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A SIMPLE FORMALISM FOR THE BPS MONOPOLE

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A B S T R A C T

A simple formalism for the BPS monopole is obtained by generalizing the ADHM construction of multi-instantons to a Hilbert space. Both the potential itself and the Green's functions for different isospin can be obtained with very little effort from the instanton formulae.

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Manton ¹⁾, Adler ²⁾, and Rossi ³⁾ have shown that the Bogomolny-Prasad-Sommerfield (BPS) monopole ^{4),5)} can be treated as a limiting case of a multi-instanton. Their description, however, is very inconvenient.

For the gauge group $SU(2)$, the ADHM construction ⁶⁾ describes configurations with instanton number k by a normed vector $v(x)$ in a $k+1$ dimensional quaternionic vector space. For the monopole, one needs infinite k . As we shall see, a good choice for the vector space is the complex Hilbert space $\mathcal{L}^2(-\frac{1}{2}, \frac{1}{2})$, tensored with the quaternions.

Let us denote a scalar product over the $\mathcal{L}^2$ part by

$$\langle w | v \rangle = \int_{-\frac{1}{2}}^{\frac{1}{2}} w^+(z) v(z) dz, \quad (1)$$

where the conjugation applies both to complex numbers and to quaternions. For the ADHM construction we use the notation of Ref. 7). The potential takes the form

$$A_\mu = \langle v(x) | i \partial_\mu v(x) \rangle. \quad (2)$$

Here

$$X = x^\mu q_\mu \quad (3)$$

is the position in quaternionic notation. The q_μ are orthogonal unit quaternions with $q_0 = 1$.

We also need the spatial part

$$X' = X - X^0 \quad (4)$$

and its absolute value

$$r_x = |X'|. \quad (5)$$

The conditions on $|v(x)\rangle$ are

$$\langle v(x) | v(x) \rangle = 1 \quad (6)$$

and

$$\Delta^+(x) |v(x)\rangle = 0. \quad (7)$$

Here $\Delta(x)$ is an operator of the form

$$\Delta(x) = A + Bx \quad (8)$$

with a one-dimensional co-kernel. Moreover, $\Delta^+(x)\Delta(x)$ must be real and invertible.

For a BPS monopole at the origin we just have to put

$$\Delta^+(x) = i\partial_z + cx^+. \quad (9)$$

For convenience, we set the scale parameter c equal to 1. The equation

$$\Delta^+(x)\Delta(x)F(x, z, z') = \delta(z - z') \quad (10)$$

is satisfied by

$$F(x, z, z') = -\frac{1}{2r_x} \exp(ix^0(z - z')) \times \\ \times \left(\sinh r_x |z - z'| + \coth r_x/2 \sinh r_x z \sinh r_x z' - \tanh r_x/2 \cosh r_x z \cosh r_x z' \right). \quad (11)$$

Note that $\Delta(x)$ has a derivative going to the left. Partial integration leaves boundary terms, such that Eq. (10) implies

$$F(x, \pm \frac{1}{2}, z') = 0. \quad (12)$$

Up to gauge transformations, Eqs. (6) and (7) have the unique solution :

$$v(x, z) = N(x) \exp(ix^+ z), \quad (13)$$

with

$$N(x) = \langle \exp(ix^+ z) | \exp(ix^+ z) \rangle^{-1/2} = (r_x^{-1} \sinh r_x)^{-1/2}. \quad (14)$$

In Eq. (2) the x^0 dependence of the potential obviously drops out, and we indeed obtain the well-known result of BPS.

Let us now calculate the scalar Green's functions, from which spinor and vector Green's functions can be easily obtained. According to Ref. 7) the scalar isospinor Green's function is given by

$$G(x, y) = \frac{\langle v(x) | v(y) \rangle}{4\pi^2 |x - y|^2} \quad (15)$$

for arbitrary multi-instantons. For monopoles, one is more interested in the Fourier transform

$$g(x', y', \omega) = \int G(x, y) \exp(i\omega y^0) dy^0, \quad (16)$$

where the dummy co-ordinate is integrated out. We obtain for example directly the result ^{2), 3)}:

$$g(x', y', \sigma) = \frac{N(x)N(y)}{2\pi d} \int_0^{1/2} e^{-dz} \left(\cosh r_x z \cosh r_y z - \frac{x' y'}{r_x r_y} \sinh r_x z \sinh r_y z \right) dz, \quad (17)$$

where

$$d = |x' - y'|. \quad (18)$$

For the SU(2) tensor product $\frac{1}{2} \times \frac{1}{2}$, we represent the quaternions by 2x2 matrices, and write down their indices explicitly. The Green's function is given by ⁸⁾

$$G_{abcd}(x, y) = \frac{\langle v(x) | v(y) \rangle_{ac} \langle v(y) | v(x) \rangle_{db}}{4\pi^2 |x - y|^2} + \frac{1}{4\pi^2} C_{abcd}, \quad (19)$$

with

$$C_{abcd} = \int_{-1/2}^{1/2} (v^+(x, z_1) v(x, z_2))_{ab} (v^+(y, z_3) v(y, z_4))_{dc} M(z_1, z_2, z_3, z_4) dz_1 dz_2 dz_3 dz_4, \quad (20)$$

In general, M is defined by

$$\begin{aligned} & \frac{1}{2} \text{tr} \left[(A^+ A)_{il} (B^+ B)_{mj} + (B^+ B)_{il} (A^+ A)_{mj} - 2(A^+ B)_{il} (B^+ A)_{mj} \right] M_{rsij} = \\ & = \delta_{rl} \delta_{sm}. \end{aligned} \quad (21)$$

The trace is over a 2×2 matrix. In our case, everything is proportional to the unit matrix, and we obtain

$$\begin{aligned}
 & - \left(\frac{\partial}{\partial z_3} + \frac{\partial}{\partial z_4} \right)^2 M(z_1, z_2, z_3, z_4) = \\
 & = \delta(z_1 - z_3) \delta(z_2 - z_4) = 2 \delta(z_1 - z_2 - z_3 + z_4) \delta(z_1 + z_2 - z_3 - z_4), \quad (22)
 \end{aligned}$$

and

$$M(z_1, z_2, \pm \frac{1}{2}, z_4) = M(z_1, z_2, z_3, \pm \frac{1}{2}) = 0. \quad (23)$$

This yields

$$\begin{aligned}
 M(z_1, z_2, z_3, z_4) = & -\frac{1}{4} \delta(z_1 - z_2 - z_3 + z_4) \times \\
 & \times \left(|z_1 + z_2 - z_3 - z_4| - 1 + |z_1 - z_2| + \frac{(z_1 + z_2)(z_3 + z_4)}{1 - |z_1 - z_2|} \right). \quad (24)
 \end{aligned}$$

The delta function guarantees that $G_{abcd}(x, y)$ is invariant under a common translation of the zero components of x and y .

As before, we are primarily interested in the Fourier transform $g_{abcd}(x', y', \omega)$ of $G_{abcd}(x, y)$. The transform of the integrand of G_{abcd} includes the factor

$$\delta(z_4 - z_3 + \omega) \delta(z_1 - z_2 - z_3 + z_4) = \delta(z_4 - z_3 + \omega) \delta(z_1 - z_2 - \omega), \quad (25)$$

such that effectively only two integrations remain, which can easily be carried out. For $\omega=0$ they reproduce Adler's formula ⁹⁾ for $g_{abcd}(x', y', 0)$, with much less effort.

One obviously should try to extend this formalism to different gauge groups, and to multi-monopoles.

REFERENCES

- 1) N.S. Manton, Nuclear Phys. B135 (1978) 319.
- 2) S. Adler, Phys.Rev. D18 (1978) 411.
- 3) P. Rossi, Nuclear Phys. B149 (1979) 170.
- 4) M.K. Prasad and C.M. Sommerfield, Phys.Rev.Letters 35 (1975) 760.
- 5) E.B. Bogomolny, Sov.J.Nucl.Phys. 24 (1976) 861.
- 6) M.F. Atiyah, N.J. Hitchin, V.G. Drinfeld and Yu.I. Manin, Phys.Letters 65A (1978) 185.
- 7) E.F. Corrigan, D.B. Fairlie, P. Goddard and S. Templeton, Nuclear Phys. B140 (1978) 31.
- 8) E.F. Corrigan, P. Goddard and S. Templeton, Nuclear Phys. B151 (1979) 93.
- 9) S. Adler, Phys.Rev. D19 (1979) 2997.