wave amplitude for real values of $\beta > R_T$. However, along the line in the complex β plane, where

$$\text{Im } S_g(\beta, R_T) = 0 \tag{34}$$

the partial wave amplitude (33) has an infinite series of poles at complex values

$$\beta_m = (\alpha_m(k) + 1/2) / k \tag{35}$$

defined by

$$S_g(R_T, \beta_m) = (m - 1/4) \pi \quad \text{for } m = 1, 2, 3, \dots \tag{36}$$

If we parametrize the complex pole positions $\alpha_m(k)$ in the λ -plane by

$$\alpha_m + 1/2 = k R_T \sqrt{1 + \gamma_m^2} \tag{37}$$

γ_m is determined from the transcendental equation

$$\begin{aligned}
 S_g(R_T, \beta_m) &\approx i \frac{k R_T^2}{2 R_L} \left[\gamma_m - \frac{1 + \gamma_m^2}{2i} \log \frac{1 + i \gamma_m}{1 - i \gamma_m} \right] \\
 &= (m - 1/4) \pi; \quad m = 1, 2, 3, \dots
 \end{aligned} \tag{38}$$

resulting from Eq. (32) for $(R_T^2/R_L^2)|\gamma_m|^2 \ll 1$. For $R_L/kR_T^2 \rightarrow 0$ one may explicitly solve¹⁹⁾ for α_m

$$\alpha_m(k) + 1/2 = k\beta_m \underset{k \rightarrow \infty}{\approx} kR_T + x_m e^{i\pi/3} \left(\frac{k\varrho_L}{2}\right)^{1/3} \quad (39)$$

for $m = 1, 2, 3, \dots$,

where

$$\varrho_L = \frac{R_L^2}{R_T} \quad (40)$$

is the longitudinal radius of curvature of the spheroid at the point of maximal transverse extension and

$$x_m \approx \left(\frac{3}{2}\pi(m - 1/4)\right)^{2/3}; \quad (41)$$

[more exactly the values x_m are given by the zeros of the Airy function: $\text{Ai}(-x_m) = 0$ with $x_1 = 2.33811$, $x_2 = 4.08795$, ...].

These complex poles of the partial wave amplitudes are moving poles in the λ -plane, i.e. *Regge-like poles*^{*}) in the direct channel. Remember, however, that λ is *not* the angular momentum; [only in the limit of a sphere we obtain genuine direct channel Regge poles^{20,21)} in the angular momentum plane; see Eq. (25)]. The way to disentangle^{20,21)} the real and the complex semiclassical contributions to the full scattering amplitude $f(k, \theta)$ is by means of a Sommerfeld-Watson transformation in the complex λ plane (see Fig. 4 and Appendix A)

$$f(k, \theta) = \text{"background integral"} + \text{sum over residues of Regge-like poles at } \alpha_m(k) \text{ in the } \lambda\text{-plane.} \quad (42)$$

FIG. 4

^{*}) In Section 7.2 we show that one such trajectory in the complex λ plane generates (asymptotically for $k \rightarrow \infty$) an approximately linear parent trajectory and infinitely many parallel daughter trajectories in the complex angular momentum plane.

The *background integral* is evaluated as for the spherical case^{20,21)} [for $\theta \gtrsim (k\rho_L/2)^{-1/3}$] by means of the saddle point method; it obtains its contributions from a *real* saddle point at

$$0 \leq \beta_S(\theta) = \frac{R_T \cos \frac{\theta}{2}}{\sqrt{1 + \left(\frac{\Delta R}{R_T} \sin \frac{\theta}{2}\right)^2}} \lesssim R_T \quad (43)$$

The result is identical with the usual WKB approximation for $f(k, \theta)$; it corresponds to the *real classical* path of an incoming ray with impact parameter $\beta < R_T$ being reflected off the surface of the spheroid into the angle $\theta = \theta(\beta_S)$ (geometrical-optical reflection). It provides the *classically allowed* contribution (in a complex semiclassical approximation) of the amplitude and is due to rays with impact parameters $\beta < R_T$.

The contributions from the *Regge-like poles* have also a very intuitive geometrical interpretation¹⁹⁻²³⁾.

FIG. 5

The Regge-like poles describe *resonance phenomena* being excited by "complex rays" with (quantized) impact parameters [cf. Eq. (39) and Figs. 4 and 5]

$$\text{Re } \beta_m = \frac{\text{Re } \alpha_m + 1/2}{k} \approx R_T + \delta R(m); \quad m=1,2,3,\dots \quad (44)$$

with monotonously increasing values for $\delta R(m)$. Thus the first few resonances are excited close to the surface by rays of approximately tangential incidence ($\beta \approx R_T$). The eigenfunctions corresponding to the eigenvalues α_m may be shown to be concentrated along *longitudinal geodesics of the scatter's surface*. Thus the intensity carried by the (complex) rays of near grazing incidence propagates along longitudinal geodesics of the curved surface into the geometrical shadow behind the scatterer, i.e. into a classically forbidden region; hence the name for this *tunnelling phenomenon*: "surface creep waves¹⁹⁻²³⁾" (see Fig. 5).

These resonance phenomena close to the surface are of course due to the enhanced probability of finding the (complex) ray in the vicinity of the surface on account of its multiple reflections between the two turning points $\rho = R_T$ and $\rho = \beta$. Thus the formulation in terms of *complex multiple reflections* taking place near the caustic (= surface) provides an intuitive understanding of the *dynamical origin* of these resonance phenomena.

Analytically the sum over the contributions from the Regge-like poles at α_m is given by

$$f_{\text{tunnelling}}(k, \theta) = \sum_{m=1}^{\infty} r_m \frac{S_{0\nu_m}^{(1)}(-\cos \theta)}{\sin \pi \nu_m} \quad (45)$$

where the residues r_m are easily calculated from Eq. (33) to be

$$\begin{aligned} r_m &\equiv -\frac{2\pi(\alpha_m + 1/2)}{(kR_T)^2} \lim_{\lambda \rightarrow \alpha_m} (\lambda - \alpha_m) f_\lambda(kR_T, kR_L) \\ &= \frac{-2\pi R_L / (kR_T^2)}{\log \frac{1+i\gamma_m}{1-i\gamma_m}} \end{aligned} \quad (46)$$

and where according to Eq. (24)

$$\nu_m + 1/2 = n(\alpha_m) + 1/2 \approx \frac{1}{\pi} \int_0^\pi d\theta' \sqrt{(\alpha_m + 1/2)^2 + (k\Delta R \sin \theta')^2} \quad (47)$$

A uniform approximation of the spheroidal angular functions $S_{0\nu_m}^{(1)}(-\cos \theta)$ in terms of the Legendre functions has been derived in Appendix C. For $0 \leq \theta \leq \pi$ and ν_m large we obtain

$$\begin{aligned} \frac{S_{0\nu_m}^{(1)}(-\cos \theta)}{\sin \pi \nu_m} &\approx \left[\frac{\sin \Omega(\theta) (\nu_m + 1/2)}{\sin \theta \sqrt{(\alpha_m + 1/2)^2 + (k\Delta R \sin \theta)^2}} \right]^{1/2} \\ &\quad \times \frac{P_{\nu_m}(-\cos \Omega(\theta))}{\sin \pi \nu_m} \end{aligned} \quad (48)$$

with "action angle"

$$\Omega(\theta) = \frac{1}{\nu_m + 1/2} \int_0^\theta d\theta' \sqrt{(\alpha_m + 1/2)^2 + (k\Delta R \sin \theta')^2} \quad (49)$$

such that

$$\Omega(0) = 0, \quad \Omega\left(\frac{\pi}{2}\right) = \frac{\pi}{2} \quad \text{and} \quad \Omega(\pi) = \pi. \quad (50)$$

Using the large $|\nu_m|$, $\text{Im } \nu_m > 0$ approximation of the Legendre functions and Eq. (49) we may further simplify Eq. (48) to

$$\frac{S_{0\nu_m}^{(1)}(-\cos \theta)}{\sin \pi \nu_m} \approx \quad (51)$$

$$\left\{ \begin{aligned} &\frac{P_{\alpha_m}(-\cos \theta)}{\sin \pi \alpha_m} \approx -i H_0^{(1)}((\alpha_m + 1/2)\theta) \quad \text{for } k\theta \ll \frac{|\alpha_m + 1/2|}{\Delta R}, \\ &- \sqrt{\frac{2}{\pi}} \left[\sin \theta \sqrt{(\alpha_m + 1/2)^2 + (k\Delta R \sin \theta)^2} \right]^{-1/2}, \\ &+ \left[e^{i(\nu_m + 1/2)\Omega(\theta) + i\pi/4} + e^{i(\nu_m + 1/2)(2\pi - \Omega(\theta)) - i\pi/4} \right] \end{aligned} \right. \quad (52)$$

for $\epsilon \leq \theta \leq \pi - \epsilon$, $\epsilon > 0$

The lower expression, Eq. (52), is the familiar WKB approximation of the angular functions, where the second exponential in the bracket is negligible for $\theta \neq \pi$.

Due to the curvature of the surface the surface waves radiate off tangentially intensity into the geometrical shadow (see Fig. 5) leading to the strongly decreasing angular distribution (51, 52) governed by the exponential

$$e^{-\text{Im}[(\gamma_m + 1/2)\Omega(\theta)]} = e^{-\text{Im} \int_0^\theta d\theta' \sqrt{(\alpha_m + 1/2)^2 + (k\Delta R \sin\theta')^2}} \quad (53)$$

$$\approx e^{-\text{Im}(\alpha_m + 1/2)\theta} \quad (54)$$

$k\theta \ll |\alpha_m + 1/2|/\Delta R$

Thus the decrease is controlled by the imaginary parts of the positions α_m of the Regge-like poles. For sufficiently large angles, $\text{Im}(\alpha_1 - \alpha_2)\theta \gg 1$, the *lowest surface wave dominates* the sum of all of them, since according to Eqs. (39) and (41)

$$\text{Im} \alpha_1 < \text{Im} \alpha_2 < \text{Im} \alpha_3 < \dots \quad (55)$$

To sum up: the tunnelling component due to Regge-like poles in the λ -plane results from contributions due to complex classical paths in a complex semiclassical approximation to the amplitude. It represents the *classically forbidden* part of the amplitude coming from "complex rays" with impact parameters $\beta \gtrsim R_T$.

For small angles, $\theta \ll (k\rho_L/2)^{-1/3}$, the saddle point and the complex poles come to lie too close to each other in the complex λ -plane to be separable any more; they have to be treated simultaneously^{20, 22}). Such a treatment leads to the Fraunhofer diffraction formula

$$0 \leq \theta \ll \left(\frac{k\rho_L}{2}\right)^{-1/3}: \quad f(k, \theta) \approx 2i \frac{J_1(kR_T\theta)}{kR_T\theta} \quad (56)$$

This formula is independent of the longitudinal extension of the spheroid; not so, however, its region of validity: the larger $\rho_L = R_L^2/R_T$, the smaller its angular region of validity.

It is highly instructive to see how the classical and the tunnelling terms collaborate to lead to this near forward result. (The derivation is in strict analogy to the spherical case discussed in Refs. 20 and 21.) For $\theta \ll (k\rho_L)^{-1/3}$ the background integral turns out to behave as

$$f_{\text{classical}}(k, \theta) \approx i \frac{H_1^{(2)}(kR_T\theta)}{kR_T\theta} \quad (57)$$

while the Regge-like pole contributions sum up to give [see Eqs. (45) and (51)]

$$f_{\text{tunnelling}}(k, \theta) \approx -i \sum_{m=1}^{\infty} \tau_m H_0^{(1)}((\alpha_m + 1/2)\theta) \quad (58)$$

$$\text{and with } \tau_m = \frac{\alpha_m + 1/2}{(kR_T)^2} \frac{d(\alpha_m + 1/2)}{dm} \quad (59)$$

from Eqs. (38) and (46)

$$f_{\text{tunnelling}}(k, \theta) \approx \frac{-i}{\text{Im}(\alpha_1 - \alpha_2)\theta \ll 1} \int_{\alpha_1 + 1/2}^{\infty} e^{i\varphi} (\alpha + 1/2) d(\alpha + 1/2) H_0^{(1)}((\alpha + 1/2)\theta) \\ = i \frac{\alpha_1 + 1/2}{kR_T} H_1^{(1)}((\alpha_1 + 1/2)\theta) / (kR_T\theta) \quad (60)$$

$$\theta \ll (kg_L/2)^{-1/3} \quad i H_1^{(1)}(kR_T\theta) / (kR_T\theta) + \dots \quad (61)$$

with $H_1^{(1,2)}(z) = J_1(z) \pm Y_1(z)$. Due to the singularity of $Y_1(z)$ at $z = 0$, both terms (57) and (61) become singular at $\theta = 0$. Their singularities cancel in the sum and one ends up with the imaginary Fraunhofer amplitude, where each of the two terms (57) and (61) contributes exactly half. (This is very intuitive: the classically allowed contribution gives the geometrical total cross-section πR_T^2 , while the tunnelling component promotes it to $\sigma_{\text{total}} = 2\pi R_T^2$.) This is a *typical example of how quantum effects due to the contributions of infinitely many complex classical paths may get enhanced to give sizeable contributions of the order of magnitude of (real) classical contributions.*

Neither $|f_{\text{classical}}|^2$ nor $|f_{\text{tunnelling}}|^2$ show oscillations; the typical pattern of diffraction minima and maxima in $|J_1(kR_T\theta)/(kR_T\theta)|^2$ arises from interference of the two terms.

Let us next consider the general case of an arbitrary refraction index $N(k)$. For $\theta \gtrsim (kp_L/2)^{-1/3}$, i.e. outside the Fraunhofer diffraction region, the classical component becomes strongly N -dependent. The *tunnelling component*, however, turns out to be essentially independent of N , i.e. *independent of the strength of the potential*. This has been shown in Ref. 21 for the spherical case; the proof may be straightforwardly extended to the spheroidal case. This result is intuitively easy to understand: the tunnelling phenomenon of surface creep waves is induced by the discontinuity of the potential; it takes place, however, in a spatial region, where the potential vanishes (see Fig. 5). The only restriction on N is that it is not allowed to come too close to 1 (which would invalidate the effect of the discontinuity in generating the tunnelling phenomenon), more precisely that

$$|\sqrt{N^2 - 1}| \gg (kg_{\text{min}})^{-1/3} \quad (62)$$

where $\rho_{\min}$ is the minimum value taken by the longitudinal radius of curvature ρ along the scatterer, which is equal to

$$\rho_{\min} = \frac{R_T^2}{R_L} \quad (63)$$

the longitudinal radius of curvature at the tip, for a prolate spheroid. The Regge pole positions α_m as well as the residues are then independent of N to leading order with corrections of the order of $O(1/|(k\rho_{\min})^{1/3} \sqrt{N^2 - 1}|)$. In this sense the tunnelling phenomenon of surface creep waves is *nonperturbative*. Let us emphasize that the *tunnelling amplitude depends only on the geometry of the surface-caustic*, i.e. in this example of a spheroid on the two parameters R_T and R_L .

Altogether this example of a potential with a discontinuity at its periphery has lead to a nonperturbative tunnelling phenomenon of geometrical nature. It arises from complex classical trajectories with multiple reflections between turning points, is described by Regge-like poles in the λ -plane (of the direct channel), and has the physical interpretation of surface creep waves. It is caused by incoming rays with impact parameters $\beta \gtrsim R_T$. Together with the classical contribution due to rays with $\beta \approx R_T$, i.e. the classical contribution for $\theta \approx 0$, the tunnelling component constitutes the component we have set out to study in Section 2.1.

Let us make a final remark: surface waves are not only mathematical fiction. They have been verified experimentally²⁶⁾: for example, by sending a well-collimated laser-beam tangentially on the surface of a suspended water droplet²⁶⁾, by sending ultrasound waves tangentially on the surface of a cylinder²⁷⁾ with a size of the order of magnitude of centimetres, and by emitting long radio waves tangentially to the surface of the earth²⁸⁾; in all cases the product $k \times$ extension of the scatterer is of the order of 30 to 1000, i.e. $\gg 1$. Also the beautiful natural phenomenon of the "glory-effect"^{21,26)} has been shown to be due to surface waves (see Section 7.4).

4. TUNNELLING IN HADRONIC SCATTERING; THE GEOMETRY OF THE "HADRONIC CAUSTIC"

In this section we return to our picture outlined in Section 2: apart from $\theta \approx 0$ hadronic scattering is interpreted as a tunnelling phenomenon along a geometrical caustic near the interface of interacting hadronic matter and the surrounding vacuum. Let us start from the *working hypothesis* that a tunnelling mechanism of the kind discussed in the previous subsection provides already a good first approximation. Accordingly, we shall restrict ourselves for the time being to a surface-caustic due to a cut-off potential. (This will be justified to a large extent in Section 5.2, where a more general complex caustic is discussed.)

The main question is then: which is the shape of the hadronic caustic, characterizing the spatial proportions of the interaction region? To this end we first have to discuss the tunnelling amplitude for a "surface-caustic" of general shape.

4.1 Surface waves along a boundary with general shape

The general problem of scattering off a totally reflecting body with sharp boundary and a smooth convex shape has been discussed in Ref. 23. The resulting tunnelling amplitude exhibits explicitly the dependence on the geometrical properties of the surface-caustic (and only on those!). Since this general result has been, however, obtained rather heuristically, we shall only use it for qualitative arguments; all our following quantitative results will arise from rigorous calculations.

The general tunnelling amplitude of Refs. 19 and 23 -- specialized to a shape which is rotationally symmetric with respect to the beam direction, i.e. the z-axis -- may be cast*) into the following form, which is more adequate for our purposes

$$f_{\text{tunnelling}}(k, \theta) = \sum_{m=1}^{\infty} f^{(m)}(k, \theta) \quad \text{with} \quad (64)$$

$$f^{(m)}(k, \theta) \approx (k g_L(\theta)/2)^{-1/3} C_m \frac{1}{\sqrt{k R_T \sin \theta}} \frac{[k g_L(\theta) k g_L(\theta)]^{1/6}}{k R_T} \times e^{ik \int_0^\theta d\theta' b_{\text{out}}(\theta')} e^{ie^{i\pi/3} x_m (k/2)^{1/3} \int_0^\theta d\theta' (g_L(\theta'))^{1/3}} \quad (65)$$

valid for $k g_L(\theta) \gg 1$ (and k -independent $p_L(\theta)$) with complex constants C_m .

FIG. 6

*) After straightforward but tedious applications of elementary differential geometry.

$b_{out}(\theta)$ is sort of an outgoing impact parameter, the distance between the tangent in $P(\theta)$ forming the angle θ with the z -axis and its parallel through the origin; $\rho_L(\theta)$ is the longitudinal radius of curvature in $P(\theta)$ (see Fig. 6).

For sufficiently small angles one obtains the approximation

$$f^{(m)}(k, \theta) \underset{\theta \text{ small}}{\approx} C_m \frac{1}{\sqrt{k R_T \sin \theta}} \frac{(k \rho_L(0))^{1/3}}{k R_T} e^{i(\alpha_m + 1/2) \theta} \quad (66)$$

where

$$\alpha_m + 1/2 \underset{k \text{ large}}{\approx} k R_T + e^{i\pi/3} x_m \left(\frac{k \rho_L(0)}{2} \right)^{1/3} \quad (67)$$

is the generalization of Eq. (39). Thus, in complete generality, surface waves are excited by "quantized, complex" rays with impact parameters $(\text{Re } \alpha_m + 1/2)/k \geq R_T$, the first few having near grazing incidence, i.e. impact parameters close to R_T . They propagate along the surface (-caustic) and, due to its curvature, are radiated off tangentially. The amplitude for tunnelling intensity into the angle θ depends on the values of $\rho_L(\theta)$ and $b_{out}(\theta)$ taken along the segment of the longitudinal geodesic on the surface, which the surface wave has run through (see Fig. 6).

The strong angular decrease

$$e^{-\sin \pi/3 x_m (k/2)^{1/3} \int_0^\theta (\rho_L(\theta'))^{1/3} d\theta'} \quad (68)$$

is exclusively controlled by the longitudinal radius of curvature, $\rho_L(\theta)$. The Fraunhofer diffraction formula -- involving only the transverse extension R_T -- is again valid only in a small angular region, $0 \leq \theta < [k \rho_L(0)/2]^{1/3}$, which itself strongly depends on the longitudinal extension of the scatterer.

Again the tunnelling amplitude remains unaltered if instead of a totally reflecting scatterer ($N \rightarrow i\infty$) a complex refraction index N satisfying

$$|\sqrt{N^2 - 1}| \gg (k \min_{\theta} \rho_L(\theta))^{-1/3} \quad (69)$$

is introduced [cf. the inequality (62)].

An important message from this investigation of scattering off a scatterer with general shape is: the tunnelling component of the scattering amplitude reflects the three-dimensional geometry of its characterizing caustic and nothing else!

Of course the results for a prolate spheroid obtained in Section 3.2 may easily be recovered from the general formula (65) with

$$b_{out}^{\text{spheroid}}(\theta) = (R_T^2 + \Delta R^2 \sin^2 \theta)^{1/2}$$

$$\rho_L^{\text{spheroid}}(\theta) = \frac{(R_T R_L)^2}{(R_T^2 + \Delta R^2 \sin^2 \theta)^{3/2}} \quad (70)$$

in the limit $R_L/(kR_T^2) \rightarrow 0$, i.e. for k -independent dimensions R_T and R_L .

4.2 The "hadronic caustic": a "fat string"

We have seen within the restricted framework of a cut-off potential how the tunnelling component depends on the three-dimensional shape of the caustic. Returning to our working hypothesis that such a tunnelling mechanism is responsible for elastic hadron scattering, we are now ready to determine the shape of the hadronic caustic from a few key features in high-energy elastic data.

A conspicuous qualitative feature of elastic scattering in the FNAL-ISR range at moderate fixed $p_T \approx k \sin \theta \approx k\theta$ is its *energy independence* (apart from possible weak $\log s$ -dependences). $pp \rightarrow pp$ scattering for $p_T \gtrsim 1.5$ GeV/c is a particularly good example^{15,29,30}).

According to Eq. (66) for a caustic with general shape each surface wave gives for moderate p_T a contribution

$$\left| f_{\text{tunnelling}}^{(m)} \right| = |C_m| \frac{(k \rho_L(0))^{1/3}}{k R_T} \frac{1}{\sqrt{R_T p_T}} e^{-\text{const.} \frac{(k \rho_L(0))^{1/3}}{k R_T} R_T p_T} \quad (71)$$

Hence energy independence of the elastic scattering amplitude requires an *energy dependent shape of the caustic*, i.e. an energy-dependent form of the interaction region. More precisely we need in the *centre-of-mass system* with $k \approx \sqrt{s}/2$

$$\rho_L(0) \underset{k \text{ large}}{\propto} k^2 \quad (72)$$

and

$$R_T \underset{k \text{ large}}{\propto} \text{constant}$$

(apart from possible $\log k$ -dependences), which is, of course, consistent with the general interpretation of strong interaction data that R_T should be roughly energy independent and of the order

$$R_T \approx 1 \text{ fermi.} \quad (73)$$

Notice: a spherical interaction region with $R \approx \text{constant}$ would lead to an absurd antishrinkage like $\exp(-\text{const.} k^{-2/3} p_T)$; an even worse behaviour results for the shape of a Lorentz-contracted pancake with $R_T \approx \text{constant}$.

The *simplest* (rotationally symmetric) shape, which fulfils the two requirements is a *prolate spheroid* with minor (transverse) half axis R_T and major (longitudinal) half axis R_L (in the c.m.s.)

$$R_L \propto_{k \text{ large}} k \quad \text{such that} \quad g_L(0) = \frac{R_L^2}{R_T} \propto_{k \text{ large}} k^2. \quad (74)$$

This is the shape of a "cigar" or a "fat string", which stretches out longer and longer along the beam direction for increasing momentum k . As we shall see in Sections 5.1 and 6.2, this spheroidal shape -- apart from being the simplest one -- meets all phenomenological requirements from data on a very satisfactory semi-quantitative level; in particular it will lead to *power law behaviour for large angle scattering*. Finally and most importantly, such a shape for the interaction region emerges also from most current theoretical approaches; this is discussed in detail in the next subsection.

The example of scattering off a prolate spheroid, treated in Section 3.2, finds now its application. The tunnelling amplitude for hadronic scattering (for $\theta \neq 0$) is given by Eqs. (45), (51), and (52) with

$$R_T \approx 1 \text{ fermi} \quad \text{and} \quad R_L \approx \Delta R = c k R_T^2 \quad (75)$$

to be inserted, where $\Delta R/(k R_T^2) = c$ is now a fixed number (apart from possible log k -dependences). Correspondingly the pole positions α_m should be determined numerically from Eqs. (37) and (38) rather than from Eq. (39).

For our semiclassical treatment it is, however, essential that this number c , which is a measure for the excentricity of the spheroid, be numerically small: the action (38) is obviously only large if $k R_T^2/R_L \approx 1/c$ is large. This constant number c is the small parameter in the game. Indeed, in comparison with data, it will turn out to be of the order $c \approx 0.1$. Furthermore, in the next subsection, we shall find a possible interesting interpretation for its small size.

In this paper we shall disregard the possible interesting log k -dependences in R_T and R_L , like for example, $R_T \propto \log k$.

4.3 Possible relation to colour electric flux tubes

Within the framework of our analysis, we may well content ourselves with having determined the geometrical shape of the caustic surface, which -- in our view -- characterizes the tunnelling dynamics of hadronic scattering. It is nevertheless interesting to ask -- as an aside in this paper -- whether such an elongated interaction region is plausible, and in particular, which dynamics might lead to it. We find it very encouraging that in fact most current theoretical approaches -- though differing considerably in their input -- suggest exactly such an elongated shape. Expressions like "colour electric flux tube", "fat string", "firesausage", and "cigar" have been introduced to characterize it.

In particular, QCD ideas, adorned with an *ad hoc* colour confinement mechanism⁵⁾ provide an interesting phenomenological basis for a treatment of extended hadrons and their interactions. Let us for instance consider bag-bag scattering as pictured by Low⁷⁾, Nussinov⁸⁾, Kogut and Susskind⁶⁾, 't Hooft⁶⁾, Johnson and Thorn⁹⁾. After the collision of two initial hadron bags, two objects*) carrying colour charge race away from each other with relative velocity $2v \approx 2c$. Thus -- due to colour confinement -- a fused single bag in a highly excited state with mass $M = \sqrt{s}$ evolves for intermediate times and stretches along the beam direction. Colour electric flux lines terminate on the concentration of colour at the ends of the intermediate bag. An *ad hoc* confining mechanism [e.g. a volume pressure^{1,7,9)}, a surface tension^{3,5)}, or a colour electric Meissner effect⁶⁾] counterbalances the outward pressure of the colour flux lines. As a result the intermediate bag develops into a *colour electric vortex tube* for sufficiently large separation of the two colour charges. The transverse extension $2R_T$ of the tube becomes constant with

$$R_T \approx 1 \text{ fermi.} \quad (76)$$

The energy ϵ deposited in the tube, i.e. the sum of the colour electric field energy and the confining energy grows linearly proportional to the separation of the charges^{3,5-9,31)}. Stretching the flux lines can result in

- i) breaking the flux tube into segments^{6,7)} by creation of (light) quark-antiquark pairs along the way (multihadron production); these segments will have finite length for centrally produced hadrons with small p_L ;
- ii) increasing the length until an appreciable finite fraction ϵ of the available energy $\sqrt{s}$ has been deposited⁶⁾ in the flux tube. In the c.m.s. a longitudinal extension $2R_L$ with

$$R_L \propto \sqrt{s} \quad (77)$$

($\sqrt{s} \approx 2k$) is reached before the tube breaks up into two or a few fragments with high p_L .

This alternative seems to be appropriate for the elastic and low multiplicity channels. The dimensions (76) and (77) of the intermediate bag apparently support our result of an elongated interaction region. Johnson and Thorn⁹⁾ for example, obtain

$$R_T \approx 1.58 \text{ fermi}, \quad c \approx \frac{R_L}{k R_T^2} = \frac{1}{C^2} \frac{1}{2\pi\alpha_c}, \quad (78)$$

*) Two colour octets, if a gluon has been exchanged between the quarks of the initial hadron bags [Low⁷⁾, Nussinov⁸⁾], two colour triplets, if a quark of one initial hadrons annihilates with an antiquark of the other one [Johnson and Thorn⁹⁾].

where C^2 is the eigenvalue of the (quadratic) Casimir operator for the colour at either end and $\alpha_c = g^2/4\pi \approx 0.55$ is the quark gluon coupling constant. For colour triplets one obtains

$$c = \frac{3}{16} \frac{1}{2\pi\alpha_c} = 0.054 \quad (79)$$

[with a parametrization $c \approx R_L/(k (1 \text{ fermi})^2)$ one obtains $c \approx 0.136$]. Hence -- even though c is inversely proportional to α_c -- c is numerically small.

All the above mentioned results are essentially independent of the details of the confining mechanism. Only the specific shape of the ends of the flux tube is quite sensitive to it: volume pressure leads to a cusp at each end³¹⁾, surface tension to a smooth flattish behaviour^{3,5)}.

The (heavy) quark bag model for extended hadrons and their interactions of Ref. 4 also leads to the universal spatial extensions $R_T \approx 1 \text{ fermi}$ and $R_L \propto \sqrt{s}$ of highly excited interacting hadronic matter, the so-called "firesausage", which plays a key role in that approach.

In fact it seems more generally as if any model for multiparticle production, where the emitted hadrons carry limited transverse momenta³²⁻³⁵⁾, leads to a long interaction region with $R_L \propto \sqrt{s}$. Let us follow the argument by Low and Gottfried³⁵⁾. After the collision of the two incoming hadrons in their c.m.s. two pulses of hadronic matter recede at the speed of light in opposite directions. The time Δt it takes a secondary particle with mass m , energy ω , and longitudinal momentum p_L to separate from one of these pulses and behave like a free particle is Lorentz dilated:

$$\Delta t \approx \frac{\omega}{m} \tau_0, \quad (80)$$

where τ_0 is a characteristic hadronic time scale ($\sim 10^{-23} \text{ sec}$). Correspondingly the longitudinal distance from the collision point to the point of separation is

$$\Delta x_L \approx \frac{p_L}{m} \tau_0. \quad (81)$$

Thus highly energetic secondaries with large p_L (and small p_T) detach from the interaction region after having traversed a longitudinal distance $\Delta x_L \propto \sqrt{s}$. It seems very likely that elastic scattering -- "producing" two high p_L particles in the final state and in addition being the shadow of all inelastic channels -- has also such a long interaction region along the beam direction. Incidentally, due to the Lorentz-dilatation a spherical interaction region appropriate for scattering near threshold is transformed into a long spheroidal one at high energies.

These arguments have been put forward³⁶⁾ and used to explain the striking lack of cascading³⁷⁾ in high energy (~ 200 GeV) hadron-nucleus scattering experiments. The length of the interaction region for primary reactions as determined in the laboratory system is of the order of 200 fermi, i.e. much larger than the diameter of the nucleus. Correspondingly the highly energetic secondaries from a primary reaction taking place inside the nucleus become only free for a further reaction outside the nucleus; thus there is no cascading.

5. A REALISTIC TUNNELLING PICTURE FOR HADRONIC DIFFRACTION

5.1 Qualitative discussion of the tunnelling amplitude

Let us first discuss the qualitative behaviour of the tunnelling amplitude -- Eqs. (45), (51), and (52) -- as obtained from a spheroidal "fat string" with sharp boundary and dimensions $R_T \approx 1$ fermi, $R_L \approx ckR_T^2$ $\xrightarrow{k \rightarrow \infty} \infty$ and c small in different regions of p_T . The aim is to demonstrate that already this simple prototype of a tunnelling amplitude naturally accounts for the key features of elastic hadron scattering at small and large p_T (of course with the exception of $p_T \approx 0$, the realm of Fraunhofer diffraction). Recall that the tunnelling amplitude depends (in a nonperturbative way) only on the geometry of its characterizing caustic which we shall find reflected in the features of the data.

In the next subsection we shall treat a more realistic case of a complex caustic for the tunnelling rays.

i) $0 \leq p_T < p_T^{(0)} = 1/(\text{Im } \alpha_1/k)$, where $p_T^{(0)}$ will turn out to be ≈ 0.3 GeV/c with $p_T = k \cdot \sin \theta \approx k\theta$ [corresponding to the interval $0 \leq -t < 0.1$ (GeV/c)²]. This is the region of validity of the Fraunhofer diffraction formula (56), which is only sensitive to the transverse dimension R_T . As discussed in Section 3.2, the tunnelling and the classical component of the scattering amplitude contribute an equal amount at $p_T = 0$. However, due to the enormous length of the interaction region [resulting in $\text{Im } \alpha_1 \propto k$, see Eqs. (37) and (38)], the domain of validity has shrunk to an unimportantly small region around $p_T \approx 0$!

ii) $p_T^{(0)} \leq p_T \leq p_T^{(1)} = 1/[\text{Im } (\alpha_2/k) - \text{Im } (\alpha_1/k)]$, where $p_T^{(1)}$ will turn out to be of the order of 1.1 GeV/c. In this region *all surface waves* contribute and *conspire* -- according to Eq. (60) -- to give an enormous forward peak.

iii) $p_T^{(1)} \leq p_T \leq p_T^{(2)} = |\alpha_1(k) + 1/2|/\Delta R$, where $p_T^{(2)}$ will turn out to be of the order of 2.5 GeV/c. For $p_T \gtrsim p_T^{(1)}$ the *lowest surface wave* dominates the sum of all of them according to Eq. (55). For p_T sufficiently small as compared to $p_T^{(2)}$, we may approximate the exponent in the spheroidal angular functions Eqs. (52) and (53)

$$\int_0^\theta d\theta' \sqrt{(\alpha_1 + 1/2)^2 + (k \Delta R \sin \theta')^2} \approx (\alpha_1 + 1/2) \theta \approx \frac{\alpha_1 + 1/2}{k} p_T \quad (82)$$

Hence we obtain

$$\frac{d\sigma}{dt} = \frac{\pi R_T^4}{4} \left| f_{\text{tunnelling}}(k, \theta) \right|^2 \propto \frac{1}{p_T} e^{-A p_T} \quad (83)$$

with $A = 2 \operatorname{Im} \alpha_1(k)/k$. A is a constant as we recall from Eq. (38) or from the approximation (39) for α_m and approximately given in terms of $\rho_L(0)/k^2$, where $\rho_L(0) = R_L^2/R_T \propto k^2$. Thus an *energy independent Orear type exponential behaviour*³⁸⁾ $\exp(-A p_T)$ is predicted, the *slope* A of which is *sensitive to the longitudinal extension* R_L of the interaction region as well as to the transverse one.

Figures 7 and 8a show elastic $pp \rightarrow pp$ data for a comparison with the predicted Orear behaviour. Indeed, for $1.5 \lesssim p_T \lesssim 3.8$ GeV/c (the highest available value of p_T) all ISR data^{15,29)} at five energies from $\sqrt{s} = 23.4$ GeV to 62.1 GeV and the preliminary FNAL data³⁰⁾ at $p_{\text{lab}} = 400$ GeV/c follow a straight line in a plot of $\log(p_T d\sigma/dt)$ versus p_T . Figure 8 demonstrates furthermore the energy independence of all data in the FNAL-ISR energy range for $p_T \gtrsim 1.5$ GeV/c. Even at $p_{\text{lab}} = 19$ GeV/c (which is not yet asymptotic in the sense that $R_L \gg R_T$), the data³⁹⁾ show already an Orear-type behaviour (83) with the *same slope* as the high-energy data over the same p_T range, which at this energy reaches out to $\theta \approx 90^\circ$ (see Fig. 7).

iv) θ fixed, $k \rightarrow \infty$ and $p_T \gtrsim p_T^{(2)} = |\alpha_1(k) + 1/2|/\Delta R$. This is the kinematical region which is generally believed to be insensitive to confinement. The idea is rather that the substructure of hadrons, i.e. quarks and gluons are probed. As is well known, the simple minded parton models for elastic scattering lead typically to a power law behaviour $d\sigma/dt \propto (p_T^2)^{-N} f(\theta)$, where the power N is a measure for the compositeness of the hadrons and is given, for example, by the counting rule⁴⁰⁾ to be equal to the sum over the number of valence quarks in all four hadrons minus two (for $pp \rightarrow pp$ one thus expects $N = 10$).

In fact, Fig. 8b shows that all elastic $pp \rightarrow pp$ data in the FNAL³⁰⁾-ISR^{15,29)} energy range seem to merge into a power law behaviour for $p_T \gtrsim 2$ GeV/c with a power N somewhere between 8 and 10.

In the following we shall find the at first sight surprising result that the tunnelling amplitude -- due to the length of the spheroidal caustic -- leads to a power law behaviour at fixed angles as well. In order to obtain the fixed angle behaviour of the tunnelling amplitude (45), we have to study the exponent in Eqs. (52) and (53) in the limit $k \rightarrow \infty$, θ fixed

$$I_m(\theta) = \int_0^\theta d\theta' \sqrt{(\alpha_m + 1/2)^2 + (k \Delta R \sin \theta')^2}, \quad (84)$$

which is an incomplete elliptical integral of the second kind. Using well-known identities of elliptical integrals, we obtain

$$I_m(\theta) \underset{k \rightarrow \infty, \Delta R \propto k \rightarrow \infty}{\approx} \quad (85)$$

$$\text{with } 2k\Delta R \sin^2 \frac{\theta}{2} + \mathcal{K}_m \left[1 + \log \left(\frac{4k\Delta R}{\mathcal{K}_m} \right) + \log \left(\tan^2 \frac{\theta}{2} \right) \right] + o(1)$$

$$\mathcal{K}_m = \frac{k}{4\Delta R} \left(\frac{\alpha_m + 1/2}{k} \right)^2. \quad (86)$$

Thus we find the following power law behaviour

$$\begin{aligned} |f_{\text{tunnelling}}^{(m)}(k, \theta)|^2 &\propto \frac{1}{(k \sin \theta)^2} e^{-2 \text{Im}(I_m(\theta))} \\ &\propto (p_T^2)^{-N_m} g_m(\theta) \end{aligned} \quad (87)$$

where the powers N_m are given by

$$N_m \approx 2 \text{Im } \mathcal{K}_m + 1 = \frac{\text{Im}(\alpha_m + 1/2)^2}{2k\Delta R} + 1 \quad (88)$$

and the angular distributions $g_m(\theta)$ by

$$g_m(\theta) = D_m (1 + \cos \theta)^{2N_m - 2} \quad (89)$$

with constants D_m .

The powers N_m do in fact not depend on the energy k : from Eq. (37) we see that $\text{Im}(\alpha_m + 1/2)^2/(k\Delta R)$ is asymptotically for large k given by $(kR_T^2/R_L) \text{Im}(1 + \gamma_m^2)$, γ_m depends only on the product $(m - 1/4)R_L/(kR_T^2)$ according to Eq. (38) and $R_L/(kR_T^2) \approx c$ is constant. Thus for small values of m the power N_m depends on c and m ; however, for sufficiently large values of $c(m - 1/4)\pi$ we find

$$N_m \underset{c(m-1/4)\pi \text{ large}}{\approx} 2m + 1/2 \quad (90)$$

depending exclusively on the index m , which counts the surface wave orbits. To the extent that the numbers c and m enter the power N_m as well as the fixed p_T slope $A_m = 2 \text{Im } \alpha_m(k)/k$, these two quantities are related (in contrast to parton models!).

The functions $g_m(\theta)$ look reasonable; they are roughly constant for $\theta \lesssim 45^\circ$. Their precise form cannot, however, be compared immediately to data at very large angles. One first has to clarify the question of s-u crossing properties, which differ for different reactions. Incidentally, it is, however, interesting that the s-u crossed version of the tunnelling amplitude (87) becomes almost angle-independent for fixed p_T

$$|f_{\text{tunnelling}}^{(m)}(k, \theta)|^2 \Big|_{s \leftrightarrow u} \propto p_T^{-2N_m} (1 + \cos \theta)^3 \quad (91)$$

where $(1 + \cos \theta)^3$ varies only by a factor 8 from $\theta = 0$ to $\theta = 90^\circ$.

Finally let us recall that also in this kinematical region of large p_T the lowest surface wave contribution dominates the cross-section.

We emphasize that the requirements $R_T \approx \text{const.}$ and $R_L \propto k$ are at the same time responsible for energy-independence at fixed p_T in region (iii) and the power law behaviour for fixed angles in region (iv).

Can we understand intuitively how a power law can result from our approach? We notice: for diffraction into any fixed angle θ and for $k \rightarrow \infty$, the incoming ray of neargrazing incidence will have travelled along the surface (as surface creep wave) all the way up to the tip of the spheroid; the coordinates of the point $P(x_0, z_0; \theta)$, where it is radiated off tangentially (see Fig. 6), are easily calculated:

$$\begin{aligned} x_0(\theta, k) &\approx \cot \theta \frac{1}{ck} \xrightarrow{k \rightarrow \infty} 0 \\ z_0(\theta, k) &\approx R_L \left(1 - O\left(\frac{1}{k^2}\right)\right) \xrightarrow{k \rightarrow \infty} R_L \end{aligned} \quad (92)$$

and the longitudinal radius of curvature in $P(x_0, z_0; \theta)$ is [see Eq. (70)]

$$\rho_L(\theta, k) \approx \frac{1}{\sin^3 \theta} \frac{1}{ck} \xrightarrow{k \rightarrow \infty} 0. \quad (93)$$

This means, for fixed θ the outgoing rays will be radiated off the tip of the spheroid, where the transverse coordinate x_0 , as well as the longitudinal radius of curvature ρ_L , have natural dimension and tend to zero like $1/k$ for $k \rightarrow \infty$. Thus, in the fixed angle limit the complex surface "ray" (wave) probes the pointlike tip of the spheroid before being radiated off. Let us remember from Eq. (65) that the integral $I_m(\theta)$, Eq. (85), is in fact approximately given in terms of the integral

$$k^{1/3} \int_0^\theta d\theta' (\rho_L(\theta'))^{1/3}. \quad (94)$$

We then realize that the integration between $\theta_0 \neq 0$ and θ leads to a scale invariant result $[\rightarrow g_m(\theta)]$; while the contribution from the lower integration limit retains some "memory" of the region $\theta' \approx 0$, where the scale R_T is important, resulting in breaking the scale invariance by the powers $(R_T^2 p_T^2)^{-N_m}$.

To the extent that there exists a possible relation between the spheroidal interaction region and the long intermediate colour electric flux tube in bag-bag scattering (as speculated on in Section 4.3), we can vaguely "reconcile" our geometric ideas with parton ideas: probing the pointlike tip of the spheroid means probing the tip of the intermediate flux tube, where indeed the quarks are situated. In this sense, the scale invariant portion of the integral $I_m(\theta)$ obtained from integrating over $\theta' \geq \theta_0 > 0$ simulates a scale free "pointlike interaction with

quarks". If this is a correct interpretation, we may infer that the *mechanism which confines the quarks at large energies to the pointlike tips of the intermediate fat string is of the same crucial importance at large p_T as it is at small p_T .*

In summary, it is the geometry of the stringlike "confining walls" with pointlike tips and not the interaction of quasi-free pointlike constituents, which gives rise to the power law behaviour in our approach.

5.2 "Blurring" the boundary

Up until now we have only discussed the idealized case of a surface-caustic generated by a discontinuity of the potential at its boundary $\xi = \xi_B$, i.e. at $\rho = \rho_B = \Delta R \sinh \xi_B = R_T$. In this section let us investigate the more general case of a complex caustic, which allows the potential to have a smooth drop along the real ξ viz. ρ axis ("blurred" boundary).

We consider a potential $U(k, \vec{x})$ exhibiting the same spheroidal symmetry as our previous sharp boundary potential with focal distance

$$2\Delta R = 2ckR_T^2 \xrightarrow{k \rightarrow \infty} \infty \quad \text{with } R_T \approx 1 \text{ fermi, } c \ll 1. \quad (95)$$

The wave equation (1) is separable with the following form of the potential

$$U(\vec{x}, k) = \frac{u(k, \Delta R \sinh \xi)}{(\Delta R)^2 (\sinh^2 \xi + \sin^2 \theta)} = \frac{u(k, \Delta R \sinh \xi)}{2\Delta R \cosh \xi} \left(\frac{1}{r_1} + \frac{1}{r_2} \right), \quad (96)$$

where r_1 and r_2 are the distances of the point $\vec{x}$ from the two foci, respectively [thus $U(k, \vec{x})$ exhibits a Coulomb type behaviour with respect to each focus at short distances]. $u(k, \Delta R \sinh \xi)$ is some short range, spheroidally symmetric, reduced potential. The resulting angular equation remains equal to the free equation (19), while the "radial" equation leads to a "radial" action

$$S_\xi \approx \int d\xi \sqrt{(k\Delta R \sinh \xi)^2 - (\lambda + 1/2)^2 - u(k, \Delta R \sinh \xi)} \quad \text{or} \\ S_\xi \equiv S_\rho \approx \frac{k}{\Delta R} \int d\rho \frac{1}{\sqrt{\rho^2 + \Delta R^2}} \sqrt{\rho^2 - \beta^2 - \frac{u(k, \rho)}{k^2}} \quad (97)$$

in terms of the transverse variable $\rho = \Delta R \sinh \xi$.

It is obvious that in general the potential $u(k, \rho)$ will introduce (if at all) turning points in ρ (i.e. caustics in $\vec{x}$), which do not only depend on the geometrical extensions but also on the coupling strength of the potential. In order to obtain a turning point close to the periphery of the potential independent of its coupling, we need a *strong variation of the potential* nearby, as typically provided

by a (complex) singularity (or discontinuity). The simplest example for a singularity, admitting a blurred, fuzzy boundary is a complex pole in the variable ξ or equivalently ρ . Let the pole position be at

$$\xi = \xi_0 \quad \text{with} \quad \Delta R \sinh \xi_0 = \rho_0 \quad (98)$$

situated close to the periphery of the potential

$$\begin{aligned} \text{Re } \rho_0 &= R_T \approx 1 \text{ fermi} \quad \text{and} \\ \text{Im } \rho_0 &\approx R_T, \end{aligned} \quad (99)$$

where R_T is now the mean transverse range of the potential and $\text{Im } \rho_0$ a measure for the fuzziness of the boundary. We then may write near the pole

$$u(k, \rho) \approx - \frac{u_0 \rho_0^2}{\rho^2 - \rho_0^2}. \quad (100)$$

For the sake of simplicity let us assume for the time being that $u_0 = u_0(k)$ is independent of ρ -- even though this is not really a short range potential. A potential with a complex pole has been discussed extensively in Ref. 41 for the spherical case. We proceed analogously and present here only a sketch of the arguments.

The potential introduces two turning points ρ_1, ρ_2 with

$$\rho_{1,2}^2 = \frac{\beta^2 + \rho_0^2}{2} \pm \sqrt{\left(\frac{\beta^2 - \rho_0^2}{2}\right)^2 - \rho_0^2 \frac{u_0}{k^2}}. \quad (101)$$

We are interested in the case

$$\epsilon^2 = \left| \frac{\rho_2^2 - \rho_0^2}{\rho_1^2 - \rho_0^2} \right| = \left| \frac{\rho_1^2 - \beta^2}{\rho_1^2 - \rho_0^2} \right| \ll 1 \iff \frac{|u_0|}{|k\rho_0|^2} \ll \left| \frac{(\rho_1^2 - \rho_0^2)^2}{\rho_0^4} \right|. \quad (102)$$

Then the turning point at ρ_2 is close to the pole at ρ_0 and ρ_1 close to β , i.e. the positions of the two turning points become independent of the strength of the potential. One obtains again a semiclassical series involving complex multiple reflections between the two turning points $\rho_2 \approx \rho_0$ and $\rho_1 \approx \beta$, in complete analogy to the series obtained from multiple reflections between R_T and β in the sharp boundary case (cf. Section 3.2). Hence tunnelling (generalized surface creep waves) will occur along a complex caustic $\rho(\vec{x}) \approx \rho_0$, which is again situated close to the periphery of the potential. This tunnelling phenomenon corresponds again to the contributions from complex poles in the λ -plane, or equivalently the $\beta = (\lambda + \frac{1}{2})/k$ -plane. The "radial" action between ρ_1 and ρ_2 takes the form

$$\begin{aligned}
 S_g(p_2, p_1) &\approx \frac{k}{\Delta R} \int_{p_1}^{p_2} dp \sqrt{p^2 - \beta^2 + \frac{u_0 p_0^2 / k^2}{p^2 - p_0^2}} \quad \text{for } |p_{1,2}|^2 \ll \Delta R^2 \\
 &= \frac{k}{\Delta R} \int_{p_1}^{p_2} dp \sqrt{\frac{(p^2 - p_1^2)(p^2 - p_2^2)}{p^2 - p_0^2}} \\
 &= \frac{k}{\Delta R} \int_{p_1}^{p_0} dp \sqrt{p^2 - p_1^2} + O(E^2 \log E^2) + S_g(p_2, p_0) \\
 &= S_g^{\text{free}}(p_0, \beta) + O(E^2 \log E^2) + O(E^2)
 \end{aligned}
 \tag{103}$$

where

$$S_g^{\text{free}}(p_0, \beta) = \frac{k}{\Delta R} \int_{\beta}^{p_0} dp \sqrt{p^2 - \beta^2}. \tag{104}$$

The Bohr-Sommerfeld type quantization condition analogous to Eq. (36), which determines the complex pole positions α_m in the λ -plane, is given by⁴¹⁾

$$\begin{aligned}
 S_g(p_2, p_1) &\approx S_g^{\text{free}}(p_0, \beta) \\
 &= (m - 1/2)\pi + \frac{i}{2} \log \left[i F\left(\frac{1}{\pi} S_g(p_0, p_2)\right) \right]
 \end{aligned}
 \tag{105}$$

for $m = 1, 2, \dots$

Compared to the quantization condition (36) for the sharp boundary case, we find i) the real position of the discontinuity, $\rho = R_T$, replaced by the complex pole position $\rho = p_0$; ii) $m - 1/4$ replaced by $m - 1/2$; and iii) the additional correction term $\log iF$, which is due to the presence of a pole in addition to the two turning points. $F(\eta)$ is the following function⁴¹⁾

$$\begin{aligned}
 F(\eta) &= \frac{2\pi i \eta \exp(-2\eta \log(e^{-i\pi} \eta)/e)}{\Gamma(1-\eta)^2} \\
 &\begin{cases} \approx -i & \text{for } |\eta| \gg 1 \\ \approx 2\pi i \eta & \text{for } \eta \rightarrow 0 \end{cases} \quad \text{and } 0 < \arg \eta < 2\pi
 \end{aligned}
 \tag{106}$$

Furthermore, using Eq. (102), we obtain

$$S_g(p_0, p_2) \approx \frac{u_0}{k \Delta R} \frac{p_0}{\sqrt{\beta^2 - p_0^2}} \frac{\pi}{4}. \tag{107}$$

For very small values of this action, the log term in Eq. (105) becomes important; (this leads to the Born approximation, where the amplitude becomes proportional to the coupling strength u_0 of the potential). In order to obtain independence of the coupling u_0 , we shall require

$$\frac{1}{2} |\log i F| \ll (m - 1/2)\pi \tag{108}$$

This is the "nonperturbativity" requirement analogous to the ones (62) and (69) in case of the sharp boundary. With this additional constraint the pole positions $\alpha_m(k)$ in the complex λ -plane are situated at

$$\alpha_m(k) + 1/2 = k\beta_m \underset{k \text{ large}}{\approx} k g_0 \sqrt{1 + \tilde{\gamma}_m^2}, \quad (109)$$

where $\tilde{\gamma}_m$ is defined like γ_m in Eq. (38) with $(m - 1/4)R_L/(kR_T^2)$ substituted by $(m - 1/2)R_L/(k\rho_0^2)$ with $R_L = \sqrt{\Delta R^2 + \rho_0^2}$. With this equation the two inequalities (102) and (108) for u_0 become tightest for $m = 1$. Assuming that the bound (108) is well fulfilled by $.25 \ll (m - 1/2)\pi$, we obtain (for a favourable choice of phases and for realistic values of $k|\rho_0|^2/\Delta R$ and $\tilde{\gamma}_1$)

$$0.045 \lesssim \left| \frac{u_0}{(k g_0)^2} \right| \ll 1. \quad (110)$$

Within this range of coupling strength u_0 the tunnelling amplitudes, given by the contributions of the Regge-like poles at α_m , are *nonperturbative*; they depend only on the parameters characterizing the complex pole position ρ_0 of the potential close to its periphery and not on u_0 . Roughly speaking, the *complex pole at $\rho = \rho_0$ acts like a complex sharp boundary at $\rho = \rho_0$* .

It is obvious from this analysis that a genuine short range potential with a complex pole at $\rho = \rho_0$ will lead to similar results ($\rho_1 \approx \beta$ for $\text{Re } \beta > R_T$ will even be obtained without any conditions on the potential strength).

After this exercise the following generalization is very plausible: given a short range potential which

- i) exhibits spheroidal symmetry as given in Eq. (96), and
- ii) generates a turning point at $\rho = R$ with $\text{Re } R = R_T \approx 1$ fermi, which only depends on geometrical properties of the potential (like extension and fuzziness of the boundary given in terms of $\text{Re } R$ and $\text{Im } R$, respectively) and not on its strength;

then we obtain a tunnelling phenomenon along a caustic $\rho(\vec{x}) \approx R$, which is due to complex multiple reflection between the complex turning points R and $\beta = (\lambda + 1/2)/k$. This tunnelling phenomenon is *nonperturbative* if a lower limit for the potential strength is observed. This limit depends on the nature of the singularity of the potential, which is responsible for the caustic. The intuitive interpretation of tunnelling by means of surface creep waves can be maintained.

All our results obtained in the sharp boundary case -- Eqs. (45), (51), and (52) as well as their qualitative features extracted in Section 5.1 (Orear-type behaviour, power law behaviour, etc.) -- remain valid, if ΔR remains unchanged,

R_T is substituted by R and R_L is substituted by $\sqrt{\Delta R^2 + R^2}$. This in fact justifies our extensive treatment of the sharp-boundary case. Furthermore, we have seen that the dependence of the tunnelling amplitude on the quantization index m may vary slightly according to the generating mechanism of the caustic. So we should not take it too literally in a comparison with data.

Of course this whole discussion may be extended to potentials with more than one complex pole in ρ ; however, the one situated closest to the real axis will dominate the behaviour for $p_T \neq 0$.

5.3 Hadronic tunnelling -- the shadow of copious particle production

We are now ready to make our hypothesis of "maximal importance" of particle production (see Section 2) more concise. So far we have almost exclusively discussed the tunnelling component of the quantum mechanical scattering amplitude $f(k, \theta)$ and established a very promising agreement with the characteristics of high energy hadronic diffraction data out to large p_T .

Two points are crucial for the following discussion: i) Remember that the tunnelling component is due to incoming rays with (quantized) impact parameters

$$\text{Re } \beta_m \gtrsim R_T \quad (111)$$

corresponding to complex poles with positions $\beta_m = (\alpha_m + 1/2)/k$ in the β plane. R_T is the mean transverse range of the potential, i.e. the mean transverse extension of the interaction region. ii) The remaining "classical" component of the amplitude $f(k, \theta)$ arises from contributions of saddle points $\beta_S(\theta)$ with

$$\text{Re } \beta_S(\theta) \lesssim R_T \quad (112)$$

where $\text{Re } \beta_S(\theta)$ specifies the impact parameter of the "classical" ray scattered into the angle θ . More precisely, in the case of a spheroidal cut-off potential (and refraction index $N \rightarrow i\infty$), we have a real saddle point at

$$\beta_S(\theta) = \frac{R_T \cos \theta/2}{\sqrt{1 + (\Delta R/R_T)^2 \sin^2 \theta/2}} \lesssim R_T \quad (113)$$

being near R_T only for $\theta \approx 0$. For a caustic due to a complex pole at $\rho = \rho_0$ the saddle point becomes complex; it is obtained from β_S [Eq. (113)], by substituting R_T by ρ_0 (as has been shown in Ref. 41 for the spherical case, $\Delta R = 0$).

In a strict quantum mechanical framework with complex potentials, even for so-called *total absorption* ($\text{Re } N \rightarrow 1$, $\text{Im } N \rightarrow 0$ such that $k \text{ Im } N \rightarrow \infty$ for $k \rightarrow \infty$; N = refraction index), this "classical" term dominates the amplitude for sufficiently large angles θ (i.e. outside the Fraunhofer diffraction domain).

Our hypothesis of "maximal importance" of particle production, which leads *beyond the quantum mechanical framework*, states: whenever the two incoming hadrons interpenetrate each other classically, i.e. whenever an incoming ray "impinges" on the scatterer with the shape of the hadronic interaction region, the intensity goes into particle production. This means that the (real and complex) *semiclassical contributions* due to incoming rays with impact parameter $\text{Re } \beta < R_T$ are totally absorbed. This leaves us with the following *amplitude for elastic hadron scattering*:

- i) It consists of the *full (spheroidal) tunnelling component at all angles* -- as rigorously calculated in the quantum mechanical framework.
- ii) It obtains a contribution from the saddle point component *only near* $\theta \approx 0$ (where $\text{Re } \beta_S \approx R_T$) to make up for a Fraunhofer type behaviour by conspiring with the tunnelling component (cf. Section 3.2).

There is an immediate consistency check. We have seen in Section 3.2: as long as the saddle and the tunnelling component are of comparable order of magnitude they interfere and produce Fraunhofer diffraction minima at angles, where

$$J_1(kR_T\theta) = 0, \text{ i.e. } p_T \approx k\theta \approx \frac{1}{R_T} (3.84 + n\pi) \quad (114)$$

for $n=1,2,3,\dots$

Diffraction minima are then -- according to our interpretation -- signals of a residual effect of the saddle point contribution, which is dying off rapidly for increasing angles.

High-energy $pp \rightarrow pp$ data do indeed show a single dip^{15,29} at $p_T \approx 1.15$ GeV/c, which is compatible with being the first diffraction minimum for $R_T \approx 0.7$ fermi. This is consistent with an amplitude, where the saddle point term starts from its full contribution at $\theta = 0$, then decreases at least as fast as the tunnelling component, which fully takes over beyond $p_T \approx 1.5$ GeV/c; hence the *absence of further minima* in the data^{15,29,30}.

Let us finally emphasize very strongly: our "absorption prescription" to account for multiparticle channels is *completely different* from the traditional one in hadron physics, where impact parameter amplitudes

$$f(k,b) \sim \int_0^\infty d(k\theta) J_0(k\theta b) f(k,\theta) \quad (115)$$

are absorbed for real impact parameters $b < R_T$.

6. SEMIQUANTITATIVE COMPARISON WITH DATA

6.1 Scattering off nuclei at intermediate energies: tunnelling along a spherical caustic

In proton scattering off heavy nuclei the effective radii are much larger than in hadron-hadron scattering. Typically

$$R_A \approx 1.13 \text{ fermi} \cdot A^{1/3} \quad (116)$$

where A is the atomic number of the heavy nucleus. Correspondingly we might expect three energy regions with different physics:

- i) $k \lesssim 100$ MeV, where $kR_A \lesssim 1$, such that the semiclassical approximation is not yet applicable (k = c.m.s. momentum);
- ii) $k \sim$ a few GeV, where $kR_A \gg 1$; the interaction region will still be about spherical (with radius R_A) due to the low energy;
- iii) $k \gtrsim 50$ -100 GeV; the long hadronic interaction region will have developed³⁶⁾ as in hadron-hadron scattering.

Region (i) certainly has nothing in common with high-energy hadron-hadron scattering; region (iii) should show similar features as hadron-hadron physics, which is so far nicely borne out by the data³⁷⁾. Region (ii), where particle production starts to become important, should show a *significant tunnelling component* coming up at larger angles. However, the *caustic* characterizing such a "nuclear tunnelling" should be *spherical with radius R_A* . Assuming to first approximation a surface-caustic, leaves us with a *parameter-free* description of the tunnelling amplitude as obtained from Eqs. (45), (51), and (52) in the spherical limit, $\Delta R \rightarrow 0$, with $R_T \equiv R_L \equiv R_A$ given by Eq. (116).

There exist high precision data⁴²⁾ for $pA \rightarrow pA$ scattering at one energy $E_{\text{lab}} = 1.98$ GeV for $A = {}^{12}\text{C}$, ${}^{40}\text{Ca}$, ${}^{48}\text{Ca}$, ${}^{58}\text{Ni}$, ${}^{208}\text{Pb}$, where kR_A varies roughly from 20 to 60. Figure 9 shows the elastic data plotted versus $kR_A\theta$. The forward peak is followed by an exponential decrease in this variable, modulated by regular oscillations, which follow the diffraction pattern (114): minima at $kR_A\theta \approx 3.84 + n\pi$ for $n = 0, 1, 2, \dots$. They signalize an interference between the tunnelling and the saddle point contributions; the decrease in amplitude of these oscillations for increasing $kR_A\theta$ is -- according to our interpretation -- due to the tunnelling component becoming dominant. It is quite consistent that we should find a saddle-point component, which is non-negligible, since at these energies absorption into particle production should be weaker than at very high energies.

A crucial test for a strong presence of the tunnelling component is the peculiar (kR_A) -dependence of the exponential slope predicted from Eqs. (39) and (54) to be

$$e^{-\text{constant } (k R_A)^{1/3} \theta} \quad (117)$$

in the domain of dominance of the lowest (or the lowest few) surface wave^{*}), i.e. more or less right after the diffraction peak; the constant has to be universal for all data. In a plot $\log d\sigma/d\Omega$ versus the variable $(k R_A)^{1/3} \theta$, Fig. 10, we see that each data set may be averaged by a straight line with a universal slope! In Fig. 9, versus $k R_A \theta$, the same data showed a considerable antishrinkage; so this is a highly nontrivial result.

A last test is to compare the average behaviour of the data with the *parameter-free prediction* for the tunnelling amplitude with respect to *shape and normalization*. Figure 11 shows the data in a plot of $R_A^{-2} d\sigma/d\Omega$ versus $(k R_A)^{1/3} \theta$, where -- according to our prediction -- the different data sets should (on the average) fall on top of each other [this prediction is true to the extent that $(\alpha_m + 1/2)/(k R_A) \approx$ independent of $k R_A$]. Figure 11 shows the prediction for ^{12}C of

$$\left| \sum_{m=2}^{\infty} \text{surface waves due to a surface-caustic with radius } R_A \right|^2$$

which perfectly accommodates the average behaviour of all data! (The lowest surface wave was excluded, since it gave a somewhat too flattish behaviour at large angles; we remember, however, from Section 5.2, that the starting value for the surface wave index is not sacred, since it may vary slightly with the nature of the mechanism generating the caustic.)

6.2 High-energy $pp \rightarrow pp$ scattering

A qualitative comparison with key features in $pp \rightarrow pp$ scattering has already been performed in Section 5.1. Let us now compare the $pp \rightarrow pp$ data^{29, 30)} at representative energies $p_{\text{lab}} \gtrsim 400$ GeV/c with the predictions for the tunnelling amplitude on a semi-quantitative level. We ignore the contribution from the "classical" saddle-point component completely, since its parametrization would have to depend on all details of the interaction (and thus "pollute" this comparison); correspondingly, we cannot expect to describe the $p_T = 0$ behaviour, nor the dip region (see Section 5.3). We do expect, however, a quantitative description of the data beyond the second maximum, i.e. for $p_T \gtrsim 1.5$ GeV/c.

For a spheroidal surface-caustic there are only *two free parameters* of purely geometrical origin, R_T and c , where the constant c is a measure for the excentricity of the spheroid (remember: the focal distance is $2\Delta R$ with $\Delta R = c k R_T^2$). For a complex spheroidal caustic $\rho(\vec{x}) \approx R$ as discussed in Section 5.2, we have *three free*

^{*}) This behaviour is expected for all sufficiently large angles; for a spherical caustic there is of course no power law behaviour.

parameters: $\text{Re } R = R_T$, $\text{Im } R$ and c as defined by $\Delta R = ckR_T^2$. We typically expect

$$R_T \approx 1 \text{ fermi}, \quad c \ll 1 \quad \text{and} \quad \arg R \approx \frac{\pi}{4}. \quad (118)$$

We performed "eye-ball" fits to the data.

The more realistic prediction from a *complex spheroidal caustic* (for the sum over *all* surface wave contributions) gives an excellent solution for $p_T \gtrsim 1.5 \text{ GeV}/c$ and reproduces the strong forward peak semi-quantitatively, as displayed in Fig. 12. Thus *shape and normalization* of the data may be understood just in terms of three "geometrical" parameters, which come out to be of the expected order of magnitude

$$R_T = 1.1 \text{ fermi}, \quad \arg R = 0.45 \quad \text{and} \quad c = 0.086 \quad (119)$$

The asymptotic power

$$N_1 = \frac{1}{2} \frac{\text{Im}(\alpha_1 + \frac{1}{2})^2}{k \Delta R} + 1 \quad (120)$$

in the power law $d\sigma/dt \propto (p_T^2)^{-N_1} g(\theta)$ for $p_T \gg 2.5 \text{ GeV}/c$ comes out to be

$$N_1 = 10.2 \quad (121)$$

[surprisingly close to its value $N = 10$, expected from the constituent counting rule⁴⁰⁾].

An example for an "eye-ball" fit to the predictions for the *spheroidal surface-caustic* has already been presented in Ref. 10; it is of almost comparable quality, if the first two surface waves are omitted, and R_T and c are of similar order of magnitude as above ($R_T = 0.81 \text{ fermi}$ and $c = 0.158$). A satisfactory fit may also be obtained with only the lowest surface wave omitted.

Let us emphasize that R_T and c are of the same order of magnitude as the corresponding parameters calculated for a colour flux tube with colour triplets at its ends⁹⁾ [cf. Section 4.3, Eqs. (78) and (79)].

In summary, one can state that the tunnelling amplitude has just the right kind of angular decrease and normalization as shown by the data; it provides a uniform semi-quantitative description from small p_T out to the largest p_T available.

7. EXTENSIONS AND INTERESTING CONSEQUENCES

7.1 Extension to quantum number exchange reactions

An extension to $2 \rightarrow 2$ quantum number exchange reactions is most direct and intuitive. Each such reaction amplitude may be approximately written as difference of two elastic amplitudes [using isospin invariance, invariance under $SU(3)$ or some higher symmetry], which become identical for $k \rightarrow \infty$. Both elastic amplitudes are

given as sums over all surface waves corresponding to a spheroidal interaction region with essentially the same dimensions. Slight differences in these elastic amplitudes arise, since they are shadows of slightly different inelastic channels. Inelastic channels obtain contributions from rays with impact parameters $\beta \lesssim R_T$, surface waves from those with $\beta \gtrsim R_T$. Thus the difference of the two elastic amplitudes will naturally be in the amount of contribution from the lowest surface wave (or the lowest few ones) coming from $\beta \approx R_T$. Hence the exchange reaction will receive (energetically suppressed) contributions from the first (or the first few) surface waves. Correspondingly, one expects that i) it does not show the enormous forward peak of elastic reactions, which is due to the conspiracy of infinitely many surface waves; ii) instead the Orear type behaviour of the cross-section, $\exp(-2 \operatorname{Im}(\alpha_1/k) p_T)/p_T$ continues from intermediate p_T down to small values of p_T ; and iii) for sufficiently large values of p_T , where also the elastic amplitudes are dominated by the lowest surface wave, elastic and exchange cross-sections should show the same p_T dependence (apart from normalization). If one compares the data⁴³⁾ for $\pi^-p \rightarrow \eta n$ at 40 GeV/c with the elastic ISR data for $pp \rightarrow pp$ in Fig. 7, one finds all these predictions confirmed.

Thus we have made plausible that the $2 \rightarrow 2$ quantum number exchange reactions are dominated by the lowest surface wave, i.e. by *one single direct channel Regge-like pole* $\alpha_1(k)$ (in the λ -plane and not in the J plane!), which is peripheral: $\operatorname{Re}(\alpha_1(k) + 1/2)/k = \operatorname{Re} \beta_1 \approx R_T \approx 1$ fermi. This leads to a peripheral impact parameter profile for the imaginary part of the amplitude. A similar reaction mechanism had already been proposed previously by us⁴⁴⁾ and other authors⁴⁵⁾ and had been successfully compared with the data of a large number of reactions.

7.2 Relation to dual models

In dual models $2 \rightarrow 2$ quantum number exchange reactions are built up by a sum over a parent Regge pole $\alpha(s)$ in the direct channel angular momentum (J) plane with approximately linear trajectory and infinitely many parallel daughter trajectories.

In the previous subsection we have seen that in our framework such an exchange reaction is described by *one single Regge-like pole in the λ -plane*, $\alpha_1(k)$ [remember: $k \approx \sqrt{s}/2$]. It is very instructive to investigate (asymptotically for $k \rightarrow \infty$) the spectrum in J generated by one such Regge-like pole, $\alpha(k)$ say, in the λ -plane.

To this end we need the expansion²⁵⁾ of the spheroidal angular functions $S_{on}^{(1)}$ in terms of Legendre polynomials

$$S_{on}^{(1)}(kAR, \cos \Theta) = \sum_{J=0,1}^{\infty} d_J^{on}(kAR) P_J(\cos \Theta) \quad (122)$$

where the sum $\sum'$ extends over $\{\text{even}\}_{\text{odd}}$ J for $\{\text{even}\}_{\text{odd}}$ n. Using Eqs. (22) and (122) we obtain the relation between the spherical ($f(J, k)$) and the spheroidal ($f_{\lambda(n)}(kR_T, kR_L)$) partial wave amplitudes

$$\begin{aligned} f(J, k) &= \frac{(kR_T)^2}{2} \int_{-1}^{+1} dz P_J(z) f(k, \theta) \quad [z = \cos \theta] \\ &= \sum'_{n=0,1} \frac{(\lambda(n)+1/2)}{J+1/2} \frac{d\lambda(n)}{dn} f_{\lambda(n)}(kR_T, kR_L) d_J^{\text{on}}(k\Delta R), \end{aligned} \quad (123)$$

where the sum $\sum'$ runs over $\{\text{even}\}_{\text{odd}}$ n for $\{\text{even}\}_{\text{odd}}$ J.

A single pole of $f_{\lambda(n)}$ at $\lambda = \alpha(k)$ may be written as

$$f_{\lambda} /_{\lambda \approx \alpha(k)} \approx - \frac{1}{\pi} \frac{(kR_T)^2}{2} \frac{r(k)}{\alpha+1/2} \frac{1}{\lambda - \alpha(k)}, \quad (124)$$

where $r(k)$ is the residue as defined in Eq. (46). Expanding λ near the pole

$$\lambda \approx \alpha + (n - \nu(\alpha)) \frac{d\lambda}{dn} /_{n=\nu(\alpha)} \quad (125)$$

we obtain

$$(J+1/2) f(J, k) \approx \sum'_{n=0,1} d_J^{\text{on}}(k\Delta R) \frac{\tilde{r}_n}{n - \nu(\alpha(k))}. \quad (126)$$

Using Eqs. (47) and (85) [for $\theta = \pi/2$] we obtain

$$\begin{aligned} \nu(\alpha(k)) \equiv \nu(s) &\underset{k \approx \sqrt{s}/2 \rightarrow \infty}{\approx} \nu_0 + \nu' s \\ &+ \frac{2}{\pi} (\text{Re } \mathcal{K}) \log(\nu' s) + i \frac{N-1}{\pi} \log(\nu' s), \end{aligned} \quad (127)$$

where $\kappa (= \kappa_1)$ is given in Eq. (86) and the large p_T -power $N (= N_1)$ according to Eq. (88) by

$$N-1 = 2 \text{Im } \mathcal{K} \quad (128)$$

The intercept ν_0 is

$$\nu_0 = -\frac{1}{2} + \frac{2}{\pi} \mathcal{K} \log \frac{2\pi e}{\mathcal{K}} \quad (129)$$

the slope ν'

$$\nu' = \frac{\Delta R}{2\pi k} = \frac{c R_T^2}{2\pi} \approx 0.5 \text{ GeV}^{-2} \quad (130)$$

[using the "best fit" values for c and R_T as obtained in Section 6.2, Eq. (119)]. Thus the n -trajectory $\nu(s)$ is asymptotically a linear trajectory with reasonable slope and with an imaginary part increasing like $\log s$. From Ref. 25 one knows that the series of coefficients d_J^{on} decreases like

$$d_J^{on}(k\Delta R) \sim \text{const.} \frac{(\frac{1}{2}k\Delta R)^J}{\Gamma(J+\frac{3}{2})} \quad \text{for } J > n. \quad (131)$$

Hence we obtain an approximately *linear parent trajectory* at $J \approx \nu(s)$ accompanied by infinitely many parallel *even daughter trajectories*. Although there are *ancestors*, they *decouple* stronger than exponentially!

Thus one single Regge-like pole in the λ -plane at the same time

- i) simulates [according to Eqs. (28) and (51)] for sufficiently small angles a single, effective, peripheral Regge pole in the J plane with $\alpha(s) \propto \sqrt{s}$;
- ii) leads to a power law behaviour of the amplitude at large angles;
- iii) and gives approximately the kind of spectrum as expected from dual models.

7.3 Extension to multiparticle physics

This will be the subject of a subsequent paper¹⁴). Let us just briefly sketch the main idea of how a tunnelling phenomenon responsible for elastic reactions could also become operative in inelastic reactions.

We have argued that elastic scattering of two hadrons

$$h_1 + h_2 \longrightarrow h_1 + h_2 \quad (132)$$

with masses m say, is (apart from $\theta \approx 0$) a pure tunnelling phenomenon. It is described by the sum of all surface waves, which are excited by all (complex) rays with impact parameters $\text{Re } \beta \gtrsim R_T(m) \approx 1$ fermi. All semiclassical contributions with $\text{Re } \beta < R_T(m)$ are supposed to go into inelastic multiparticle channels.

It is quite suggestive to explore the possibility that these "inelastic" rays with $\text{Re } \beta < R_T(m)$ may again be subdivided into tunnelling components. Let us imagine first an elastic multiparticle reaction $h_1 + h_2 + \dots + h_n \rightarrow h_1 + h_2 + \dots + h_n$ roughly arranged as

$$h_1 + \text{cluster}(h_2, \dots, h_n)_{\text{of mass } M} \longrightarrow h_1 + \text{cluster}(h_2, \dots, h_n)_{\text{of mass } M}. \quad (133)$$

This fictitious elastic particle cluster reaction possesses again a spheroidal tube-like interaction region with transverse dimension $R_T(M)$. In general, we expect for increasing cluster mass $M > m$ a decreasing transverse extension

$$\begin{aligned} R_T(M) &< R_T(m) && \text{for } M > m \\ \text{such that } R_T(M) &\propto \frac{1}{M} && \text{for } M \gg m \end{aligned} \quad (134)$$

behaves according to dimensional analysis for large masses M . In analogy to the tunnelling picture for reaction (132), this reaction would be due to tunnelling

along a tube with transverse radius $R_T(M)$. Here the surface waves receive contributions from all rays with impact parameters $\text{Re } \beta \geq R_T(M)$. Hence, roughly speaking: to the extent that elastic hadron-hadron scattering (132) is the literal shadow of all inelastic channels $h_1 + h_2 \rightarrow \text{multiparticle states}$, elastic hadron-cluster (M) scattering (133) becomes the shadow of all inelastic channels $h_1 + \text{cluster (M)} \rightarrow \text{multiparticle states with mass } \geq M$.

Similarly an inelastic reaction of the type

$$h_1 + \text{cluster}_2(M) \longrightarrow h_3 + \text{cluster}_4(M) \quad (135)$$

will be described in analogy to $2 \rightarrow 2$ exchange reactions

$$h_1 + h_2 \longrightarrow h_3 + h_4 \quad (136)$$

by some fraction of the lowest surface wave, tunnelling along the tube with $R_T(M) < R_T(m)$, i.e. the lowest surface wave contributing to reaction (133).

In order to describe typical realistic multiparticle reactions

$$h_1 + h_2 \longrightarrow h_3 + \text{cluster}_4(M) \quad \text{with } M \gg m, \quad (137)$$

we shall combine the ideas obtained above with the fact that a single surface wave corresponds to a factorizable Regge-like pole in the λ -plane. This *factorization property in the direct channel* allows to express reaction (137) in terms of the lowest surface waves of reactions (132) and (133).

In an application¹⁴⁾ to inclusive reactions, we predict a factorized (scaling) form

$$E \frac{d^3 \sigma}{d p^3} = f(p_T) \cdot f\left(A \frac{p_T}{M}\right) g\left(\frac{M^2}{s}\right); \quad A = \text{constant} \quad (138)$$

valid over the whole kinematical range of all variables (with the exception of $p_T \approx 0$, $\theta \approx 0$). Furthermore, the analytic form of $f(p_T)$ is predicted to be related to the square root of the lowest surface wave contribution to reaction (132), i.e. to $\sqrt{(d\sigma/dt)(h_1 + h_2 \rightarrow h_1 + h_2)}$ for $p_T \gtrsim 1.5 \text{ GeV}/c$ (where only the lowest surface wave dominates the sum of all of them). Finally an interesting relation between hadronic inclusive and e^+e^- induced inclusive reactions arises. The predictions are successfully compared to a large number of experimental data.

7.4 Glory effect in hadron physics?

In scattering of visible light with wave number k off water droplets with radius R (in a cloud or in fog) one observes, for $kR \approx 30-1000$, the beautiful natural phenomenon²⁶⁾ of the so-called glory effect. The glory effect is a very strong enhancement of the backscattered intensity (by several orders of magnitude) resulting in a steep narrow backward peak in the observed angular distribution. This phenomenon has been understood as late as 1969 by Nussenzweig²¹⁾ in terms of a subtle enhancement due to *surface waves* creeping around the surface of the

water droplet. Unfortunately, the normalization of the backward peak depends on the refraction index²¹⁾, i.e. the strength of the potential; however, its angular dependence (in the spherical case) is given by²¹⁾

$$\text{intensity} \quad \theta \approx \pi \quad |J_0(kR(\pi - \theta))|^2 \quad (139)$$

only depending on kR .

Backward peaks in scattering off atoms and nuclei have already been attributed to glory and have been successfully compared with the predicted angular dependence (138) (see, for example, Ref. 46).

In hadron reactions we have backward peaks as well [and many of them show in fact a dip at $-u \approx k^2(\pi - \theta)^2 \approx 0.2 \text{ (GeV/c)}^2$ which, for $R \approx 1$ fermi, coincides nicely with the position of the first zero of the J_0 Bessel function]. More recently data⁴⁷⁾ at rather high energies are available with $kR \gtrsim 30$. They show very steep backward peaks which cannot at all be understood by conventional u -channel Regge exchanges. It is quite likely⁴⁸⁾ that one is observing *hadronic glory* there.

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APPENDIX A

In this Appendix we shall summarize the essential steps in the derivation of the exact partial wave solution of the problem treated in Section 3.2: scattering of a plane wave off an infinitely deep (prolate) spheroidal potential well (with half axes R_T and R_L). The appearing spheroidal wave functions^{24,25} will be defined and the Sommerfeld-Watson transformation used in Section 3.2 will also be given.

The stationary wave (Schrödinger) equation

$$[\Delta + k^2 - U(\vec{x}, k)] \psi(\vec{x}, k) = 0 \quad (A1)$$

reduces for this case to the free equation ($U(\vec{x}, k) \equiv 0$) outside the spheroidal boundary and the boundary condition $\psi(\vec{x}, k) \equiv 0$ on the boundary. For the sake of generality, in the following derivation of the spheroidal partial wave expansion, we shall allow for a plane wave of arbitrary incidence. The special case of axial incidence -- relevant to Section 3.2 -- will be considered at the end. In terms of the (prolate) spheroidal coordinates (17) the (free) wave equation separates into the following "radial" and angular equations:

$$\left[\frac{d^2}{d\xi^2} + (k \Delta R \sinh \xi)^2 - (\lambda + 1/2)^2 - \frac{m^2 - 1/4}{\sinh^2 \xi} \right] \cdot \sqrt{\sinh \xi} R(k \Delta R, \cosh \xi) = 0 \quad (A2)$$

$$\left[\frac{d^2}{d\theta^2} + (k \Delta R \sin \theta)^2 + (\lambda + 1/2)^2 - \frac{m^2 - 1/4}{\sin^2 \theta} \right] \cdot \sqrt{\sin \theta} S(k \Delta R, \cos \theta) = 0. \quad (A3)$$

The equation for the azimuthal angle (ϕ) dependence is identical to the spherical case with solutions

$$\cos m\phi, \sin m\phi, \quad m = 0, 1, 2, \dots \quad (A4)$$

The quantity $\lambda + 1/2$ is a separation parameter analogous (but not identical) to the angular momentum in the spherical case. For an infinite, discrete set of real non-integer eigenvalues $\lambda_{mn}(k \Delta R)$, $n = 0, 1, \dots$ $|m| \leq n$ of this parameter the angular Eq. (A3) has solutions

$$S_{mn}^{(1)}(k \Delta R, \cos \theta) = (-1)^{n-m} S_{mn}^{(1)}(k \Delta R, -\cos \theta) \quad (A5)$$

regular at $\theta = 0$ and $\theta = \pi$. We choose to normalize them to the Legendre functions at $\theta = 0$

$$\lim_{\theta \rightarrow 0} S_{mn}^{(1)}(k \Delta R, \cos \theta) / P_n^m(\cos \theta) = 1. \quad (A6)$$

Then, multiplying Eq. (A3) by $S_{mn}^{(1)}$ and integrating we find the orthogonality relation

$$\int_{-1}^{+1} dz S_{mn}^{(1)}(k\Delta R, z) S_{mn'}^{(1)}(k\Delta R, z) = \delta_{nn'} \frac{(n+m)!}{(n-m)!} \frac{1}{(\lambda_{mn} + 1/2) d\lambda_{mn}/dn} \quad (A7)$$

Three types of solutions of the "radial" Eq. (A2) are of interest here: the solution $R_{mn}^{(1)}(k\Delta R, \cosh \xi)$ regular at $\xi = 0$ and the solutions $R_{mn}^{(3)}$ and $R_{mn}^{(4)}$ behaving as purely $\left\{ \begin{smallmatrix} \text{outgoing} \\ \text{incoming} \end{smallmatrix} \right\}$ spherical waves for $k\Delta R \cdot \cosh \xi \rightarrow \infty$, with corresponding normalization to the spherical Hankelfunctions $h_n^{(1/2)}(k\Delta R \cdot \cosh \xi)$ at infinity^{24,25)}

$$\lim_{k\Delta R \cosh \xi \rightarrow \infty} R_{mn}^{(3)}(k\Delta R, \cosh \xi) / h_n^{(1/2)}(k\Delta R \cdot \cosh \xi) = 1 \quad (A8)$$

with

$$R_{mn}^{(1)} = \frac{1}{2} (R_{mn}^{(3)} + R_{mn}^{(4)}) \quad (A9)$$

and Wronskian determinant^{24,25)}

$$\{ R_{mn}^{(1)}, R_{mn}^{(3)} \}_{\cosh \xi} = \frac{\pm i}{k\Delta R \sinh^2 \xi} \quad (A10)$$

We are now ready to expand the sum of the incident plane wave

$$\psi_i = e^{i\vec{k}\vec{x}} \text{ with wave vector } \vec{k} = k(\sin \gamma \cos \phi', \sin \gamma \sin \phi', \cos \gamma) \quad (A11)$$

and the scattered wave ψ_s in terms of product solutions of Eq. (A1)

$$R_{mn}^{(1)}(k\Delta R, \cosh \xi) S_{mn}^{(1)}(k\Delta R, \cos \theta) \begin{cases} \cos m\phi \\ \sin m\phi \end{cases} \quad (A12)$$

The boundary condition $\psi = \psi_i + \psi_s \equiv 0$ on the spheroidal boundary $\cosh \xi_B = R_L/\Delta R$ fixes the coefficients. Taking the limit $|\vec{x}| \rightarrow \infty$, we then arrive at the following spheroidal partial wave expansion of the scattering amplitude $f(k, \theta, \phi)$ [using Eqs. (2) and (20)]

$$f(k, \theta, \phi) = \frac{2}{(kR_T)^2} \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} (2 - \delta_{0m}) (\lambda_{mn} + 1/2) \frac{d\lambda_{mn}}{dn} \frac{(n-m)!}{(n+m)!} \times \quad (A13)$$

$$\times f_{\lambda_{mn}}(kR_T, kR_L) S_{mn}^{(1)}(k\Delta R, \cos \gamma) S_{mn}^{(1)}(k\Delta R, \cos \theta) \cos m(\phi - \phi')$$

with partial wave amplitudes

$$f_{\lambda_{mn}}(kR_T, kR_L) = 2i \frac{R_{mn}^{(1)}(k\Delta R, R_L/\Delta R)}{R_{mn}^{(3)}(k\Delta R, R_L/\Delta R)} \quad (A14)$$

Let us now specialize to the case of a plane wave incident in z direction (Section 3.2), i.e. we put $\eta = 0$ in Eq. (A13). Then, only terms with $m = 0$ survive in Eq. (A13) and the form (22) of Section 3.2 results.

For this case the Sommerfeld-Watson transformation of Eq. (22) into the complex λ -plane is obtained in the standard way, using Eq. (A5):

$$f(k, \theta) = \frac{i}{(kR_T)^2} \int_{C_\lambda} \frac{(\lambda + 1/2) d\lambda}{\sin \pi n(\lambda)} f_\lambda(kR_T, kR_L) \times S_{on(\lambda)}^{(1)}(k\Delta R, -\cos \theta), \quad (A15)$$

where C_λ encircles the real eigenvalues λ_n ($n(\lambda_n) = 0, 1, \dots$) in clockwise direction. It may be shown from the wave equation (A2) -- in complete analogy to the spherical case⁴⁹) -- that $f_\lambda(kR_T, kR_L)$ is a meromorphic function of λ and also that $S_{on(\lambda)}^{(1)}(k\Delta R, -\cos \theta)$ and $\sin \pi n(\lambda)$ are entire functions of λ (cf. Ref. 24).

The Regge-like moving poles of $f_\lambda(kR_T, kR_L)$ in the complex λ -plane at $\lambda = \alpha_m(kR_T, kR_L)$, $m = 1, 2, \dots$ are the solutions of

$$R_{0\nu(\alpha_m)}^{(3)}(k\Delta R, R_L/\Delta R) = 0 \quad (A16)$$

according to Eq. (14).

APPENDIX B

In this Appendix we use the method of *uniform approximations*^{17,18)} to derive an approximate expression for the spheroidal partial waves $f_\lambda(kR_T, kR_L)$, Eq. (23) -- valid near its λ -poles -- for the case of a sharp spheroidal boundary with $R_L \propto k \rightarrow \infty$. We take special account of the peculiarity of a strongly k -dependent boundary, not treated in the literature so far. Moreover, the uniform method provides more accurate approximations in general than the semiclassical one, allows for the estimation of errors, and may finally serve as an independent verification of the results (31) and (33) obtained from the multiple reflection method using the rules of Ref. 12.

The first problem one encounters in approximating the "radial" wave functions $R_{on(\lambda)}^{(j)}(k\Delta R, R_L/\Delta R)$ in Eq. (23) on the boundary with $R_L \simeq \Delta R \propto k \rightarrow \infty$, is that the boundary point $\sinh \xi_B = R_T/\Delta R \propto 1/k$ tends to zero and hence approaches a singularity of the "radial" Eq. (18). Therefore -- as a first step -- we perform a Langer-type substitution¹⁷⁾, which for spherical problems is well known to give improved approximations near the origin. The substitution

$$\cosh \xi - 1 = \frac{1}{2} \left(\frac{R_T}{\Delta R} \right)^2 e^\sigma; \quad -\infty \leq \sigma \leq \infty, \quad (B1)$$

$$R_{on}(k\Delta R, \cosh \xi) = \frac{1}{\cosh \xi/2} Y(k\Delta R, \sigma) \quad (B2)$$

transforms Eq. (18) into

$$\left[\frac{d^2}{d\sigma^2} + P^2(\sigma) \right] Y = 0 \quad (B3)$$

with

$$P^2(\sigma) = \left(\frac{k R_T^2}{2\Delta R} \right) e^\sigma \left[e^\sigma - \frac{(\beta/R_T)^2}{1 + \left(\frac{R_T}{2\Delta R} \right)^2 e^\sigma} \right] - \epsilon(\sigma), \quad (B4)$$

impact parameter $\beta = (\lambda + 1/2)/k$ and

$$\epsilon(\sigma) = \left(\frac{R_T}{4\Delta R} \right)^2 e^\sigma / \left(1 + \left(\frac{R_T}{2\Delta R} \right)^2 e^\sigma \right)^2 < O\left(\frac{1}{\Delta R^2} \right) \quad (B5)$$

for all real σ .

We observe

- i) the singularity at $\xi = 0$ is mapped into a turning point at $\sigma_1 = -\infty$;
- ii) the image of the boundary ξ_B , at $\sigma_B = \log (2\Delta R/R_T^2)(R_L - \Delta R) \xrightarrow{k \rightarrow \infty} 0$ is now well separated from $\sigma_1 = -\infty$;
- iii) the second relevant turning point of $P^2(\sigma)$ is located at

$$\sigma_2 = \log \left[2 \left(\frac{\Delta R}{R_T} \right)^2 \left(\sqrt{1 + (\beta/\Delta R)^2} - 1 \right) \right] \approx 2 \log \frac{\beta}{R_T}$$

[assuming $|\beta/\Delta R| \ll 1$ and $|\arg \beta| < \pi/2$]; finally

iv) in the limit $e^\sigma \ll 4\Delta R^2/R_T^2 \propto k^2$, with "excentricity" parameter $kR_T^2/\Delta R = 1/c$ held fixed, Eq. (B3) approaches the confluent hypergeometric equation

$$\left[\frac{d^2}{d\sigma^2} + \left(\frac{kR_T^2}{2\Delta R} \right)^2 e^{2\sigma} - \left(\frac{kR_T^2}{\Delta R} \right) \left(\frac{k}{4\Delta R} \beta^2 \right) e^\sigma \right] Y_{as.} = 0 \quad (B6)$$

with solutions in terms of the well studied Whittaker functions⁵⁰⁾

$$Y_{as.} = \begin{cases} N_1 \cdot M_{-\frac{i}{4} \frac{k}{\Delta R} \beta^2, 0} \left(e^{-i\pi/2} 4k\Delta R \sinh^2 \xi/2 \right) / \sinh \xi/2 \\ \quad \text{being regular at } \xi = 0; \\ N_2 \cdot W_{-\frac{i}{4} \frac{k}{\Delta R} \beta^2, 0} \left(e^{-i\pi/2} 4k\Delta R \sinh^2 \xi/2 \right) / \sinh \xi/2 \\ \quad \text{being purely outgoing for large } k\Delta R \sinh^2 \xi/2. \end{cases} \quad (B7)$$

This limiting equation (B6) -- having the same qualitative structure as the original one (relevant turning points, etc.) -- serves as an excellent comparison equation^{17,18)} for deriving a uniform approximation of the "radial" functions $R_{on}^{(1)}$ and $R_{on}^{(3)}$ over the whole range of σ viz. ξ for $\Delta R \propto k \rightarrow \infty$. For space reasons we cannot, however, present the derivation in full mathematical rigour here. An excellent discussion of the uniform approximation method may be found in Refs. 17 and 18.

We map the original "radial" Eq. (B3) onto a confluent hypergeometric equation of the type (B6)

$$\left[\frac{d^2}{d\tau^2} + Q^2(\tau) \right] Z(\tau) = 0 ; \quad Y(\sigma) = \frac{1}{\sqrt{\frac{d\tau}{d\sigma}}} Z(\tau(\sigma)) \quad (B8)$$

with

$$Q^2(\tau) = \left(\frac{kR_T^2}{2\Delta R} \right)^2 e^{2\tau} - \frac{kR_T^2}{\Delta R} \kappa e^\tau \quad (B9)$$

and κ left as a parameter to be determined later. The exact mapping $\tau = \tau(\sigma)$ of Eq. (B3) onto Eq. (B6) is the solution of the differential equation^{17,18)} (with $\dot{\tau} = d\tau/d\sigma$)

$$\dot{\tau}^2 Q^2(\tau) = P^2(\sigma) \left\{ 1 + \frac{\sqrt{\dot{\tau}}}{P^2(\sigma)} \frac{d^2}{d\sigma^2} \frac{1}{\sqrt{\dot{\tau}}} - \frac{E(\sigma)}{P^2(\sigma)} \right\}, \quad (B10)$$

which is hard to solve in general. However, since our comparison function $Q^2(\tau)$ was chosen to be qualitatively very similar to the original $P^2(\sigma)$, $\tau(\sigma)$ will be slowly varying and hence from Eq. (B10) the mapping is approximately determined by

$$\int_{\tau_2}^{\tau(\sigma)} d\tau' \sqrt{Q^2(\tau')} \approx \int_{\sigma_2}^{\sigma} d\sigma' \sqrt{P^2(\sigma')} \quad (B11)$$

and

$$Y(\sigma) = \frac{1}{\sqrt{\dot{\tau}}} Z(\tau(\sigma)) \approx \left[\frac{Q^2(\tau(\sigma))}{P^2(\sigma)} \right]^{1/4} Z(\tau(\sigma)) \quad (B12)$$

with range of validity specified by the "error control" function¹⁷⁾

$$E(\sigma) \equiv \frac{1}{2} \int_{\sigma}^{\sigma'} d\sigma' \left| \frac{\sqrt{\dot{\tau}}}{P^2(\sigma')} \frac{d^2}{d\sigma'^2} \frac{1}{\sqrt{\dot{\tau}}} - \frac{E(\sigma')}{P^2(\sigma')} \right| \ll 1. \quad (B13)$$

With the convenient change of variables in Eq. (B11)

$$v = \left(\frac{k R_T^2}{\Delta R} \right) e^{\tau} \quad \text{and} \quad \varrho = R_T e^{\sigma/2} \left(1 + \left(\frac{R_T}{2 \Delta R} \right)^2 e^{\sigma} \right)^{1/2} = \Delta R \sinh \xi \quad (B14)$$

the transformed mapping $v(\xi)$ reads^{*)}

$$\frac{1}{2} \int_{v_2}^{v(\xi)} dv \sqrt{\frac{v-4\kappa}{v}} = k \int_{\beta}^{\Delta R \sinh \xi} d\varrho' \frac{\sqrt{\varrho'^2 - \beta^2}}{\sqrt{\varrho'^2 + \Delta R^2}} \equiv S_{\varrho}(\varrho, \beta) \quad (B15)$$

where the r.h.s. just equals the "radial" part of the classical action, as defined in Eq. (32) of Section 3.2.

Next, we fix the lower integration limit v_2 and the free parameter κ in Eq. (B9) by the requirement that the factor

$$\frac{1}{\sqrt{\dot{\tau}}} \approx \left[\frac{Q^2(\tau(\sigma))}{P^2(\sigma)} \right]^{1/4} = \left[\frac{v(\xi)}{4 k \Delta R \tanh^2 \xi/2} \right]^{1/4} \left[\frac{v(\xi) - 4\kappa}{k \Delta R \sinh^2 \xi - \frac{k}{\Delta R} \beta^2} \right]^{1/4} \quad (B16)$$

in Eq. (B12) remains continuous across i) the turning point $\rho_2 = k \Delta R \sinh \xi_2 = \beta$, and ii) the singular point $\xi = 0$: i) requires $v_2 = v(\xi_2) = 4\kappa$, ii) $v(0) = 0$, or using Eq. (B15)

$$\begin{aligned} \frac{1}{2\pi} \int_0^{4\kappa} dv \sqrt{\frac{4\kappa-v}{v}} &= \kappa \\ &= \frac{k}{\pi} \int_0^{\beta} d\varrho \frac{\sqrt{\beta^2 - \varrho^2}}{\sqrt{\varrho^2 + \Delta R^2}} \approx \frac{k}{4 \Delta R} \beta^2 \end{aligned} \quad (B17)$$

thus fixing κ to

*) We fix all appearing square roots in Eq. (B15) to be real and positive if their arguments are real and positive. Moreover, we require $|\arg v/4\kappa| < \pi$.

$$\kappa \approx \frac{k}{4\Delta R} \beta^2 \quad (|\beta| \ll \Delta R) \quad (B18)$$

and

$$v(\xi) \approx 4k\Delta R \sinh^2 \xi/2 \quad (B19)$$

both as expected from the limiting Eq. (B6).

Using Eqs. (B2), (B12), and (B15) it is now a simple matter to write down the "radial" functions $R_{on}^{(1)}$ and $R_{on}^{(3)}$ in terms of the uniform Whittaker approximants

$$M_{-i\kappa,0}(e^{-i\pi/2} v(\xi)) \text{ and } W_{-i\kappa,0}(e^{-i\pi/2} v(\xi)) \quad (B20)$$

respectively, with limit $|\xi| \ll 1$ given in Eq. (B7). Moreover, a complete set of simple uniform approximation formulae for the Whittaker functions has been derived by Taylor^{50,51}) in terms of the variable [cf. Eq. (B15)]

$$\xi_T(v) \equiv \frac{1}{2} \int_{4\kappa}^{v(\xi)} dv' \sqrt{\frac{v'-4\kappa}{v'}} = S_g(\xi, \beta); \quad g = \Delta R \sinh \xi \quad (B21)$$

$$\frac{W_{-i\kappa,0}(e^{-i\pi/2} v)}{\Gamma(1/2 - i\kappa)} = G(e^{-i\pi/2} \kappa, e^{-i\pi/2} v) \times \begin{cases} 2e^{-i\pi/4} \cosh(i\xi_T + i\frac{\pi}{4}); & -2\pi < \arg \xi_T < 0 \quad (B22a) \\ e^{i\xi_T}; & -\pi < \arg \xi_T < 2\pi \quad (B22b) \\ 2e^{i\pi/4} \cosh(i\xi_T - i\frac{\pi}{4}); & \pi < \arg \xi_T < 3\pi \quad (B22c) \\ e^{-i\xi_T + i\pi/2}; & 2\pi < \arg \xi_T < 5\pi \quad (B22d) \end{cases}$$

valid for sufficiently large v and/or κ , $|\arg v/4\kappa| < \pi$, with

$$G(-i\kappa, -iv) = \frac{e^{-i\kappa \log(-i\kappa/e)}}{\Gamma(1/2 - i\kappa)} \left[1 - \frac{4\kappa}{v}\right]^{-1/4} \approx \left[1 - \frac{4\kappa}{v}\right]^{-1/4} \quad (B23)$$

for $|\arg(-i\kappa)| < \pi$.

Using these results, we easily obtain the correct normalization of $R_{on}^{(3)}$ in terms of $W_{-i\kappa,0}(e^{-i\pi/2} v)$ for ξ , viz. $v \rightarrow \infty$ without a tedious matching procedure. The normalization of $R_{on}^{(1)}$ then follows from the Wronskians (A10) and⁵⁰)

$$\{W_{\mu,0}(z), M_{\mu,0}(z)\}_z = \frac{1}{\Gamma(1/2 - \mu)}. \quad (B24)$$

Finally, the following uniform approximation for the spheroidal partial waves results

$$f_\lambda(kR_T, kR_L) = \frac{2i R_{on}^{(1)}(k\Delta R, R_L/\Delta R)}{R_{on}^{(3)}(k\Delta R, R_L/\Delta R)} \approx -e^{-\pi\kappa} \frac{M_{-i\kappa,0}(e^{-i\pi/2} v_B)}{W_{-i\kappa,0}(e^{-i\pi/2} v_B)/\Gamma(1/2 - i\kappa)} \quad (B25)$$

with

$$\kappa \approx \frac{k}{4\Delta R} \beta^2 = \frac{(\lambda + 1/2)^2}{4k\Delta R} \quad \text{and} \quad v_B = v(\xi_B) \approx \frac{kR_T^2}{\Delta R} = \frac{1}{c} \quad (\text{B26})$$

for $|\beta| \ll \Delta R \propto k \rightarrow \infty$.

This uniform expression allows, together with Eqs. (B22), the investigation of the behaviour of $f_\lambda(kR_T, kR_L)$ in the *whole* λ -plane. Moreover, the results (31) and (33) from the complex multiple reflection method are easily verified using Eqs. (B22). For $|\beta| \gg \Delta R$ the approximation (B25) automatically approaches the spherical partial waves as it should be. Furthermore, let us emphasize the *scaling property* exhibited in Eq. (B25) for $|\beta| \ll \Delta R$:

$$f_\lambda(kR_T, kR_L) = \tilde{f} \left(\frac{(\lambda + 1/2)^2}{k\Delta R}, \frac{(kR_T)^2}{k\Delta R} \right). \quad (\text{B27})$$

The equation for the "Regge-like" poles in the complex λ -plane now takes on the form ($|\beta| \ll \Delta R$)

$$W_{-i\frac{k}{4\Delta R}\beta^2, 0} (e^{-i\pi/2} v_B) = 0. \quad (\text{B28})$$

Finally, let us demonstrate, how from this pole condition the simpler, semi-classical one, Eq. (36), is obtained using Eqs. (B22) for $v_B \approx kR_T^2/\Delta R = 1/c \gg 1$. The phase of $\xi_T(v) = S_\rho(R_T, \beta)$ in the λ viz. β domain of interest

$$0 < \arg(\lambda + 1/2) < \frac{\pi}{2}, \quad \left| \frac{\lambda + 1/2}{k} \right| = |\beta| \gtrsim R_T \quad (\text{B29})$$

crucially determines which of the approximations (B22) has to be used in Eq. (B28). Thus let us continue $S_\rho(R_T, \beta)$ from real $\beta < R_T$, where $\arg S_\rho = 0$, to the region (B29), remembering that near $\beta \approx R_T$

$$S_\rho(R_T, \beta) \propto \left(\frac{R_T^2}{\beta^2} - 1 \right)^{3/2} / \left(\frac{R_T}{\beta} \right)^2. \quad (\text{B30})$$

We choose to go around this branch point in the upper (R_T^2/β^2) -plane implying

$$\left(\frac{R_T}{\beta} \right)^2 - 1 \implies e^{i\pi} \left(1 - \left(\frac{R_T}{\beta} \right)^2 \right) \quad (\text{B31})$$

to end up in region (B29), i.e. $-\pi < \arg(R_T^2/\beta^2) < 0$. Hence, from Eq. (B30) in region (B29)

$$\frac{3\pi}{2} < \arg S_\rho(R_T, \beta) < \frac{5\pi}{2} \quad (\text{B32})$$

and Eq. (B22c) applies, verifying Eq. (36) of Section 3.2

$$\cosh \left(i S_\rho(R_T, \beta_m) - i \frac{\pi}{4} \right) = 0 \quad \text{or} \quad S_\rho \left(R_T, \frac{\alpha_m + 1/2}{k} \right) = (m - \frac{1}{4})\pi \quad (\text{B33})$$

$m = 1, 2, 3, \dots$

with $\arg S_\rho = 2\pi$ or $S_\rho > 0$.

APPENDIX C

In this Appendix we shortly outline the derivation of a uniform approximation of the spheroidal angular functions $S_{on(\lambda)}^{(1)}(k\Delta R, \cos \theta)$ (cf. Appendix A) for complex n viz. λ , $0 \leq \theta \leq \pi$, using exactly the same mapping techniques as in Appendix B. In the θ range of interest $0 \leq \theta \leq \pi$, the only relevant exceptional points are the singularities at $\theta = 0$ and $\theta = \pi$. For $0 < \arg(\lambda + 1/2) < \pi/2$ the turning points are unimportant here. Hence, we approximately map the spheroidal angular Eq. (19)

$$\left[\frac{d^2}{d\theta^2} + (k\Delta R \sin \theta)^2 + (\lambda + 1/2)^2 + \frac{1}{4 \sin^2 \theta} \right] \times \sqrt{\sin \theta} S_{on(\lambda)}^{(1)}(k\Delta R, \cos \theta) = 0 \quad (C1)$$

on the Legendre equation ($\Omega = \Omega(\theta)$)

$$\left[\frac{d^2}{d\Omega^2} + (n + 1/2)^2 + \frac{1}{4 \sin^2 \Omega} \right] \sqrt{\sin \Omega} P_n(\cos \Omega) = 0, \quad (C2)$$

which has the same singularities at $\Omega = 0$ and $\Omega = \pi$ as Eq. (C1). The analogue of the approximate mapping Eq. (B11) now reads

$$(n + 1/2) \Omega(\theta) = \int_0^\theta d\theta' \sqrt{(\lambda + 1/2)^2 + (k\Delta R \sin \theta')^2} \equiv S_\theta(\theta, 0), \quad (C3)$$

where the r.h.s. is just the angular part of the classical action and $\theta = 0$ has already been mapped on $\Omega = 0$.

The free parameter $n + 1/2$ is fixed by requiring that the second singular point at $\theta = \pi$ corresponds to $\Omega = \pi$, i.e.

$$n + 1/2 = \frac{2}{\pi} \int_0^{\pi/2} d\theta' \sqrt{(\lambda + 1/2)^2 + (k\Delta R \sin \theta')^2} \quad (C4)$$

Then we obtain as in Appendix B the uniform approximation for $|n| \propto k\Delta R \gg 1$, $0 \leq \theta \leq \pi$

$$S_{on(\lambda)}^{(1)}(k\Delta R, \cos \theta) \approx \left[\frac{\sin \Omega(\theta)}{\sin \theta} \frac{(n + 1/2)}{[(\lambda + 1/2)^2 + (k\Delta R \sin \theta)^2]^{1/2}} \right]^{1/2} P_{n(\lambda)}(\cos \Omega(\theta)) \quad (C5)$$

in terms of the Legendre functions, with $\Omega(\theta)$ given by Eq. (C3), and proper normalization

$$S_{on}^{(1)}(k\Delta R, 1) = P_n(1) = 1. \quad (C6)$$

Since clearly

$$\Omega(\pi - \theta) = \pi - \Omega(\theta) \quad (C7)$$

the eigenvalues λ_n , for which $S_{on}^{(1)}$ is continuous at $\theta = \pi$ and satisfies Eq. (A5) are determined by integer values of n in Eq. (C4) [remembering $P_n(-z) = (-1)^n P_n(z)$, $n = 0, 1, \dots$]. Thus, indeed, $n(\lambda)$ in Eq. (C5) equals the index $n(\lambda)$ of the angular functions $S_{on(\lambda)}^{(1)}$.

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FIGURE CAPTIONS

Figure 7 Test for an Orear type behaviour $\exp(-Ap_T)$ in $pp \rightarrow pp$ and $\pi^- p \rightarrow \eta n$ at intermediate p_T as predicted from the hadronic tunnelling amplitude. Data from Refs. 29,30 and 43.

Figure 8 $pp \rightarrow pp$ data at six energies: $p_{Lab} = 400$ GeV/c, FNAL³⁰⁾ and $\sqrt{s} = 23.4, 30.5, 44.6, 52.8, 62.1$ GeV, ISR²⁹⁾ demonstrating energy-independence of $d\sigma/dt$ for $p_T \geq 1.5$ GeV/c.

8a) The plot of $\log(p_T d\sigma/dt)$ versus p_T exhibits the Orear type behaviour $\exp(-Ap_T)$ for intermediate p_T ;

8b) the plot of $\log(d\sigma/dt)$ versus $\log(p_T)$ indicates a transition to a power law behaviour $(p_T^2)^{-N}$, $N = 8-10$, for large $p_T \geq 2$ GeV/c

with a considerable overlap region between the two behaviours. The straight lines are to guide the eye.

Figure 9 Data⁴²⁾ of elastic scattering of protons off heavy nuclei (with atomic number A) at $E_{Lab} = 1.98$ GeV versus $kR_A \theta$ are shown ($R_A = 1.13$ fermi $A^{1/3}$ = radius of the nucleus). The oscillations follow a diffraction pattern with minima at the positions of the zeros of $J_1(kR_A \theta)/(kR_A \theta)$. The forward peak is -- on the average -- followed by an exponential decrease which exhibits strong antishrinkage with increasing kR_A .

Figure 10 Same data as in Figure 9 plotted versus $(kR_A)^{1/3} \theta$. They exhibit -- on the average -- an exponential behaviour $\exp(-B(kR_A)^{1/3} \theta)$ with *universal* slope B as predicted by the "nuclear" tunnelling amplitude (corresponding to a spherical caustic with radius R_A). The straight parallel lines are to guide the eye.

Figure 11 The same data as in Figure 9 are shown. The parameter-free prediction of the "nuclear" tunnelling amplitude (solid line) is compared with the average behaviour of all data with respect to shape and normalization.

Figure 12 Semi-quantitative comparison of the prediction (solid line) from the hadronic tunnelling amplitude (corresponding to a fat string-like caustic with $R_T \approx 1$ fermi and $R_L \propto \sqrt{s}$) with ISR data²⁹⁾ at $\sqrt{s} = 52.8$ and 62.1 GeV and the FNAL data³⁰⁾ at $p_{Lab} = 400$ GeV/c for $pp \rightarrow pp$. The amplitude exhibits an Orear type behaviour for intermediate p_T and turns automatically into a power law at large p_T .

FIG.7

FIG.9

FIG.10

FIG.11

FIG.12