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NON-ABELIAN ELECTRIC AND MAGNETIC FLUX IN UNIFIED SU(5) GAUGE THEORY

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ABSTRACT

We study the long-distance behaviour of pure unified SU(5) gauge theory in the limit when the electroweak subgroup is unbroken. We show that the symmetry breaking pattern $SU(5) \rightarrow SU(3)_c \times SU(2) \times U(1)_Y$, with $SU(3)_c$ and $SU(2) \times U(1)_Y$ realized, respectively, in confining and Coulombic phases, is a possible dynamical phase of the SU(5) theory. The proof relies in showing that the duality equation of 't Hooft, relating the electric and magnetic flux, is exactly satisfied for the above symmetry breaking pattern. The infra-red structure of SU(5), broken down to $SU(3)_c \times SU(2) \times U(1)_Y$, is not self-dual.

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Unified gauge theories of strong, weak and electromagnetic interactions^{1),2)} must undergo spontaneous symmetry breaking in order to explain the observed disparity in the strength of the three interactions. Until now, the only known method of explicitly implementing the desired symmetry breaking pattern has been through the use of elementary Higgs fields. Unfortunately, the inclusion of such field in a gauge theory increases the number of free dimensionless couplings, something that should preferably be avoided in the context of a unified theory.

Perhaps the most attractive way of implementing symmetry breaking is via a dynamical Higgs mechanism. Although this goal still eludes us, techniques have recently become available which enable one to investigate more quantitatively, at least, the consistency of the hypothesis of the dynamical realization of a pure gauge theory in a particular phase. If a gauge theory based on a gauge group G breaks dynamically down to some subgroup H , which is realized in some particular phase, then information about this symmetry breaking gets coded in an exact duality equation found by 't Hooft³⁾. The equation relates the properties of gauge invariant non-Abelian electric and magnetic flux.

The main purpose of this note is to show that the symmetry breaking pattern $SU(5)^*) \rightarrow SU(3)_C \times SU(2) \times U(1)_Y$ with $SU(3)_C$ and $SU(2) \times U(1)_Y$ realized in confining and Coulombic phases, respectively, is a possible dynamical phase of the pure (i.e., without fermions) $SU(5)$ gauge theory. The duality equation of 't Hooft is exactly satisfied in this phase. The result is interesting because it puts the problem of dynamical realization of the $SU(5)$ gauge theory in the above phase on the same level as the problem of confinement in QCD. As a by-product of our investigation we find that the infra-red structure of pure $SU(5)$, broken down to $SU(3)_C \times SU(2) \times U(1)_Y$, is not self-dual.

Following Ref. 3), we employ the $A_0 = 0$ gauge and consider the $SU(5)$ gauge system at a fixed time $t = 0$ enclosed in a rectangular three-dimensional box with sides a_i ($i = 1, 2, 3$). The following generalized boundary conditions are imposed:

$$A_i(x_1 = a_1, x_2, x_3) = \Omega_i(x_2, x_3) A_i(x_1 = 0, x_2, x_3) \quad (1)$$

together with the conditions obtained by cyclic permutations of the indices 1, 2, 3. Here $\Omega(x) A_i(x) = \Omega^{-1}(x) A_i(x) \Omega(x) + (ig)^{-1} \Omega^{-1}(x) \partial_i \Omega(x)$.

*) We take the $SU(5)$ gauge model²⁾ (without quarks and leptons) since it provides the simplest example of a unified gauge theory. Our considerations can probably be extended to other models.

Consistency of the boundary conditions at the corners of the box implies (say for $x_1 = 0$ plane)

$$\Omega_3^{-1}(0,0) \Omega_2^{-1}(a_3,0) \Omega_3(0,a_2) \Omega_2(0,0) = e^{\frac{2\pi i}{5} m_1} \mathbb{1}_5 \mathbb{Z}_5 \quad (2)$$

with corresponding relations for the $x_2 = 0$ and $x_3 = 0$ planes. $\mathbb{Z}_5$ is the centre of $SU(5)$ and m_i ($i = 1,2,3$) are integers taking values between 0 and 4. The Hilbert space of the quantum states of the $SU(5)$ gauge system is spanned by the vectors $|A_i(\underline{x})\rangle$, where $|A_i(\underline{x})\rangle$ is identified with $|\Omega(\underline{x}) A_i(\underline{x})\rangle$ if $\Omega(\underline{x})$ is a continuous, homotopically trivial gauge transformation.

The 't Hooft disorder parameter $B(C_i)$ (C_i is some curve in the box which starts from the plane $x_i = 0$ and ends at the plane $x_i = a_i$) creates a bare magnetic flux line along C_i , carrying one unit of magnetic flux. The integer m_i represents the number of magnetic flux units in the box along the x_i direction. The order parameter $A(C_i) = (1/5) \text{Tr} \mathcal{P} \exp(i \oint_{C_i} A_i(\underline{x}) d\underline{x})$ creates a bare electric flux line along C_i which carries one unit of electric flux. The quantum states of the system $|\psi_{\underline{e}}\rangle$ ($\underline{e} = (e_1, e_2, e_3)$, $e_i \in \mathbb{Z}$, $0 \leq e_i \leq 4$) containing e_i units of electric flux along the x_i direction are defined by

$$\Omega_{\underline{k}}(\underline{x}) |\psi_{\underline{e}}\rangle = e^{\frac{2\pi i}{5} (\underline{e} \cdot \underline{k})} |\psi_{\underline{e}}\rangle \quad (3)$$

where $\Omega_{\underline{k}}(\underline{x})$ ($\underline{k} = (k_1, k_2, k_3)$, $k_i \in \mathbb{Z}$) is a continuous, but not homotopically trivial, gauge transformation satisfying

$$\Omega_{\underline{k}}(x_i = a_i) = e^{\frac{2\pi i}{5} k_i} \Omega_{\underline{k}}(x_i = 0), \quad i = 1, 2, 3. \quad (4)$$

In order to show that a pure $SU(5)$ gauge theory can dynamically break down to $SU(3)_C \times SU(2) \times U(1)_Y$ we must prove that the following duality equation³⁾ is satisfied

$$\begin{aligned} & \exp \left[-\beta F_M(m_1, m_2, 0; \alpha_1, \alpha_2, \alpha_3, \beta) \right] \\ &= \frac{1}{5} \sum_{e_1, e_2=0}^4 \exp \left[\frac{2\pi i}{5} (e_1 m_1 + e_2 m_2) - \alpha_3 F_E(e_1, e_2, 0; \alpha_2, \alpha_1, \beta, \alpha_3) \right] \end{aligned} \quad (5)$$

with $\beta, a_3 \gg a_1, a_2$. Here $F_M(m_1, m_2, 0; a_1, a_2, a_3, \beta)$ ($F_E(e_1, e_2, 0; a_2, a_1, \beta, a_3)$) is the free energy of the $SU(5)$ system with $m_1, m_2, 0$ ($e_1, e_2, 0$) units of magnetic (electric) flux along the x_1, x_2, x_3 directions, enclosed in a box with sides a_1, a_2, a_3 (a_2, a_1, β) at a temperature β^{-1} (a_3^{-1}). To see whether (5) holds, we turn to the study of the electric and magnetic properties of $SU(5)$.

ELECTRIC FLUX

The massive gauge particles of $SU(5)$ give a negligible direct contribution to the partition function $Z_E(e, 0, 0; a_2, a_1, \beta, a_3) = \exp[-a_3 F_E(e, 0, 0; a_2, a_1, \beta, a_3)]$ because of their very small Boltzmann factors. Moreover, since the $SU(3)_c$ electric flux is squeezed, the contribution to Z_E of the $SU(5)$ electric fluxes which contain a non-zero $SU(3)_c$ component is also negligible. Therefore, the dominant contribution to Z_E comes from the electric fluxes which contain only $SU(2) \times U(1)_Y$ components. Later on we will see that the massive gauge particles also contribute to Z_E , but indirectly, by creating $SU(2) \times U(1)_Y$ electric fields.

In order to decompose the $SU(5)$ fluxes with respect to $SU(2) \times U(1)_Y$, we recall the definitions of the $SU(3)_c$, $SU(2)$ and $U(1)_Y$ subgroups of $SU(5)$:

$$\left(\begin{array}{c|c} U & 0 \\ \hline 0 & 1_3 \end{array} \right) \in SU(3), \quad \left(\begin{array}{c|c} 1_3 & 0 \\ \hline 0 & V \end{array} \right) \in SU(2), \quad \left(\begin{array}{c|c} e^{2i\phi} 1_3 & 0 \\ \hline 0 & e^{-3i\phi} 1_2 \end{array} \right) \in U(1)_Y, \quad (6)$$

$$0 \leq \phi \leq 2\pi.$$

An $SU(5)$ electric flux of e unites ($0 \leq e \leq 4$) corresponds to the element $e^{2\pi i e/5} 1_5$ of $\mathbb{Z}_5$, (see Eq. (3)). The equation

$$e^{2\pi i e/5} 1_5 = \left(\begin{array}{c|c} e^{2i\phi} 1_3 & 0 \\ \hline 0 & e^{-3i\phi} V \end{array} \right) \quad (7)$$

has the following two solutions^{*)}

$$\begin{aligned} \text{(i)} \quad \phi &= \frac{2\pi e}{10} \pmod{2\pi}, \quad V = e^{\frac{2\pi i}{2} (e)_2} 1_2 \in \mathbb{Z}_2 \\ \text{(ii)} \quad \phi &= 2\pi \left(\frac{e}{10} + \frac{1}{2} \right) \pmod{2\pi}, \quad V = e^{\frac{2\pi i}{2} (e+1)_2} 1_2 \in \mathbb{Z}_2 \end{aligned} \quad (8)$$

^{*)} Here $(e)_2$ is defined by $(e)_2 = e + 2k$, where $k \in \mathbb{Z}$ and $0 \leq (e)_2 \leq 1$.

Consequently, a given SU(5) electric flux of e units consists of an SU(2) electric flux of $(e)_2$ (resp. $(e+1)_2$) units and an $U(1)_Y$ electric flux equal to $e/10$ (resp. $e/10 + 1/2$)*).

Although the massive gauge particles give negligible direct contribution to Z_E , they nevertheless contribute indirectly in the following way. A pair of such particles can be created from the vacuum by thermal fluctuations. Then these particles can separate and reach the walls at $x_1 = 0$ and $x_1 = a_2$ respectively, where they can annihilate. In the following, whenever we refer to a pair creation and its subsequent annihilation, we will mean it in this sense. This process changes the electric field configuration in the box. To study this effect, we recall that the gauge bosons belong to the 24 dimensional adjoint representation of SU(5) which decomposes under the breaking $SU(5) \rightarrow SU(3)_C \times SU(2) \times U(1)_Y$ as follows:

$$24 \rightarrow (8,1)_{Y=0} + (3,2)_{Y=\frac{\sqrt{15}}{6}} + (\bar{3},\bar{2})_{Y=-\frac{\sqrt{15}}{6}} + (1,3)_{Y=0} + (1,1)_{Y=0}. \quad (9)$$

The first entry in the parentheses above refers to $SU(3)_C$, the second to $SU(2)$ and the subscript to the $U(1)_Y$ charge. The generators λ_a ($a = 1, \dots, 24$) of SU(5) are normalized according to $\text{tr}(\lambda_a \lambda_b) = 1/2 \delta_{ab}$. The massive gauge bosons (X bosons) belong to the $(3,2)_{Y=\sqrt{15}/6}$ and $(\bar{3},\bar{2})_{Y=-\sqrt{15}/6}$ representations of $SU(3)_C \times SU(2) \times U(1)_Y$. We denote them by $X_{\alpha i}$ and $\bar{X}^{\alpha i}$ respectively ($\alpha = 1, 2, 3$; $i = 1, 2$). A pair creation and subsequent annihilation of these particles produces, among other things, one unit of $SU(3)_C$ electric flux. Therefore, such a process gives a negligible contribution to Z_E . It is obvious that the relevant objects to consider are $SU(3)_C$ singlets made out of these bosons. They are: i) The "meson states" $\bar{X}^{\alpha i} X_{\alpha j}$ with $Y = 0$, which decompose under SU(2) into a singlet and a triplet X_{ij} ; and ii) The "baryon states" $\epsilon_{\alpha\beta\gamma} X_{\alpha i} X_{\beta j} X_{\gamma k}$ with $Y = \sqrt{15}/2$, which decompose under SU(2) into two doublets ψ_i , and a quartet.

A $\bar{X}X$ creation and subsequent annihilation produces two units of SU(2) electric flux. This process changes neither the number of units of SU(5) flux, e ($0 \leq e \leq 4$) nor the number of units of SU(2) flux, $(e)_2$ (resp. $(e+1)_2$). However, it changes the SU(2) electric field configuration in the box. That is, it changes the electric energy of the system. By repeating this process an arbitrary number of times, one produces all the energy eigenstates of an SU(2) electric flux with $(e)_2$ (resp. $(e+1)_2$) units. The energy eigenvalues (remembering that the SU(2) is in the Coulombic phase) are given by

*) The unit of $U(1)_Y$ flux is taken to correspond to $\phi = 2\pi$.

$$E_e^{SU(2)} = \frac{g^2 \alpha_2}{2 \alpha_1 \beta} \left(\ell + \frac{(e)_2}{2} \right)^2 \quad \left(\text{resp. } E_e^{SU(2)} = \frac{g^2 \alpha_2}{2 \alpha_1 \beta} \left(\ell + \frac{(e+1)_2}{2} \right)^2 \right), \quad (10)$$

$$\ell \in \mathbb{Z}.$$

A $\bar{\theta}\theta$ creation and subsequent annihilation ($\theta = \varepsilon_{ij} \psi_i \psi_j$, see above) produces one unit of $U(1)_Y$ electric flux, but otherwise leaves unchanged the units of $SU(5)$ flux e and the $SU(2)$ field configuration. It changes the $U(1)_Y$ electric field configuration by an amount which corresponds to one unit of $U(1)_Y$ flux. Consequently, the energy spectrum of $e/10 \pmod{1}$ (resp. $e/10 + 1/2 \pmod{1}$) units of $U(1)_Y$ electric flux is given by

$$E_k^{U(1)_Y} = \frac{g^2 \alpha_2}{2 \alpha_1 \beta} 15 \left(k + \frac{e}{10} \right)^2 \quad \left(\text{resp. } E_k^{U(1)_Y} = \frac{g^2 \alpha_2}{2 \alpha_1 \beta} 15 \left(k + \frac{e}{10} + \frac{1}{2} \right)^2 \right), \quad (11)$$

$$k \in \mathbb{Z}.$$

Similarly, a $\bar{\psi}\psi$ creation and subsequent annihilation process produces one unit of $SU(2)$ flux and half a unit of $U(1)_Y$ flux. Thus, without affecting the $SU(5)$ flux e , it changes both the $SU(2)$ and $U(1)_Y$ electric field configurations.

This process applied on an energy eigenstate of the system with $(e)_2$ units of $SU(2)$ flux and $e/10 \pmod{1}$ units of $U(1)_Y$ flux produces an energy eigenstate with $(e+1)_2$ units of $SU(2)$ flux, and $e/10 + 1/2 \pmod{1}$ units of $U(1)_Y$ flux.

It follows from the above analysis that the energy spectrum of the system with e units of $SU(5)$ electric flux is given by

$$E_{k,\ell}^{(1)} = \frac{g^2 \alpha_2}{2 \alpha_1 \beta} \left[15 \left(k + \frac{e}{10} \right)^2 + \left(\ell + \frac{(e)_2}{2} \right)^2 \right] \quad (12)$$

and

$$E_{k,\ell}^{(2)} = \frac{g^2 \alpha_2}{2 \alpha_1 \beta} \left[15 \left(k + \frac{e}{10} + \frac{1}{2} \right)^2 + \left(\ell + \frac{(e+1)_2}{2} \right)^2 \right],$$

$k, \ell \in \mathbb{Z}$, which corresponds to the two possible ways of sharing the $SU(5)$ electric flux within $SU(2) \times U(1)_Y$. Thus, the partition function Z_E is equal to

$$e^{-\alpha_3 F_E(e,0,0; \alpha_2, \alpha_1, \beta, \alpha_3)} = \sum_{k, \ell=-\infty}^{\infty} \left(e^{-\alpha_3 E_{k, \ell}^{(1)}} + e^{-\alpha_3 E_{k, \ell}^{(2)}} \right). \quad (13)$$

MAGNETIC FLUX

We turn now to the computation of the partition function $Z_M(m, 0, 0; a_1, a_2, a_3, \beta) = \exp [-\beta F_M(m, 0, 0; a_1, a_2, a_3, \beta)]$ of an $SU(5)$ gauge system at a temperature β^{-1} with m ($0 \leq m \leq 4$) units of $SU(5)$ magnetic flux along the x_1 direction. In contrast to the electric case, an $SU(5)$ magnetic flux can contain an $SU(3)_C$ component, since the colour magnetic fields are screened at the quantum level. Thus, such a component carries negligible energy.

Now we must decompose an $SU(5)$ magnetic flux of m units with respect to the whole unbroken subgroup $SU(3)_C \times SU(2) \times U(1)_Y$. Such a flux corresponds to $e^{2\pi i m/5} 1_5 \in \mathbb{Z}_5$ (see Eq. (2)). The equation

$$e^{\frac{2\pi i}{5} m} 1_5 = \left(\begin{array}{c|c} U e^{2i\phi} & 0 \\ \hline 0 & V e^{-3i\phi} \end{array} \right), \quad U = e^{\frac{2\pi i}{3} \alpha} 1_3, \quad (14)$$

has the following solutions:

$$\begin{aligned} \text{i) } \phi &= -\frac{2\pi m}{15} \pmod{\frac{2\pi}{3}}, \quad V = 1_2, \quad \alpha = (m)_3 \text{ or } (m+1)_3 \text{ or } (m+2)_3 \\ \text{ii) } \phi &= -\frac{2\pi m}{15} + \frac{2\pi}{6} \pmod{\frac{2\pi}{3}}, \quad V = -1_2, \quad \alpha = (m)_3 \text{ or } (m+1)_3 \text{ or } (m+2)_3. \end{aligned} \quad (15)$$

Hence, an $SU(5)$ magnetic flux of m units consists of an $SU(3)_C$ flux with $(m)_3$ or $(m+1)_3$ or $(m+2)_3$ units, an $SU(2)$ flux with 0 (resp. 1) units and $-m/15 \pmod{1/3}$ (resp. $-m/15 + 1/6 \pmod{1/3}$) units of $U(1)_Y$ flux.

The $SU(5)$ theory contains massive magnetic monopoles. These particles do not directly contribute to Z_M , but they give an indirect contribution due to pair creation and subsequent annihilation, just like the X bosons in the electric case. The $SU(5)$ monopoles could carry simultaneously $SU(3)_C$, $SU(2)$ and $U(1)_Y$ magnetic charges. The explicit values of the $SU(3)_C$ magnetic charges are of no interest since the $SU(3)_C$ magnetic fields are screened at the quantum level. Thus, we may write the quantization condition⁴⁾ of the magnetic charges:

$$\exp(igB) = h \in \mathbb{Z}_3 \subset SU(3)_c, \quad (16)$$

where $B = xt_3 + yY$, with

$$t_3 = \frac{1}{2} \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 & -1 \end{pmatrix}$$

and

$$Y = \frac{1}{2\sqrt{15}} \begin{pmatrix} 2 & & \\ & 2 & \\ & & 2 & -3 & -3 \end{pmatrix}.$$

Here x and y are the $SU(2)$ and $U(1)_Y$ magnetic charges respectively. The relevant solutions of Eq. (16) are:

$$x=0, \quad |y| = \frac{4\pi}{g} \sqrt{15} \frac{1}{3}; \quad (17)$$

$$|x| = \frac{4\pi}{g} \cdot \frac{1}{2}, \quad |y| = \frac{4\pi}{g} \sqrt{15} \frac{1}{6}.$$

Using the magnetic monopoles which carry these charges in a way similar to X bosons in the electric case, we can construct the energy spectrum of the $SU(5)$ system with m units of magnetic flux. This gives rise to the following expression for the partition function Z_M

$$e^{-\beta F_M(m, 0, 0; a_1, a_2, a_3, \beta)} = \sum_{k, l=-\infty}^{\infty} \left\{ e^{-\frac{8\pi^2 \beta a_1}{g^2 a_2 a_3} \left[15 \left(\frac{k}{3} + \frac{m}{15} \right)^2 + l^2 \right]} + e^{-\frac{8\pi^2 \beta a_1}{g^2 a_2 a_3} \left[15 \left(\frac{k}{3} + \frac{m}{15} + \frac{1}{6} \right)^2 + \left(l + \frac{1}{2} \right)^2 \right]} \right\}. \quad (18)$$

DUALITY EQUATION

We can now proceed to check the duality equation (5). We first notice that, since the long-distance properties of the $SU(5)$ system depend essentially only on the Coulombic fields, the following factorization holds:

$$F_E(e_1, e_2, 0; a_2, a_1, \beta, a_3) = F_E(e_1, 0, 0; a_2, a_1, \beta, a_3) + F_E(0, e_2, 0; a_2, a_1, \beta, a_3), \quad (19)$$

$$F_M(m_1, m_2, 0; a_1, a_2, a_3, \beta) = F_M(m_1, 0, 0; a_1, a_2, a_3, \beta) + F_M(0, m_2, 0; a_1, a_2, a_3, \beta).$$

Thus, Eq. (5) reduces to

$$\exp[-\beta F_M(m, 0, 0; a_1, a_2, a_3, \beta)] = \frac{1}{\sqrt{5}} \sum_{e=0}^4 \exp\left[\frac{2\pi i}{5} em - a_3 F_E(e, 0, 0; a_2, a_1, \beta, a_3)\right], \quad (20)$$

where $\exp(-\beta F_M)$ and $\exp(-a_3 F_E)$ are given by Eqs (18) and (13) respectively.

One can now express the right-hand side of Eq. (13) in terms of the $\mathcal{V}_3$ function

$$\mathcal{V}_3(\nu, \kappa) = \sum_{\ell=-\infty}^{\infty} e^{-\pi \kappa \ell^2 + 2\pi \ell \nu i} \quad (21)$$

Using the following property of this function

$$\mathcal{V}_3(\nu, \kappa) = \kappa^{-\frac{1}{2}} e^{-\frac{\pi \nu^2}{\kappa}} \mathcal{V}_3\left(\frac{\nu}{\kappa}, \frac{1}{\kappa}\right) \quad (22)$$

and performing the sum over e in Eq. (20), one can easily check that the duality equation (20) is satisfied up to an irrelevant normalization constant. Thus, dynamical symmetry breaking of $SU(5)$ down to $SU(3)_C \times SU(2) \times U(1)_Y$ is an allowed phase of the $SU(5)$ gauge theory.

It is worth noticing that, in contrast to the Georgi-Glashow phase of the $SU(2)$ gauge theory studied in Ref. 3), the phase of the $SU(5)$ system considered here satisfies the duality equation in a non-self-dual way. This means that $\exp[-\beta F_E(e, 0, 0; a_1, \beta)]$ does not reduce to $\exp[-\beta F_M(m, 0, 0; a_1, \beta)]$ under the replacement $g \rightarrow 4\pi/g$ and vice versa.

In the above considerations we did not discuss the second step of the symmetry breaking: $SU(2) \times U(1)_Y \rightarrow U(1)_{e.m.}$. This breaking is normally achieved by the

introduction of Higgs fields which transform non-trivially under the centre of $SU(5)$ ⁵⁾. Therefore, it cannot occur dynamically within the pure $SU(5)$ gauge theory. However, it may perhaps arise dynamically when fermions are included in the theory.

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