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EXACT RESULTS ON POTENTIAL MODELS FOR QUARKONIUM SYSTEMS

H. Grosse*)

Institute of Theoretical Physics,
University of Vienna, Austria

A. Martin

CERN, Geneva, Switzerland

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1. INTRODUCTION

As is well known, the explanation of the J/ψ and ψ' particles as states of a bound system of a charmed quark and its antiquark has been a great success, which seems to be duplicated by the T and T' states. Shortly after the first experimental discovery of ψ and ψ' [1,2], a few groups [3,4] suggested a non-relativistic potential treatment, interpreted ψ and ψ' as ground state and first excited triplet state with quantum numbers $1^{--} (J^{PC})$, and predicted the existence of three P states between them and in addition a D state around 3.77 GeV. Furthermore, the decay widths have been estimated. Remarkably, nearly all predictions were correct [5,6], the only mystery being the singlet states and $M1$ transitions. Concerning the singlet state let us remark that very recently measurements at SLAC [7] have neither confirmed the state at 2830 nor the states at 3455 and 3600. The existence of any of these three states seems, according to that experiment, to be rather questionable. On the other hand, singlet states close to the triplet states could not be resolved by experiments, since direct γ transitions between triplet and singlet states have a low rate because of small phase space and low-energy gammas are hard to detect. In addition, if spin-dependent forces are weak, the orthogonality of the space wave functions suppresses the transition between the ψ' and the singlet partner of the ψ .

Theoretically the $c\bar{c}$ system has been attacked in essentially two ways. On the one hand people tried to extract information from QCD, on the other hand people fixed a potential adjusting a few parameters and tried to predict other quantities. From QCD one gets the idea of a Coulomb-like potential near the origin modified by a running coupling constant; from lattice gauge models one gets a hint that the effective potential might grow at large distances like the distance between the quarks.

Beside these two different directions of investigations we started two years ago a more general approach than fixing definitely a potential and tried to find out general properties of classes of reasonable potentials. In that way one is able to find out rough features, like convexity properties or growth properties

for the potential, which are allowed or forbidden by the data. Clearly there are a few very good reviews available dealing with quarkonium; for a summary of all three approaches see Jackson et al. [8]. The approaches dealing with non-relativistic potentials are summarized elsewhere [9-12]. For attempts to make detailed calculations in the well-known standard model also including hidden charm channels, see the review of K. Gottfried [13]. A summary of the more general phenomenological and theoretical investigations concerning charmonium and higher quarkonium states is given by Hara [14].

To start with let us repeat the "data" [15]: essentially the spectrum and transition probabilities for the charmonium and the bottomonium system so far as they are known today (figs. 1 to 4). In this review we will always assume that a non-relativistic description of these systems is justified. In this respect we are considering an idealized situation. We also disregard spin effects and opening channels for the production of visible charm. But, after these wild assumptions are made, we shall be completely rigorous. We will concentrate on observable quantities like the spectrum and the wave functions at the origin, which are related through the Weisskopf-Van Royen formula with the leptonic decay widths. On the one hand, we will study relations between these quantities for one quarkonium system, on the other hand we will vary the parameters entering the Schrödinger equation and study the dependence of these quantities on the mass and the angular momentum.

According to the above remarks we will study in Section 2 the energy levels of one or two systems: the first question to be answered concerns the order of the levels and we will see that there are classes of potentials for which the first P state lies above the second S state or the first D state may lie below that second S state contrary to the observations. Further restrictions can be obtained by following the Regge trajectories and varying the angular momentum. Finally, in that section we will compare levels for two systems with different masses, assuming that the same potential acts between the constituents. (In the spirit of QCD the effective potential should be flavour independent for colour singlet states.)

Section 3 is devoted to the wave functions at the origin. It will be observed that the relative magnitude of these quantities gives information on the curvature of the potential. An over-all convex one (like the harmonic oscillator) does not reproduce the observed leptonic widths in charmonium. A relation for higher excited states for which we have many indications to be true is formulated as a conjecture. A numerically justified relation between the first three values of the wave function at the origin allowed us to predict the leptonic decay width for a possible T'' state. Next the mass dependence of these quantities is analysed. Certain bounds allow us to say that the charge of the bottom quark^{*)} will be

*) Recently some authors have preferred to call the constituent of the T and T' the beauty quark.

$|e_b| = 1/3$ and not twice as large. Inequalities for the quantities involved are described at the end of Section 3 and may be of some help for rough estimates.

In Section 4 we review previous work and our work on the inverse problem for confining potentials. Nearly all work on the inverse problem has been done up to now for potentials going to zero at infinity. We sketch two different ways of proving uniqueness. For further technical details and estimates we will refer to the original literature. Results about the inverse problem, given the ground-state energy for *all* angular momenta, will be included.

Finally, we will prove analyticity properties of the energy levels as a function of the angular momentum for a special class of power law potentials.

Most of the material presented is taken from our published papers or from the literature in published or preprint form. Some results in Sections 2 to 4 are new, and the results of Section 5 have not been published before.

2. THE ENERGY LEVELS

2.1 Order of levels for a given system

Let us start writing down the Schrödinger equation for the reduced wave function $u_n(r)$, $u_n(0) = 0$, for a spherically symmetric potential $V(r)$, with all relevant parameters:

$$(T+V) u_n = E_n u_n, \quad T = \frac{1}{m} \left\{ -\left(\frac{d}{dr}\right)^2 + \frac{l(l+1)}{r^2} \right\} \quad (1)$$

where m denotes the quark mass and $\hbar = 1$. The main quantum number n is identical with the number of nodes the wave function has within $(0, \infty)$. $V(r)$ is assumed to go to infinity at $r = \infty$; therefore the spectrum is purely discrete. The potential should not be singular at finite r , should be smooth everywhere, and less singular than $-1/4r^2$ at the origin (sometimes we will assume less singular than the Coulomb case), so that $T + V$ is a self-adjoint operator and domain questions do not make troubles. When no confusion is possible we set $m = 1$.

To get an idea in which way the levels might be ordered, let us first give an intuitive argument. By comparing the spectra of the Coulomb potential and the harmonic oscillator (see fig. 5) one expects by "continuity" that the levels might be ordered, for instance, for a linear combination of these two potentials $-(\alpha/r) + (1-\alpha)r^2$ like the following:

$$E(1S) < \overset{(a)}{E(1P)} < \overset{(b)}{E(2S)} < E(1D) < E(2P) < E(3S) \quad (2)$$

The same order was found by solving numerically the Schrödinger equation for the famous combination $-(\alpha/r) + (1-\alpha)r$ and we will see that a large class of potentials (although not all) will reproduce (2). Our level notation is not the usual one for the hydrogen atom, but has been used in most of the charmonium calculations.

Let us first remark that a few inequalities in (2) are trivial. Quite generally, it holds that any positive operator v (positive in the sense that all expectation values are positive) added to an operator h , such that $H = h + v$ and h are self-adjoint operators with a common dense domain of definition, shifts all discrete eigenvalues of h (denote them by e_n) in the same direction; so if we denote the eigenvalues of H by E_n , we have in the ordered sense $e_n \leq E_n$ for all n . This can easily be deduced by using the Mini-Max principle, which characterizes the eigenvalues of a lower bounded self-adjoint operator:

$$E_n = \inf_{\mathcal{H}_n} \sup_{\psi \in \mathcal{H}_n} \frac{(\psi, H\psi)}{(\psi, \psi)}, \quad (3)$$

where $\mathcal{H}_n$ denotes an n -dimensional subspace of the Hilbert space; let us denote by $\phi_1, \dots, \phi_n, \dots$ the eigenfunctions of H corresponding to E_n ; since $(\psi, \psi) \geq 0$ for all ψ we get

$$e_n \leq \sup_{\psi \in \mathcal{D}_n} \frac{(\psi, h\psi)}{(\psi, \psi)} \leq \sup_{\psi \in \mathcal{D}_n} \frac{(\psi, H\psi)}{(\psi, \psi)} = E_n, \quad (4)$$

where $\mathcal{D}_n$ denotes the linear span of $\{\phi_1, \dots, \phi_n\}$. Applying this simple result to the case in hand, we deduce the "monotonicity of the Regge trajectories" and therefore $E(nS) \leq E(nP) \leq E(nD) \leq \dots$ for all n .

The first non-trivial question concerns therefore inequality (2a) about which our first result [16] says the following:

Theorem 1: Let $V(r) = -(\alpha/r) + V_c(r)$, $W(r) = (d/dr)(r^2 V_c)$, and assume

$$\begin{cases} \left(\frac{d}{dr}\right)^2 W(r) \geq 0 \\ W(0) \leq 0 \end{cases}$$

then $E_{1P} \leq E_{2S}$.

Remark: As examples we note that any superposition of power potentials of the form

$$V_c(r) = \int_{-1}^{\infty} d\alpha r^\alpha \rho(\alpha) \epsilon(\alpha), \quad \rho > 0 \quad (5)$$

where $\epsilon(\alpha)$ denotes the sign function, fulfil condition (A) and give therefore the "canonical level order", which is indeed observed experimentally: the centre of mass of the χ states lies below the ψ' .

Proof: The main idea is to introduce a parameter λ having scale dimension -1 and studying the level difference $\epsilon = E_{2S} - E_{1P}$ for the family of potentials

$$V_\lambda(r) = -\frac{\alpha}{r} + \lambda^2 V_c(\lambda r) \quad (6)$$

as a function of λ . For $\lambda = 0$ one knows $\epsilon = 0$ due to the level degeneracy of the hydrogen atom; next one tries to find a monotonicity property for ϵ as a function of λ . Although ϵ (like all energy levels) is a function of two variables, we deduce by scaling that it has to have the form

$$\epsilon(\alpha, \lambda) = \lambda^2 \tilde{\epsilon}\left(\frac{\alpha}{\lambda}\right). \quad (7)$$

From (7) one deduces, using Euler's theorem for homogeneous functions, that

$$2\epsilon = \alpha \frac{\partial \epsilon}{\partial \alpha} + \lambda \frac{\partial \epsilon}{\partial \lambda} \quad (8)$$

On the other hand, we may use the so-called Feynman-Hellman theorem [17] to relate the variation of an energy level to the expectation value of the variation of the potential. So we have, for instance,

$$\frac{d}{d\lambda} \epsilon_n = \int_0^\infty dr |u_n|^2 \frac{d}{d\lambda} [\lambda^2 V_c(\lambda r)] = \int_0^\infty dr \frac{|u_n|^2}{r} W(\lambda r) \quad (9)$$

and a similar expression for $d\epsilon_n/d\alpha$. Using (9) for our case it is trivial to check that (8) is equivalent to the well-known Virial theorem.

Let us write more explicitly $u(r)$ for the normalized 2S wave function and $v(r)$ for the normalized 1P wave function; then we intend to get a relation between the two variations in (8). This can be obtained by taking the number of nodes of the wave functions into account and using a kind of "moment inequality". In both variations $\partial\epsilon/\partial\alpha$ and $\partial\epsilon/\partial\lambda$, the difference $u^2 - v^2$ appears; it is simple to establish:

Lemma 1: If $u(r)$ has one node at r_0 , v has none, then $u - v$ vanishes once and only once.

Proof: Since $u \sim r$, $v \sim r^2$ for $r \rightarrow 0$, we get $u - v > 0$ for small r ; from $u(r_0) = 0$ we deduce $u - v < 0$ for $r = r_0$. On the other hand, writing the Wronskian relation for u and v gives

$$v(r) u'(r) - v'(r) u(r) = - \int_0^r dr' W(r'), \quad W(r') = \left(\frac{2}{r'^2} + \epsilon \right) u \cdot v. \quad (10)$$

The integrand w on the right-hand side of (10) vanishes either once at $r = r_0$ if $\epsilon > 0$, or twice at $r = r_0$ and $r^2 = -\epsilon/2$ if $\epsilon < 0$. In both cases we get by taking (10) at $r = r_0$

$$v(r_0)u'(r_0) = \int_{r_0}^{\infty} dr' w(r') < 0 \quad (11)$$

This shows that $w(r)$ cannot be everywhere positive in $r_0 < r < \infty$. Therefore it is positive in $0 < r < r_0$. Taking now the Wronskian relation for the value r_1 for which $u(r_1) = v(r_1)$ we find

$$v(r_1)[u'(r_1) - v'(r_1)] = - \int_0^{r_1} dr' w(r') < 0 \quad (12)$$

which can be fulfilled at most once.

Lemma 2: If $\epsilon = E_{2S} - E_{1P} \geq 0$, then $u + v$ vanishes at most once.

Proof: One uses again Eq. (10), setting $u(r_2) = -v(r_2)$:

$$v(r_2)[u'(r_2) + v'(r_2)] = \int_{r_2}^{\infty} dr' w(r') \quad (13)$$

Since $r_2 > r_0$ and $\epsilon > 0$, $u' + v'$ is negative at r_2 , which excludes two zeros for $u + v$. There is no compelling reason why that sum should vanish at all.

Since from the above two lemmas we deduce (for $\epsilon > 0$) that $u^2 - v^2$ vanishes at most twice, we conclude that there exist always constants β and $\gamma > 0$, such that the inequality

$$\int_0^{\infty} dr \frac{u^2 - v^2}{r} [1 - \beta r + \gamma W(2r)] > 0 \quad (14)$$

holds; β and γ have to be adjusted in such a way that the expression in brackets changes sign where $u^2 - v^2$ changes sign. Note that we have used in an essential way property (A) to get (14). Taking the normalization condition for the wave functions into account, we get $\partial\epsilon/\partial\alpha < \gamma(\partial\epsilon/\partial\lambda)$, which introduced into eq. (8) gives

$$\epsilon < k \frac{\partial\epsilon}{\partial\lambda} \quad , \quad k = \frac{\alpha\gamma + 2}{2} > 0 \quad (15)$$

Therefore as long as ϵ is positive for some value of λ/λ_0 , it is a monotonous function for $\lambda > \lambda_0$. But for small λ it is permissible to use perturbation theory and calculate ϵ to first order

$$\epsilon_{\text{pert}} = \int_0^\infty dr (u_0^2 - v_0^2) \lambda^2 V_c(\lambda r) dr \quad (16)$$

Here u_0 and v_0 denote the unperturbed wave functions; even if the perturbation series does not converge one may deduce the sign of ϵ for small λ from (16). To justify this one may use inequalities for the energy levels of a perturbed system in terms of the levels of an unperturbed system and proceed similarly to the way we did for the ground state in an earlier work [18]. For the perturbation expression (16) we can repeat now the above concavity argument and conclude that $\epsilon > 0$ for small λ ; then ϵ remains positive for arbitrarily large λ .

Remark: Clearly one can take finally $\alpha = 0$ and put an attractive short-range potential into V_c ; so a running coupling constant can be introduced.

Generalizations of the above result to a broader class of potentials (using more sophisticated methods) have been worked out [19]. So, for instance, a potential like

$$V(r) = \int_{-2}^{-1} d\alpha r^\alpha \rho(\alpha) + \int_{-1}^{\infty} d\beta r^\beta \sigma(\beta) \epsilon(\beta) \quad (17)$$

with ρ and $\sigma \geq 0$ gives the canonical order. On the other hand, if W is convex and fulfils $W(\infty) = 0$ we get the reversed order; an example of that type is a potential like $-1/r^\delta$ with $-2 < \delta < -1$. For further examples see fig. 6.

Corollary to Theorem 1: Condition (A) on $W(r)$ implies $E_{2P} \geq E_{1D}$, $E_{2D} \geq E_{1F}$, etc.

Proof: Since we used in our derivation only the number of nodes of the wave functions involved and the fact that the angular momenta of $u(r)$ and $v(r)$ are ordered, we can apply the same procedure to the second P state and the first D state, etc.

Let us next concentrate on the level difference $\Delta = E_{1D} - E_{2S}$ [inequality (b) of (2)] and follow the same strategy as before, but using as a comparison potential the harmonic oscillator [20]. Experimentally the lowest D state first found at SPEAR is above the ψ' level at 3772 MeV.

Theorem 2: Let $V(r) = w^4 r^2 + \lambda^2 V_0(\lambda r)$ define $U(r) = (1/r)(d/dr)(r^2 V_0)$, then condition

$$B_1 \quad \left(\frac{d}{dr^2}\right)^2 U(r) \leq 0, \quad U(\infty) > -\infty \quad \text{implies} \quad \Delta \geq 0$$

while

$$B_2 \quad \left(\frac{d}{dr^2}\right)^2 U(r) \geq 0, \quad \left.\frac{d}{dr^2} U\right|_{r \rightarrow 0} \geq 0 \quad \text{implies} \quad \Delta \leq 0.$$

Proof: All is very similar to before, so that we can be very quick. The variations with respect to ω and to λ are given this time by:

$$\frac{\partial \Delta}{\partial \omega} = 4\omega^3 \int_0^\infty dr \, r^2 (w^2 - u^2) \quad (18)$$

$$\frac{\partial \Delta}{\partial \lambda} = 2 \int_0^\infty dr \, U(\lambda r) (w^2 - u^2), \quad (19)$$

where $w(r)$ denotes the 1D wave function having again no node. Using Lemmas 1 and 2 we conclude that $(w^2 - u^2)$ changes sign at most two times (for $\Delta \leq 0$), which implies now instead of the previous inequality (14)

$$\int_0^\infty dr (w^2 - u^2) \left\{ -\alpha + 4\omega^4 r^2 - \beta U(\lambda r) \right\} \leq 0 \quad (20)$$

with $\beta > 0$, depending on whether U is assumed to be concave (B_1) or convex (B_2) as a function of r^2 . Inequalities (20) imply a relation between the slope $\partial \Delta / \partial \lambda$ and Δ themselves

$$2\Delta \leq (\beta + 2) \frac{\partial \Delta}{\partial \lambda}. \quad (21)$$

For case (B_2) we use again as a last step a perturbation theory argument: for small λ the perturbation expression will determine the sign of Δ , $\partial \Delta / \partial \omega$ vanishes in perturbation theory, and $\partial \Delta / \partial \lambda$ is negative; combining all that proves the second part. The first part needs a bit more care: here perturbation theory implies that $\partial \Delta / \partial \lambda$ is positive for small λ ; following $\Delta(\lambda)$ we may obtain the situation where $\Delta = 0$; but then the first of inequalities (20) and (21) holds and implies that Δ is monotonously increasing.

Remark: A class of potentials fulfilling (B_1) is given by

$$V(r) = + \int_{-2}^2 d\alpha \epsilon(\alpha) \sigma(\alpha) r^\alpha \quad \sigma \geq 0 \quad (22)$$

while the "wrong" ordering is implied for the class

$$V(r) = \int_2^\infty d\alpha \sigma(\alpha) r^\alpha \quad \sigma \geq 0. \quad (23)$$

Corollary 1 to Theorem 2: Take $V(r) = -(\alpha/r) + \lambda^2 V_0(\lambda r)$ and let V_0 fulfil condition (A), then $\Delta(\lambda)$ has at most one zero.

Proof: Once $\Delta(\lambda) = 0$, we can apply the same reasoning leading to Theorem 1.

Corollary 2 to Theorem 2: $V(r) = -(\alpha/r) + \lambda^2 V_0(\lambda r)$. Assume V_0 fulfils conditions (A) and (B_1) [or (B_2)], then $\Delta(\lambda)$ has no (one) zero.

Proof: According to corollary 1 at most one zero occurs. To distinguish the two possibilities we follow the levels for $\lambda \rightarrow \infty$; a scaling argument shows that this limit corresponds to $\alpha \rightarrow 0$; if Δ remains positive in that limit no zero occurred, if Δ becomes negative one zero occurred. To determine the sign of Δ for $\alpha = 0$, we add a harmonic oscillator potential and use theorem 2.

Remarks: The generalization to a broader class has been worked out in Ref. [19]. For power potentials the dependence of the sign of Δ as the exponent and the coupling constant is illustrated in fig. 7. The shaded regions in figs. 6 and 7 mean that it does not make sense to speak about levels at all; the Hamiltonian is not lower bounded in these cases. According to the above result, one should avoid a confining potential rising more than the harmonic oscillator, because if that part is strong enough the D state will drop below the ψ' state.

In summary, we remark that any combination like

$$V(r) = \int_{-1}^2 d\alpha \epsilon(\alpha) r^\alpha \rho(\alpha) \quad , \quad \rho \geq 0$$

will produce the canonical order for the levels. Since our argument needs, in an essential way, the node structure of the wave functions involved, we are not able to give a more general treatment for higher excited levels.

Next let us discuss the dependence of the energy levels on the angular momentum.

2.2 Angular momentum dependence of the energy levels $E_n(\ell)$

Since the ground-state energy E_1 is given according to (3) as the infimum over expectation values of the Hamiltonian H and since the infimum of concave functions is concave, we conclude that E_1 is concave in all parameters in which H is concave (or linear). Therefore E_1 is a concave function of $\ell(\ell + 1)$, which implies for instance

$$E_{1P} \geq \frac{1}{3} E_{1D} + \frac{2}{3} E_{1S}$$

Since the sum of the first n energy levels is also given as an infimum, this time on the trace over an n -dimensional subspace of the Hilbert space

$$\sum_{k=1}^n E_k = \inf_{\mathcal{H}_n} \left\{ \sum_{k=1}^n (\psi_k, H \psi_k) \right. \\ \left. \psi_i \in \mathcal{H}_n, (\psi_i, \psi_j) = \delta_{ij}, i=1, \dots, n \right\} \quad (24)$$

That sum has also to be concave in $\ell(\ell + 1)$. It is well known that the "Regge trajectories" for the harmonic oscillator are linear; so the question arises for which class of potentials is $E_1(\ell)$ concave or convex as a function of ℓ . It is simple to prove, using the methods of theorems 1 and 2, the following.

Lemma 3: Take $V(r) = r^2 + \lambda \delta V$ and let δV satisfy conditions (B_1) or (B_2) ; then consider $\lambda \delta V$ as a perturbation of V : to first order in λ , $E(\ell)$ is then convex or concave, respectively.

Proof: We start with the expression

$$E(\lambda) = E(\ell + \Delta\ell) + E(\ell - \Delta\ell) - 2E(\ell) \quad (25)$$

which vanishes for $\lambda = 0$; to first order in λ , ϵ is given by

$$E(\lambda) = \lambda \int_0^\infty dr \delta V(r) g(r), \quad g(r) = (u_{\ell+\Delta\ell})^2 + (u_{\ell-\Delta\ell})^2 - 2(u_\ell)^2 \quad (26) \\ u_\ell = N_\ell r^{\ell+1} \exp(-(r^2/2)),$$

where the N_ℓ 's are some normalization constants; more precisely $g(r)$ is expressed as

$$g(r) = r^{2\ell+2} e^{-r^2} \left(A r^{2\Delta\ell} + B r^{-2\Delta\ell} - 2C \right) \quad (27)$$

with $A, B, C > 0$. It is easy to see that this bracket vanishes twice and is positive near $r = 0$. On the other hand, clearly

$$\int_0^\infty dr g(r) = \int_0^\infty dr r^2 g(r) = 0 \quad (28)$$

Combining (28) and (26) with the assumed convexity property for δV (as a function of r^2), we immediately get that $\varepsilon(\lambda)$ is negative (or positive).

If one considers power potentials r^α it is simple to find out numerically that $E_1(\ell)$ is convex if $\alpha \geq 2$ and concave if $\alpha \leq 2$ (see fig. 8).

In Ref. [16] we have given additional arguments which support the conjecture.

Conjecture 1: Let the potential $V(r)$ fulfil the conditions (B_1) or (B_2) , respectively; then $(d/d\ell)^2 E_1(\ell) \lesssim 0$. A hint in favour of the above conjecture is given by the following linear bound on E_1 .

Proposition 1: For the family of potentials for which

$$\left(\frac{d}{dr^2}\right)^2 \frac{1}{r} \frac{d}{dr}(r^2 V) \leq 0$$

and $V(0) \geq 0$ holds, we have

$$\frac{E(L)}{2L+3} \leq \frac{E(\ell)}{2\ell+3} \quad \text{if} \quad L \geq \ell. \quad (29)$$

Proof: We consider the family of potentials

$$V_\lambda(r) = \omega r^2 + \lambda^2 v(\lambda r) \quad (30)$$

and use the same technique as before in section 2.1 for the function

$$\Delta(\lambda) = \frac{E(\ell, \lambda)}{2\ell+3} - \frac{E(L, \lambda)}{2L+3} \quad (31)$$

$\Delta(\lambda)$ vanishes for $\lambda = 0$. Scaling gives again

$$2\Delta = \lambda \frac{\partial \Delta}{\partial \lambda} + \omega \frac{\partial \Delta}{\partial \omega} \quad (32)$$

and as before

$$\frac{\partial \Delta}{\partial \lambda} = \int_0^\infty dr f(r) \frac{\partial}{\partial \lambda} \lambda^2 v(\lambda r) \quad ; \quad f(r) = \frac{v_\ell^2}{2\ell+1} - \frac{v_L^2}{2L+1} \quad (33)$$

where u_ℓ and u_L denote the appropriate wave functions, and an analogue relation holds for $\partial\Delta/\partial\omega$. It is easy to show that $f(r)$ vanishes only once because $u'_\ell u_L - u_\ell u'_L$ is always negative; $f(r)$ is positive near $r = 0$ and negative at infinity. So we may combine the assumed concavity property with (33) and the analogue for $\partial\Delta/\partial\omega$ to conclude

$$\frac{\partial\Delta}{\partial\lambda} \geq c \frac{\partial\Delta}{\partial\omega} \quad c \geq 0 \quad (34)$$

and to prove that $\Delta(\lambda)$ must be positive increasing, which shows (29).

Unfortunately (29) does not really prove concavity, it gives only a linear upper bound of the form $E(\ell) \leq [(2\ell/3) + 1] E(0)$, but it allows a convex dependence for small ℓ values. Furthermore, (29) is a statement which depends on a particular reference energy.

Remark: If V grows less fast than r^2 at infinity [that means for any $\epsilon > 0$ there exists a constant $A(\epsilon)$ such that $V < (\epsilon r^2/4) + A(\epsilon)$], $E(\ell)/2\ell + 3$ is not only monotonically decreasing but actually goes to zero with $\ell \rightarrow \infty$. This can be seen by using the comparison theorem which has been discussed before [eq. (4)]; applied to our case it gives $E < A(\epsilon) + (2\ell + 3)\sqrt{\epsilon}$. A further hint in the direction of conjecture 1 is given by

Proposition 2: For $V(r) = r^\alpha$ we get that $E_1(\lambda)^{(2+\alpha)/\alpha}$, with $\lambda = \ell(\ell + 1)$, is a concave function of λ ; the same is true replacing E_1 by the sum of the first n energy levels.

Proof: This time we study the Schrödinger equation as a function of the mass m and of λ ; scaling gives

$$E_1(m, \lambda) = m^{-\frac{\alpha}{2+\alpha}} E_1(1, \lambda), \quad (35)$$

and E_1 is jointly concave in $1/m \equiv \sigma$ and $\lambda/m = \tau$; this means

$$\left(\frac{d}{d\sigma}\right)^2 E \left(\frac{d}{d\tau}\right)^2 E \geq \left(\frac{d^2}{d\tau d\sigma} E\right)^2 \quad (36)$$

Working out the derivatives we get

$$\left(\frac{dE}{dy}\right)^2 + \frac{\alpha}{2} E \left(\frac{d}{dy}\right)^2 E \leq 0 \quad y = \frac{\tau}{\sigma^2} \quad (37)$$

which proves proposition 2.

To proceed, let us derive and discuss what one may call a generalized "Virial theorem": Starting from the Schrödinger equation $u'' = V_E u$, we obtain

$$(F''u^2 - 2F'uu' - 2V_E F u^2 + 2F u'^2)(0) = \int_0^\infty dr u^2 \{4F'V_E + 2FV_E' - F'''\} \quad (38)$$

where F is a three times continuous differentiable function such that both sides of (38) exist. Equation (38) is obtained by firstly multiplying (1) by Fu' and integrating by parts and secondly multiplying (1) by $F'u$ integrating again and combining both expressions in such a way as to eliminate the terms $\int F'u'^2$.

Equation (38) is rather powerful; different forms will be used a few times. As special examples let us put $F = \text{const.}$ and observe that the wave function of the l^{th} partial wave behaves like $u(r) \approx C_l r^{l+1}$ at the origin; so we get

$$u'^2(0) = \int_0^\infty dr u^2 \frac{dV}{dr}, \quad l=0 \quad (39a)$$

$$\int_0^\infty dr u^2 \left\{ \frac{dV}{dr} - \frac{2l(l+1)}{r^3} \right\} = 0, \quad l \neq 0 \quad (39b)$$

Here we will use (39b) to establish

Proposition 3: Denote by $E_1(l)$ and $E_2(l)$ the first and second levels for angular momentum l ; then $E_2(l) - E_1(l)$ decreases with l if

$$\left(\frac{d}{dr}\right)^2 W(r) \leq 0 \quad W(\infty) > -\infty, \quad W = r^2 \frac{dV}{dr} \quad (40)$$

Proof: The change of an energy level by changing l is given according to the Feynman-Hellman theorem [17] by

$$\frac{d}{dl}(E_2(l) - E_1(l)) = (2l+1) \int_0^\infty \frac{dr}{r^2} (u_2^2 - u_1^2) \quad (41)$$

u_1 and u_0 being the appropriate wave functions which have one and no node in $(0, \infty)$. Similar to lemmas 1 and 2, we observe that

Lemma 4: $u_2^2 - u_1^2$ vanishes at most once in $(0, r_0]$ and at most once in $[r_0, \infty)$, where r_0 is the place where $u_2(r_0) = 0$.

Proof: The Wronskian relation reads this time

$$u_1(r) u_2'(r) - u_1'(r) u_2(r) = (E_1 - E_2) \int_0^r dr' u_1(r') u_2(r') < 0 \quad (42)$$

for $r < r_0$. Therefore if $u_1 = u_2 \Rightarrow u_1' > u_2'$, so that $u_2 - u_1$ can vanish only once in that interval. Similarly

$$u_1(r) u_2'(r) - u_1'(r) u_2(r) = (E_2 - E_1) \int_r^\infty dr' u_1(r') u_2(r') < 0 \quad (43)$$

and at the point where $u_1 = -u_2$ we have $u_1' + u_2' < 0$.

To finish the proof of proposition 3 we add to (41) two terms which actually vanish according to (39b) and the normalization condition

$$\begin{aligned} \frac{d}{d\ell}(E_2 - E_1) &= (2\ell + 1) \int_0^\infty \frac{dr}{r^2} (u_2^2 - u_1^2) \{1 - \alpha r^2 + \beta V(r)\}, \\ V(r) &= r^2 \frac{dV}{dr} - 2 \frac{\ell(\ell+1)}{r} \end{aligned} \quad (44)$$

According to assumption (40) U is concave in r^2 , so by a standard argument there exist constants α and $\beta > 0$, such that the bracket on the right of (44) vanishes at the places where $u_1^2 = u_2^2$ and the whole expression is negative.

Remark: Proposition 3 is certainly not optimal: for $V(r) = r^\alpha$ condition (40) restricts the exponent to $\alpha < 1$; on the other hand, we know that $(E_2 - E_1)(\ell) = \text{const.}$ for the harmonic oscillator.

Main conclusion of this subsection: Since we believe that the $c\bar{c}$ potential satisfies (A) and (B₁), we conclude that the centre of gravity of the P states of charmonium should be above the average of the energies of the J/ψ and the ψ'' . Experimentally, one has $3.49 > 3.43$, which is excellent.

We also predict that the mean mass of the P states of the T system will be above 9.74 GeV.

2.3 Comparison of the energy levels of charmonium and bottonium

From QCD one gets the idea that the quark-antiquark interaction should be flavour-independent. Although two such systems will be affected differently by

relativistic effects, we will take this idea seriously and compare the energy levels for two different systems with quark masses m (called J/ψ) and M (called T). Obviously since the kinetic energy decreases with increasing mass the binding energy of the T is lower than the binding energy of the J/ψ [see eq. (4)]. However, this is a very crude inequality, and it could become worse for the $t\bar{t}$ system, if the "top quark" (or "truth") turns out to be extremely heavy. What we will do in order to improve the rough estimate is to compensate the loss of kinetic energy when the quark mass is increased by an increase in the rotation energy [18].

Proposition 4: Denote by $E(M, n, l)$ the n^{th} energy level of a system with quark mass M and angular momentum L (not necessarily integer); then

$$E(M, n, L = \frac{1}{2}(\sqrt{\frac{M}{m}} - 1)) \leq E(m, n, 0) \quad (45)$$

Proof: The Hamiltonian for mass m is

$$H_m = -\frac{1}{m} \left(\frac{d}{dr} \right)^2 + V(r)$$

The operator

$$h = \left(-\frac{1}{m} + \frac{1}{M} \right) \left(\frac{d}{dr} \right)^2 + \frac{1}{4} \left(\frac{1}{M} - \frac{1}{m} \right) \frac{1}{r^2} \quad (46)$$

is positive because of the well-known inequality

$$\int_0^\infty dr \frac{u^2}{4r^2} \leq \int_0^\infty dr u'^2, \quad (47)$$

valid for an arbitrary function u vanishing at $r = 0$. Using the general property

(4), quoted in Section 2 and solving the equation

$$\frac{L(L+1)}{M} = \frac{1}{4} \left(\frac{1}{m} - \frac{1}{M} \right), \quad (48)$$

Proposition 4 follows.

Remark: This inequality has the advantage of being general; however, for usual potentials it is far from being saturated, since inequality (47) is saturated by

$$u = r^{\frac{1}{2} + \varepsilon} e^{-r} \quad E \rightarrow 0 \quad (49)$$

By restricting ourselves to a special class of potentials we can improve on (45).

Proposition 5: For potentials fulfilling condition (A) of Theorem 1, we get, restricting ourselves to the ground state ($n = 1$)

$$E(M, \sqrt{\frac{M}{m}} - 1) \leq E(m, 0) \quad (50)$$

Proof: We study the family of potentials

$$V_\lambda(r) = -\frac{q}{r} + \lambda^2 V_L(\lambda r) \quad (51)$$

and try to prove that

$$\Delta E(\lambda) = E(m, 0, \lambda) - E(M, \sqrt{\frac{M}{m}} - 1, \lambda) \quad (52)$$

remains non-negative if λ increases from zero; $L = \sqrt{M/m} - 1$ has been adjusted to have ΔE vanishing for the pure Coulomb case $\lambda = 0$. To proceed we need again a lemma:

Lemma 5: Let u and v be the ground-state wave functions fulfilling

$$\begin{aligned} \frac{u''}{m} &= (V(r) - E(m, 0)) u \\ \frac{v''}{M} &= \left(V(r) + \frac{L(L+1)}{Mr^2} - E(M, L) \right) v, \end{aligned} \quad (53)$$

then $u^2 - v^2$ vanishes twice and is positive close to $r = 0$.

Proof: The Wronskian relation following from (53) is

$$I(r) = u'v - v'u \Big|_r = - \int_0^r \frac{dr'}{r'} u v \sigma, \quad \sigma = (M-m)rV + \frac{L(L+1)}{r} - Cr \quad (54)$$

We take u and v to be positive for $0 < r < \infty$, and use the fact that condition (A) implies convexity for $rV(r)$, which implies convexity of $\sigma(r)$. Since $\sigma(0) = \sigma(\infty) = \infty$ and $I(\infty) = 0$, we conclude that there are exactly two zeros of the integrand in (54). This implies that $I(r)$ vanishes only once (say at $r = r_0$).

If $u = v$ for some $r < r_0$, $u' - v'$ is negative and hence $u - v$ vanishes at most once in $0 < r < r_0$, and similarly for $r_0 < r < \infty$. Since $u \sim r$ and $v \sim r^{L+1}$ for small r we have $u > v$. At large distances we have again $u > v$ from a standard WKB argument. Hence $u^2 - v^2$ vanishes twice.

The proof of proposition 5 follows now our standard arguments. We compare the variations of ΔE with respect to g and λ , and use the convexity assumption for $W(r)$ to establish a "moment inequality" which allows us to prove (50).

Remark: Application to the T system. To apply (50) in practice we have to use some interpolation property of the energy as a function of L , since $\sqrt{M/m} - 1$ will not be an integer in general. For general potentials one may use the concavity property of $E(M, L)$ as a function of $L(L + 1)$. For potentials fulfilling condition (B_1) we may use the proposed concavity property of the energy as a function of L : Since the quark mass ratio ($m_c \sim 1.3-1.6$ and $M_b \approx 4.7$) will be such that $L \approx 0.7$ to 0.9 , we may use

$$\left(2 - \sqrt{\frac{M}{m}}\right) E_r(l=0) + \left(\sqrt{\frac{M}{m}} - 1\right) E_r(l=1) \leq E_{J/\psi}(l=0), \quad (55)$$

if the P states of the T system have been observed. For the time being we apply the concavity property to the $l = 0$ and $l = 2$ states

$$\left(3 - \sqrt{\frac{M}{m}}\right) E_r(l=0) + \left(\sqrt{\frac{M}{m}} - 1\right) E_r(l=2) \leq 2 E_{J/\psi}(l=0) \quad (56)$$

and use theorem 2 to bind $E_{2S} < E_D$; with $M_T = 2M_b + E_T$ and $M_{J/\psi} = 2m_c + E_{J/\psi}$ we turn (56) into an inequality for quark masses

$$M_b - m_c \geq \frac{1}{2} \left\{ M_T - M_{J/\psi} + \frac{\Delta}{2} \left(\sqrt{\frac{M_b}{m_c}} - 1 \right) \right\}, \quad \Delta = E_{r'} - E_r. \quad (57)$$

The square root on the right-hand side is not very sensitive to the precise quark mass; with $M_{J/\psi} = 3.09$, $M_T = 9.46$ and $\Delta = 0.59$ we get

$$M_b - m_c \geq 3.29$$

while the naive estimate, not using the information on the spacing of the T levels, would give

$$M_b - m_c \geq 3.18.$$

Remark: The question of equal spacing. The first results in the T region suggested an equal spacing between the $J/\psi-\psi'$ and the $T-T'$ energies. Generalizing that situation to all levels and asking for a potential giving equal spacing for all S-wave states leads to the logarithmic potential (a trivial scaling argument shows the truth of that statement) which has been discussed in Ref. [21]. Meanwhile the $T-T'$ energy difference became a bit smaller than the other one.

Remark: Increasing number of "stable" levels. It has been noticed by Eichten and Gottfried and more generally by Quigg and Rosner that the number of $\ell = 0$ states stable with respect to the Zweig rule increases with the mass of the quark. We want to show here that also more and more states with increasing angular momentum become stable as the masses of the quarks increase. We shall use an adaptation of the version communicated to one of us by H. Lipkin for the $\ell = 0$ case.

The $c\bar{c}$ state with quantum number n and angular momentum zero will be stable if

$$E(m_c, n, 0) + 2m_c < 2(m_c + m_d + \Delta)$$

where m_d is the mass of the d quark and Δ its binding energy. Since $m_c \gg m_d$ the binding energy is that of a particle in a fixed potential if forces are flavour-independent. Inequality

$$E(M, n, \ell = \frac{1}{2}(\sqrt{\frac{M}{m_c}} - 1)) < E(m_c, n, 0)$$

leads to

$$E(M, n, \ell = \frac{1}{2}(\sqrt{\frac{M}{m_c}} - 1)) < 2(M + m_d + \Delta)$$

Hence at least $\frac{1}{2}[\sqrt{M/m_c} - 1]$ states, with increasing angular momenta, in the $M\bar{M}$ system, are "stable". This result is complementary to the $\ell = 0$ statement.

3. THE WAVE FUNCTIONS AT THE ORIGIN $\psi_n(0)$

3.1 Relations for a given system

We will mainly consider the $\ell = 0$ case; a few remarks concerning higher angular momenta will be given at the end of this section.

Theorem 3 [22]:

$$\left(\frac{d}{dr}\right)^2 V(r) \geq 0 \Rightarrow |\psi_2(0)|^2 \geq |\psi_1(0)|^2$$

where ψ_i denotes the full wave function: $\psi_i = (4\pi)^{-1/2} u_i/r$.

Proof:

a) Let V be convex; assume $u_2'^2(0) < u_1'^2(0)$ contrary to what we would like to prove. Then $u_2^2 - u_1^2$ vanishes once and only once (say at $r = r_1$). Equation (39a) can be written as

$$(u_2'^2 - u_1'^2)(0) = \int_0^\infty dr (u_2^2 - u_1^2) \left[\frac{dV(r)}{dr} - \frac{dV(r_1)}{dr} \right] \quad (58)$$

Since the right-hand side is positive we get a contradiction; so the first part of theorem 3 is proved.

b) The second part needs more work. First we remark that $|u_2'(0)| = |u_1'(0)|$ implies that $|u_2|$ will never become equal to $|u_1|$ within the interval $[0, r_0]$, where r_0 denotes the place at which $u_2(r_0) = 0$; this is so because $V - E_2$ is more attractive than $V - E_1$ within that region; since $u_2^2 - u_1^2$ becomes negative for large r and both wave functions are normalized, we conclude that $|u_2| = |u_1|$ happens only once (say at r_1). From all that we are able to exclude the situation where $|u_2'| = |u_1'|$, since it contradicts eq. (58) (the right-hand side becomes negative). On the other hand, there exist concave potentials for which $|u_2'| < |u_1'|$, an attractive Coulomb potential being an example. Let V_0 be such a potential; solve the Schrödinger equation for the family

$$V_\alpha(r) = (1-\alpha) V_0 + \alpha V(r) \quad (59)$$

The energy levels and wave functions are continuous functions of the parameter α ; the levels never cross. All potentials V_α are concave; if we should have $|u_2'| > |u_1'|$ for $\alpha = 1$, we would have to have a value for $\alpha \in (0, 1)$ such that the slopes are equal, which contradicts our previous statement.

One may ask whether theorem 3 could be generalized to higher levels. For the charmonium system this is a slightly academic problem for it has only two

levels stable with respect to the Zweig rule and leptonic widths may be altered by open charm channels. But for quarkonium systems with higher mass, it becomes more relevant.

There are many indications in the favour of

Conjecture 2:

$$|\psi_{n+1}(0)|^2 \geq |\psi_n(0)|^2 \text{ if } \left(\frac{d}{dr}\right)^2 V(r) \geq 0 \quad \forall r,$$

which we will summarize next. As we will show in Section 3.3, $|\psi_n(0)|$ goes to infinity with $n \rightarrow \infty$ for convex potentials, while it goes to zero for concave ones.

We illustrate conjecture 2 for the famous combination $-(\alpha/r) + (1 - \alpha)r$.

Furthermore, it has been shown in Quigg and Rosner [23] that conjecture 2 is true for potentials $V(r) = r^\alpha$ in the case where semiclassical approximation methods can be applied. More generally it holds for any smooth potential using the WKB approximation [24].

Proposition 6: Within the WKB approximation one gets

$$|u'_n(0)|^2 = \frac{1}{\pi} \frac{dE}{dn} \sqrt{E - V(0)} \quad , \quad |V(0)| < \infty \quad (60a)$$

and furthermore conjecture 2 is true for monotonous potentials; that is to say

$$\text{if } V' > 0, \frac{d}{dE} |u'_n(0)|^2 \text{ has the same sign as } V'' \quad (60b)$$

Proof:

a) For the first part we start with eq. (39a) and put in the WKB wave function

$$|u'_n(0)|^2 \simeq \frac{\int_0^{r_c} \frac{dr \frac{dV}{dr}}{(E - V)^{1/2}} \sin^2 \int_0^r dr' \sqrt{E - V(r')}}{\int_0^{r_c} \frac{dr}{(E - V)^{1/2}} \sin^2 \int_0^r dr' \sqrt{E - V(r')}} \quad (61)$$

r_c denotes here the classical turning point; note that we are allowed to consider E_n as a continuous differentiable function of n according to the Bohr-Sommerfeld quantization condition

$$n - 1/4 = \frac{1}{\pi} \int_0^{r_c} dr \sqrt{E - V(r)} = n(E) \quad (62)$$

Assuming we are in the situation where many oscillations occur in the integrals in (61), while the variation of V' and $(E - V)^{1/2}$ are smooth, we get approximately

$$|u'(0)|^2 \cong \frac{\int_0^{r_c} \frac{dr dV/dr}{(E - V)^{1/2}}}{\int_0^{r_c} \frac{dr}{(E - V)^{1/2}}} \quad (63)$$

which immediately gives (60a).

b) For simplicity let us put $V(0) = 0$; the generalization is obtained by a trivial substitution. From (63) we get by a partial integration

$$|u'(0)|^2 = - \int_0^{r_c} dr \frac{V''}{V'^2} \sqrt{1 - \frac{V}{E}} - \frac{2}{V'(0)} \quad (64)$$

Noticing that the integrand vanishes at r_c we differentiate:

$$\frac{d}{dE} (|u'(0)|^2) = - \frac{1}{2} \int_0^{r_c} dr \frac{V V''}{E^2 V'^2} \left(1 - \frac{V}{E}\right)^{-1/2} \quad (65)$$

which gives (60b).

Remark: Equation (60a) has first been discussed in another paper [25] and was used to study the duality question within the context of potential theory. It is interesting to compare (60a) with the exact results: (60a) turns out to be exact for the square well and the linear potential. For the oscillator the difference is surprisingly small:

$$|u'_n(0)|_{\text{exact}}^2 = \frac{8(2n-1)(2n-2)!}{\sqrt{2\pi} 2^{2n} ((n-1)!)^2} \quad |u'_n(0)|_{\text{WKB}}^2 = \frac{2\sqrt{2n-1/2}}{\pi} \quad (66)$$

For the ground state we get a 2.5% error, for the first excited state a 0.05% error.

Since the linear potential is the limiting situation between the two cases for conjecture 2, we studied small perturbations around it and obtained

Proposition 7: Let $V(r) = r + \lambda \delta V$, and $\Delta = u_{n+1}'^2(0) - u_n'^2(0)$, then $d\Delta/d\lambda$ has the same sign as V'' up to first-order perturbation theory.

Proof: $d\Delta/d\lambda$ is given in perturbation theory using (39a) by

$$\frac{d\Delta}{d\lambda} = \int_0^\infty dr I_n(r) V''(r) \quad , \quad I_n(r) = \int_r^\infty dr (u_{n+1}^2 - u_n^2) \quad (67)$$

where u_n denotes the solution of the unperturbed potential

$$u_n = \frac{A_i(r - E_n)}{A_i'(-E_n)} \quad , \quad A_i'(-E_n) = 0 \quad (68)$$

We have determined the minimum of $I_n(r)$ over r for small n ($n = 0$ up to 20) numerically*) and found that all minima are positive and occur after the first

*) These estimates have been worked out together with R. Bertlmann.

maximum (see table 1). On the other hand, for large values of n we used the asymptotic expansion of the Airy functions and obtained

$$I_n \underset{n \rightarrow \infty}{\sim} \frac{2}{3\pi} \frac{1}{n^2} \left[\frac{\lambda}{3} - \frac{\sin 2\lambda}{4} + \frac{2}{5} \cos 2\lambda \right] \quad (69)$$

where r has been rescaled to $r = \lambda \alpha_n^{-1/2}$ and $\alpha_n = [(3\pi/2)(n - 1/4)]^{2/3}$. Equation (69) holds if λ is finite and $n \rightarrow \infty$. For the case where $\lambda \rightarrow \infty$ we used different approximations, but nevertheless showed that $I_{n+1} - I_n$ remains positive. In table 1, it is seen that the agreement between the numerically evaluated quantities with the asymptotic formula is perfect so that $I_n(r)$ is positive for all n .

Next let us come closer to experiments. All data up to now agree with the interpretation that T , T' , and T'' are S-wave bound states $b\bar{b}$ with a coloured quark with charge $-1/3$ and mass according to Section 2.3 greater than $3.29 \text{ GeV} + m_c$, where m_c is the mass of the charmed quark. In an earlier paper [26], we estimated the leptonic decay width for the T'' , taking the widths of T and T' from experiment. The Weisskopf-Van Royen formula [27] gives the relation between the leptonic width and the wave function at the origin

$$\Gamma_e = \text{const} \frac{|\psi(0)|^2}{M^2}. \quad (70)$$

Our estimate came out of the WKB approximation for power law potentials [23], for which one finds

Proposition 8: Let $V(r) = r^\alpha$, then the WKB approximation for the wave functions at the origin gives

$$\epsilon = \frac{\ln[\psi_3(0)/\psi_2(0)]}{\ln[\psi_2(0)/\psi_1(0)]} = 0.533. \quad (71)$$

Proof: Using eq. (62) for power potentials gives

$$\begin{aligned} |\psi_n(0)|^2 &\sim (n - 1/4)^{2(\alpha - 1)/(2 + \alpha)}, \quad \alpha > 0 \\ |\psi_n(0)|^2 &\sim (n - 1/4)^{(\alpha - 2)/(2 + \alpha)}, \quad \alpha < 0. \end{aligned} \quad (72)$$

Taking the ratio (71) amounts to eliminating the exponent in (72).

Remark: We have calculated exactly the exponent ϵ by numerical solution of the Schrödinger equation for power law potentials and found a remarkable insensitivity: $0.533 < \epsilon < 0.585$, where the upper limit is reached for $\alpha \rightarrow -1$ and for $\alpha \rightarrow \infty$ (see fig. 10). We have made a numerical study of that exponent for different combinations of power potentials and still found that taking $\epsilon = 0.5$ gives an accuracy better than 12% on $\psi_3(0)$. Using the values for the leptonic width of T and T'

$$\Gamma_e^r \begin{cases} 1.3 \pm 0.2 \text{ keV (DASP, PLUTO)} \\ 1.04 \pm 0.28 \text{ keV (DESY, HD2)} \end{cases} \quad \Gamma_e^{r'} \begin{cases} 0.35 \pm 0.14 \text{ keV (DASP2)} \\ 0.32 \pm 0.13 \text{ keV (DESY-HD2)} \end{cases}$$

and the masses $M_T = 9.46 \text{ GeV}$, $M_{T'} = 10.016 \text{ GeV}$, $M_{T''} = 10.38 \text{ GeV}$, one finds

$$\Gamma_e^{r''} = 0.17 \pm 0.10 \text{ keV}.$$

The inaccuracy is linked to the large experimental uncertainty of the T' leptonic width at present. Since in the Fermilab experiment

$$\begin{aligned} B_e^{r'} \frac{d\sigma^{r'}}{dy} / B_e^r \frac{d\sigma^r}{dy} &= 0.30 \pm 0.03 \\ B_e^{r''} \frac{d\sigma^{r''}}{dy} / B_e^r \frac{d\sigma^r}{dy} &= 0.155 \pm 0.016 \end{aligned}$$

where B denotes the leptonic branching ratios, we may deduce from our estimate that the total widths of T , T' , and T'' are about equal, assuming that the production cross-sections of these three states are approximately equal. This is a strong indication that the mass of the T'' will be below the threshold for the production of visible "beauty".

Up to now we discussed only the S-wave case. Two results have been published for higher angular momenta. The first one was obtained elsewhere [28] and is given here in the formulation due to one of us (A.M.).

Proposition 9: Let V be convex; let $c_{n,\ell}$ denote the first non-vanishing derivative of the wave function at the origin for the n^{th} state with angular momentum ℓ :

$$c_{n,\ell} = \left(\frac{d}{dr} \right)^{\ell+1} u_{n,\ell}(r) \Big|_{r=0} \quad \text{then} \quad c_{1,\ell} < c_{2,\ell}$$

where $u_{n,\ell}$ denotes the corresponding reduced wave function.

Proof: Assume that the opposite of what we would like to prove holds:

$\lim |u_{1,\ell}(r)/u_{2,\ell}(r)| > 1$, for $r \rightarrow 0$, then $u_{1,\ell}^2 - u_{2,\ell}^2$ vanishes only once (say at $r = r_0$) due to a by now standard argument using the Wronskian relation.

Hence using (39b) gives

$$\int_0^\infty dr (u_{1,\ell}^2 - u_{2,\ell}^2) \left\{ \frac{dV}{dr} - \frac{2\ell(\ell+1)}{r^3} - \frac{dV}{dr}(r_0) + \frac{2\ell(\ell+1)}{r_0^3} \right\} = 0. \quad (73)$$

Now if V is convex we get a contradiction since the function in the second bracket is increasing and both change sign at r_0 . So proposition 9 is true.

Remark: Let us quote a special case of eq. (38) which may be useful to get certain bounds [27]: Take $F = r^{-2\ell}$ and replace $V(r)$ by $V(r) + \ell(\ell+1)/r^2$, then

$$(2\ell+1)^2 c_{n,\ell}^2 = \int_0^\infty dr |u|^2 \left\{ \frac{1}{r^{2\ell}} \frac{dV}{dr} + 4\ell \frac{E - V(r)}{r^{2\ell+1}} \right\} \quad (74)$$

where $c_{n,\ell}$ are the constants introduced before. A generalization of (60a) to higher angular momenta ℓ has been worked out recently [29] and says

Proposition 10: Assume $V(r)$ to be finite at $r = 0$ so for simplicity let us put $V(0) = 0$. Within the Thomas-Fermi approximation one gets

$$C_{n,l}^2 \simeq \frac{1}{\pi} \frac{dE_{n,l}}{dn} \left(\frac{l!}{(2l+1)!!} \right)^2 (E_{n,l})^{l+1/2} \quad (75)$$

Proof: Let us give here only the argument for $l = 1$; for that case one writes the kinetic energy density within the Thomas-Fermi theory

$$\sum_{\substack{m \\ E_m \leq E}} |\nabla \psi_m(r)|^2 = \int_{|\vec{k}| \leq k^F(r)} \frac{d^3k}{(2\pi)^3} k^2 \quad (76)$$

where $k^F(r) = \sqrt{E - V(r)}$ denotes the Fermi momentum. Differentiating (76) with respect to E , taking into account the fact that there are the contributions of three P states included in the sum on the left-hand side of (76), and observing the normalization of the ψ_{lm} 's gives (for $r = 0$) (75) for $l = 1$.

Remarks: For the general case see the literature [29], where also a discussion of (75) is given using a modified WKB approximation. Integrating (76) over d^3x and using a Virial type argument gives a well-known semiclassical approximation to the sum over the energy levels below some energy E [see eq. (137)]. One can even make rigorous inequalities of the same type [30].

Formula (75) becomes better for larger n values, but less accurate with increasing l and n fixed. For the ratio of the exact value for $C_{n,l}^2$ to the Thomas-Fermi approximation for the potential $V(r) = \omega^4 r^2/4$, we quote from [29] a few numerical results (see table 2). Expression (75) has been applied to the evaluation of cross-sections by summing over resonances.

3.2 Mass dependence of the wave function at the origin

Concerning the mass dependence of the wave function at the origin for the ground-state case the following has been shown in Quigg et al. [31].

Proposition 11: Let V be concave then

$$\frac{d}{dm} \frac{|\psi_i(0)|^2}{m} > 0. \quad (77)$$

Proof: Since by (39a), written with the explicit $1/m$ factor

$$\frac{\partial}{\partial m} \frac{(\psi_1(0))^2}{m} = - \int_0^\infty dr G(r) V''(r), \quad G(r) = \int_0^r dr' \frac{\partial}{\partial m} u^2(r') \quad (78)$$

it is enough to prove that $G(r) \geq 0$. Differentiating the Schrödinger equation with respect to the mass and using the Feynman-Hellman theorem for $\partial E / \partial m$, one gets similarly to the usual Wronskian relation

$$u \frac{\partial u'}{\partial m} - u' \frac{\partial u}{\partial m} = \int_r^\infty dr'' \{ \int_0^\infty dr''' V(r''') u^2(r''') - V(r'') \} u^2(r') \quad (79)$$

which is negative if V' is positive. Studying $G(r)$, note that $G'(r) = 2u(r) [\partial u(r) / \partial m]$ vanishes if $\partial u / \partial m = 0$, since $u > 0$ for the ground state and $r \in (0, \infty)$. But then $G''(r) = 2u(\partial u' / \partial m)$ is negative according to (79), which shows that G has no minimum. Since $G(0) = G(\infty) = 0$, we conclude that $G(r) \geq 0$.

Remark: This result has been applied using the Weiskopf-Van Royen relation to argue that charge $-1/3$ is more likely for the quark of the T system than charge $2/3$.

We will present a proof of a result which holds for all $\ell > 0$.

Proposition 12: Let $u_{1,2}$ be the ground-state wave functions corresponding to the masses $m_{1,2}$, then

$$\left(\frac{d}{dr} \right)^2 V(r) \leq 0 \rightarrow \lim_{r \rightarrow 0} \left(\frac{u_1^2}{u_2^2} \right) > \frac{m_1}{m_2} \quad \text{for } m_1 > m_2 \quad (80)$$

Proof: This time we use eq. (38b) and get

$$\int_0^\infty dr \left(\frac{dV}{dr} (u_1^2 - u_2^2) - 2 \frac{\ell(\ell+1)}{r^3} \left(\frac{u_1^2}{m_1} - \frac{u_2^2}{m_2} \right) \right) = 0. \quad (81)$$

Assume that the opposite of (80) holds. Since from a Wronskian argument one knows that u_1/u_2 is monotonous we conclude that $u_1^2/m_1 < u_2^2/m_2$ for all r . So the second part in (81) is positive. Concerning the first part we use the fact that $u_1^2 - u_2^2$ vanishes only once (say at r_0); by subtracting $\int_0^\infty (d/dr) V(r_0) (u_1^2 - u_2^2) dr$ from (81) proves that the first term is positive too for concave potentials which contradicts (81).

Remark: In Ref. [28] conditions on V have been found such that

$$\frac{\partial}{\partial m} \left\{ \frac{1}{m^{l+1}} c_{l,e}^2 \right\} < 0,$$

which goes in the opposite direction to (80). Unfortunately, these conditions exclude, for instance, all power potentials r^α with exponent up to $\alpha = 4$.

The natural question whether propositions 11 and 12 can be generalized to higher excited states has been answered only for special potentials [32] and within the WKB approximation.

Proposition 13:

$$\frac{d}{dm} \frac{|\psi_n(0)|^2}{m} > 0$$

for a) $V = r^\alpha$, $V'' < 0$, b) $V'' < 0$ using the WKB wave functions and c) $V = r + \delta V$, $\delta V'' < 0$.

Proof:

a) Starting from eq. (1) and making a scale transformation $r \rightarrow \lambda r$, we get back, for $m \lambda^{2+\alpha} = 1$, the same equation, but m is replaced by one; we obtain

$$\frac{u'^2(m, 0)}{m} = m^{\frac{1-\alpha}{2+\alpha}} u'^2(1, 0) \quad (82)$$

b) Combining eq. (60b) with the relation $\partial E / \partial m = -1/m \langle T \rangle \leq 0$ we prove (b).
c) For δV small use u associated to $V = r$ and scaling in (78).

Application: Since we believe that the $c\bar{c}$ potential is concave, as indicated (but not proved) by comparing the leptonic widths of the J/ψ and the ψ' we can use propositions 12 and 13 to get lower limits on the leptonic widths of the T and the T' , assuming that the potential is the same. One finds that these lower limits are incompatible with experiment if charge $\pm 2/3$ is attributed to the constituents of the T , while charges $\pm 1/3$ give a good fit.

Furthermore, mass dependence of the probability that a particle lies within a spherical shell of radius R has been investigated. For the S wave ground state we have a

Corollary to Proposition 11: Let $P_1(R) = \int_0^R dr' u_1^2(r')$ be the probability of presence inside a sphere of radius R , then $V'' < 0$ implies $G(r) \equiv (\partial/\partial m)P_1(r) > 0$.

Proof: That $G(r) > 0$ in this case has been shown already in proposition 11.

Remark: A scaling argument similar to that used in proposition 11 gives for $V(r) = \alpha/|r|^\alpha$ with $-2 < \alpha < \infty$:

$$P_n(r) = m^{\frac{2}{2+\alpha}} \int_0^r m^{\frac{1}{2+\alpha}} \psi_n^2(m=1, x) dx$$

which shows that $(\partial/\partial m)P_n(r) \geq 0$ ($\alpha = 0$ corresponds to the logarithmic potential).

An explicit calculation proves the same to be true for the finite square well;

R. Feynman (as quoted in Ref. [31]) suggested an example, on the other hand, of a monotonous potential (nested well) for which the probability to be in a sphere decreases with increasing mass for certain values of R .

Proposition 14: Within the semiclassical approximation $V'' < 0$ implies $(\partial/\partial m)P_n(r) \geq 0$.

Proof: Let

$$N = \int_0^r \frac{dr}{\sqrt{E-V}} \quad \text{and} \quad D = \int_0^{r_c} \frac{dr}{\sqrt{E-V}} \quad \text{then} \quad P_n \underset{\text{WKB}}{\simeq} \frac{N}{D}$$

Since $\partial E/\partial m < 0$ we may consider $\partial N/\partial E$ and $\partial D/\partial E$; the first derivative is negative, the second turns out to be positive for concave potentials, since after a partial integration

$$\frac{\partial D}{\partial E} = \frac{1}{V'(0)\sqrt{E-V(0)}} - \int_0^{r_c} dr \frac{V''}{V'^2} \frac{1}{\sqrt{E-V(0)}} > 0 \quad (83)$$

holds. All this together proves proposition 14.

Remark: Recently it has been shown by O. Brander [33] that the probability for the presence of a particle to be at some place inside a confining potential does not necessarily go to zero as the excitation energy goes to infinity.

3.3 Inequalities relating $\psi_n(0)$ and the energy levels

First let us prove that the wave function at the origin either goes to infinity or to zero if the energy goes to infinity for either convex or concave monotonous potential.

Theorem 4: a) $u_n'^2(0) \rightarrow \infty$ for $E_n \rightarrow \infty$ if $V'' > 0$, $V' > 0$, $V'(\infty) = \infty$
 b) $u_n'^2(0) \rightarrow 0$ for $E_n \rightarrow \infty$ if $V'' < 0$, $V' > 0$, $V'(\infty) = 0$.

Proof:

a) We start with the obvious chain of inequalities

$$u_n'^2(0) \geq \int_R^\infty dr u_n^2 \frac{dV}{dr} \geq \frac{dV(R)}{dr} \int_R^\infty dr u_n^2 = \frac{dV(R)}{dr} \left[1 - \int_0^R dr u_n^2 \right] \quad (84)$$

Since $u'^2(r) + u^2(r) [E - V(r)]$ is a monotonous decreasing function of r , we have

$$u_n'^2(0) \geq u_n^2(r) [E - V(r)] \quad (85)$$

which inserted into (84) gives

$$u_n'^2(0) \geq \frac{\frac{dV}{dr}(R)}{1 + \frac{dV(R)}{dr} \int_0^R \frac{dr}{E - V(r)}} \quad (86)$$

Since the denominator goes to 1 for $E \rightarrow \infty$, we have shown part (a).

b) For the second part (84) is replaced by

$$u_n'^2(0) = \int_0^R dr \frac{dV}{dr} u_n^2 + \int_R^\infty dr \frac{dV}{dr} u_n^2 \leq \int_0^R dr \frac{dV}{dr} u_n^2 + \frac{dV(R)}{dr} \quad (87)$$

Next using (85) again gives for E large enough

$$u_n'^2(0) \leq \frac{\frac{dV}{dr}(R)}{1 + \ln[E - V(R)/(E - V(0))]} \quad (88)$$

The denominator goes again to 1 for $E \rightarrow \infty$.

For power potentials one can use (86) and (88) and optimize with respect to R . We obtained different two-sided bounds by noting that in this case the energy and the wave function at the origin are different moments of the same measure. Remember that for $V = r^\alpha$

$$E_n = \frac{\alpha+2}{2} \langle r^\alpha \rangle, \quad u_n'^2(0) = \alpha \langle r^{\alpha-1} \rangle, \quad \langle r^\alpha \rangle = \int_0^\infty dr r^\alpha |u_n|^2 \quad (89)$$

Bounds relating both quantities are given by:

Proposition 15: Let

$$R = \langle r^{\alpha-1} \rangle / \langle r^\alpha \rangle^{\frac{\alpha-1}{\alpha}}$$

then we find

$$\begin{aligned}
 \alpha) \quad \alpha > 2: & \left(\frac{\alpha+4}{2\alpha+4} \right)^{\frac{\alpha-1}{\alpha}} < R < 1, \\
 \beta) \quad 1 < \alpha < 2: & \left(\frac{3}{2+\alpha} \right)^{1/\alpha} < R < 1, \\
 \gamma) \quad \frac{1}{2} < \alpha < 1: & 1 < R < \frac{3}{2+\alpha}, \\
 \delta) \quad 0 < \alpha < \frac{1}{2}: & 1 < R < \left(\frac{3}{2+\alpha} \right)^{\frac{1-\alpha}{\alpha}}.
 \end{aligned} \tag{90}$$

Proof: Since $\ln \langle r^\alpha \rangle \equiv I_\alpha$ is convex as a function of α , half of the bounds in (90) are immediately obtained. For the lower bounds (if $\alpha > 1$) and upper bounds (if $\alpha < 1$), we start from eq. (38) and put $F = 1/(\beta + 1)r^{\beta+1}$. One gets

$$4E \langle r^\beta \rangle = -\beta(\beta-1) \langle r^{\beta-2} \rangle + \left(\frac{2\alpha}{\beta+1} + 4 \right) \langle r^{\alpha+\beta} \rangle \tag{91}$$

$\beta = 0$ reproduces the Virial theorem; the first non-trivial case ($\beta = 1$) gives:

$$I_{1+\alpha} - I_1 = I_\alpha - I_0 + \ln \frac{2\alpha+4}{\alpha+4}. \tag{92}$$

To get for the more general case optimal bounds on R seems to be cumbersome. Using as input I_α , I_0 , and (92), one can find the minimizing configuration compatible with convexity by a graphical construction. This leads to a lower bound on R in case α . Case β is obtained from (92), which gives for $1 < \alpha < 2$:

$$\frac{\alpha+2}{3} \langle r^\alpha \rangle \langle r^{\alpha-1} \rangle \geq \langle r^{2\alpha-1} \rangle \tag{93}$$

and an application of Hölder inequality. The last two missing bounds are obtained similarly.

Remark: For potential $V = r^n$ with integer exponent eq. (91), together with moment inequalities, gives stringent restrictions; e.g. for the harmonic oscillator $V = r^2/4$ the first four identities ($\beta = 0, \dots, 3$) together with

$$\begin{vmatrix} J_0 & J_1 & J_2 \\ J_1 & J_2 & J_3 \\ J_2 & J_3 & J_4 \end{vmatrix} \geq 0, \quad \begin{vmatrix} J_1 & J_3 \\ J_3 & J_5 \end{vmatrix} \geq 0, \quad \text{with } J_n = \langle r^n \rangle, \tag{94}$$

give the following two-sided bounds

$$\frac{405}{128} \frac{(1+J_2)^2}{(9+4J_2^2)J_2^2} \leq \frac{J_1^2}{J_2} \leq \frac{27+9J_2^2}{27+11J_2^2} \tag{95}$$

which are extremely accurate.

4. THE INVERSE PROBLEM FOR CONFINING POTENTIALS

4.1 Reviewing previous results

After having established a number of relations between the relevant quantities observed in quarkonium systems, one can idealize further and study the inverse problem. The inverse problem for potentials going to zero at infinity in the one-dimensional and the radial symmetric case is well worked out and a large amount of literature about the subject is available. For a very recent book see Chadan and Sabatier [34]. Here we are in a situation which has not been treated so extensively in the literature. Assume now not only a finite number of levels and slopes for the reduced wave function at the origin are given, but also an infinite sequence. Is the potential determined uniquely? Is there a way to reconstruct it from the data?

Such a program has been undertaken already in a practical way by Quigg, Rosner and Thacker (QRT) [35a,b]. Maybe it is worth summarizing their work right now. First they consider one-dimensional confining potentials symmetric with respect to the origin and solve an approximate problem: they give themselves the first n levels, choose a zone of energy somewhere between the n^{th} level and the $(n + 1)^{\text{th}}$ one, and solve the inverse problem as if the potential was vanishing at infinity. The reflection coefficient at positive energies, which is needed to solve the classical one-dimensional problem, is set equal to zero; the potential then can be obtained explicitly and originates from a pure superposition of solitons. The hope is that by increasing n one will get a sequence of potentials closer and closer to the true one; QRT have made impressive numerical experiments for typical potentials and found a rapid convergence with n . In the second part of their work they deal with radial potentials $V(r)$ and S waves, noticing that if one uses the reduced wave function $u = r\psi$, the energy levels coincide with the odd parity levels inside a symmetric one-dimensional potential $V(|x|)$. The even levels are missing, but on the other hand $u'_n(0)$ are given; for small n , the algebraic equations relating the missing energies and $u'_1(0)$ can be solved numerically.

The problems, in principle, remain and are:

- i) Does the knowledge of the infinite sequence of $\ell = 0$ energy levels E_i and $|u'_i(0)|^2$ fix uniquely the potential?
- ii) Can one prove the convergence of the QRT procedure?
- iii) Can one get the even parity energy levels from the knowledge of E_i and $u'_i(0)$ associated with the odd levels, which are the physical ones?

We have answered questions (i) and (iii) in another paper [36]. Question (ii) seems to be still open.

To solve the problems we have found two possibilities: the first one was to use as much as possible previous work as it is summarized by Gasymov and Levitan [37]. Here we should mention that already in one of the first works on potentials with equivalent spectra [38] the discrete case has been discussed. There the explicit construction of such potentials is given.

In [37] the uniqueness question for confining potentials with the "wrong" boundary conditions $u'(r=0) = hu(r=0)$ with h finite is discussed, while we are interested in $u(0) = 0$. We shall show, however, how it is possible to relate our problem to the other one.

We have also tried to avoid the use of the older work and succeeded at the price of a "technical" assumption that acceptable potentials should fulfil.

Finally we will make a few comments about the inverse problem if all the ground-state energies as a function of the angular momentum are given.

From the Gasymov-Levitan review we learn

Theorem 5: Let $u(r, E)$ be a solution of the Schrödinger equation with boundary conditions

$$-u'' + (V-E)u = 0, \quad u'(0, E) = 0, \quad u(0, E) = 1 \quad (96)$$

Given now normalization constants γ'_i and energies E_i with

$$(\gamma'_i)^{-1} = \int_0^\infty dr \, u^2(r, E'_i) \quad (97)$$

one may associate a spectral density $\sigma(E)$

$$\sigma(E) \begin{cases} \rho(E) & \text{for } E < 0 \\ \rho(E) - \frac{2}{\pi} \sqrt{E} & \text{for } E > 0 \end{cases} \quad d\rho(E) = dE \sum \gamma_i' \delta(E - E_i') \quad (98)$$

and one is able to reconstruct the potential by solving the Gelfand-Levitan integral equation

$$F(r, t) + K(r, t) + \int_0^r ds K(r, s) F(s, t) = 0 \quad (99)$$

$$F(r, t) = \lim_{N \rightarrow \infty} \int_{-\infty}^N d\sigma(E) \cos(\sqrt{E}t) \cos(\sqrt{E}r)$$

for the kernel K . The potential is then given by

$$V(r) = 2 \frac{d}{dr} K(r, r) \quad (100)$$

4.2 Uniqueness proof

In order to solve the questions raised we propose to study the logarithmic derivative of $u(r, E)$ taken at the origin for a solution with suitable boundary conditions. First we make sure of the existence of such a solution.

Lemma 6: Assume the existence of

$$\int_R^\infty dr \frac{V'^2}{V^{5/2}} \quad , \quad \int_R^\infty dr \frac{V''}{V^{3/2}} \quad (101)$$

for some R ; then there exists a solution of $u'' = (V - E)u$ with asymptotic behaviour

$$u(r, E) \underset{r \rightarrow \infty}{\sim} \frac{\exp\left(-\int_R^r dr' \sqrt{V(r') - E}\right)}{(V - E)^{1/4}} = u_0(r, E) \quad (102)$$

for all complex values of E .

Proof: $u_0(r, E)$ fulfils a Schrödinger equation with a modified potential; the Schrödinger equation for u can be transformed into an integral equation, the inhomogeneous term of which is precisely u_0 . The finiteness of (101) allows the bounding of the resulting kernel, such that one can prove uniform convergence of the iteration procedure; furthermore, $u(r, E)$ and $(d/dr)u(r, E)$ possess the analyticity properties of the individual terms of that series with respect to E .

Theorem 6: Assume $V(r)$ is locally integrable and fulfills (101); then

$$R(E) = \frac{\frac{d}{dr} u(r, E)}{u(r, E)} \Big|_{E=0} \quad (103)$$

is a Herglotz function and admits the following representations

$$-\frac{1}{R(E)} = \sum_{i=0}^{\infty} \frac{|u(0, E'_i)|^2}{E'_i - E}, \quad \int_0^{\infty} dr u^2(r, E'_i) = 1, \quad (104)$$

$$R(E) = C + E \sum_{i=0}^{\infty} \frac{|u'(0, E_i)|^2}{E_i(E_i - E)}, \quad \int_0^{\infty} dr u^4(r, E_i) = 1, \quad (105)$$

where E'_i is the unphysical and E_i the physical spectrum.

Proof: Since we assumed V to be locally integrable (unfortunately that excludes the Coulomb case), it is allowed to integrate down to $r = 0$. For large values of r we use lemma 6, for small values we use Poincaré's theorem to conclude that $R(E)$ is a meromorphic function of E . By combining (1) with the complex conjugate we get

$$\frac{\text{Im } R(E)}{\text{Im } E} = \frac{\int_0^{\infty} dr |u|^2}{|u(0)|^2} > 0; \quad (106)$$

than means $R(E)$ is a Herglotz function. *A priori* such a function grows at most like $|E|$ and decreases at most like $1/|E|$ at ∞ . It has only poles and zeros, interlaced on the real axis. The zeros give the unphysical spectrum E'_i , the poles the physical one. To calculate the residues at the poles it is sufficient to construct the Wronskian of the solution at two nearby energies:

$$u'(0, E + \Delta E) u(0, E) - u'(0, E) u(0, E + \Delta E) = -\Delta E \int_0^{\infty} dr u(r, E) u(r, E + \Delta E) \quad (107)$$

For $E = E_i$ we get

$$\frac{d}{dE} R^{-1}(E) \Big|_{E=E_i} = \frac{\int_0^{\infty} dr u^2(r, E_i)}{u'^2(0, E_i)} \quad (108)$$

and similarly for $E = E'_i$

$$\left. \frac{d}{dE} R(E) \right|_{E=E'_i} = - \frac{\int dr u^2(r, E'_i)}{u^2(0, E'_i)} \quad (109)$$

To proceed we need

Lemma 7: Let V be locally integrable; then the asymptotic behaviour of R is given as

$$|R(E) + \sqrt{-E}| \rightarrow 0 \quad E \rightarrow -\infty.$$

For a proof see Section 2, pt. (iii), and Appendix A of [36]. The main idea is to show that the WKB approximation is valid for $E \rightarrow -\infty$; we proved it in two steps: first for V lower bounded, then for the general case.

Since $R(E)$ behaves like $-\sqrt{-E}$ for $E \rightarrow -\infty$, we can write an unsubtracted representation for $-1/R(E)$ [eq. (104)] and a once subtracted representation for $R(E)$ [eq. (105)].

Remarks: If we write (104) as a Stieltjes' integral, the measure is exactly the spectral measure mentioned in (98). The fact that u/u' is entirely fixed from the E'_i and the $|u_i(0)|^2$ without subtraction explains why this case was treated first. There is a close analogy between the quantity we consider and the Wigner R matrix which is $u(R)/u'(R)$ obtained by integrating from $r = 0$ till the boundary of the nucleus, while we integrate the Schrödinger equation from ∞ down to zero.

Lemma 7 shows that the subtraction constant is actually not free. Now we give an explicit procedure to obtain it:

Lemma 8: Let V be lower bounded, monotonous for large r going to infinity faster than r^E and less fast than r^M . The constant in eq. (105) is then given by

$$C = \lim_{N \rightarrow \infty} C_N \quad C_N = \sum_{i=0}^N \frac{|u'(0, E_i)|^2}{E_i} - \frac{2}{\pi} \sqrt{E'_{N+1}} \quad (110)$$

Proof: Let us mention that a semiclassical approximation for the wave functions and energy levels would immediately give (110). To prove (110) we tried to compute $R(0)$ by computing a contour integral of $R(E)/E$ along circles Γ_N of radius $|E'_{N+1}|$ centred at the origin containing the poles $E_0, \dots, E_N$. So we get

$$\frac{1}{2\pi i} \int_{\Gamma_N} \frac{dE R(E)}{E} = - \sum_{i=0}^N \frac{|u'(0, E_i)|^2}{E_i} + C \quad (111)$$

and next we proved that $R(E)$ is well approximated by $\sqrt{-E}$ for $|\arg(-E)| < \pi - \varepsilon$, ε small enough so that the left-hand side is close to $2\sqrt{E'_{N+1}}/\pi$ (see Ref. [36]). We have tested (110) by explicit calculations and found a very good agreement $C \sim C_N$ even for small N .

Theorem 7: Let V be locally integrable and fulfil the assumption of lemma 8; then the $\ell = 0$ bound state energies E_i and the $|u'(0, E_i)|^2$ fix uniquely the potential.

Proof: Given the constants as stated fixes the subtraction constant (110); therefore $R(E)$ in (105) is determined. Then if one takes the inverse (104) one gets the spectral measure which is needed to solve the inverse problem according to Gasymov and Levitan.

Remark: Since we know now that the knowledge of E_i and $|u'_1(0)|^2$ fix also the E'_1 , it is enough to prove the convergence of the QRT procedure for the one-dimensional case for a symmetric potential in order to solve the radial problem.

Theorem 8: Let V fulfil the assumptions of theorem 6; let $\delta V(r)$ be a continuous function, the zeros of which have no accumulation points for finite r ; then the E_i and $|u'_i(0)|$ determine uniquely the potential ($\delta V = 0$).

Proof: Let $u(r, E)$ and $v(r, E)$ be the decreasing solutions of the Schrödinger equation for the potential $V(r)$ and $V(r) + \delta V(r)$. We know from theorem 5 that with both u and v is associated the same function $R(E)$:

$$R(E) = \frac{u'(0, E)}{u(0, E)} = \frac{v'(0, E)}{v(0, E)}. \quad (112)$$

Let r_1 be the first zero of $\delta V(r)$; then combining the Wronskian for u and v and using (106) gives

$$-u'(r_1)v(r_1) + u(r_1)v'(r_1) = \int_0^{r_1} dr \delta V(r) u(r, E) v(r, E). \quad (113)$$

For E large and negative we have, generalizing lemma 7 to arbitrary r ,

$$\frac{u(r, E)}{u(0, E)} = \exp(-\sqrt{-E} r) [1 + o(1)] \quad (114)$$

and a similar equation for v , as well as

$$\frac{u'(r, E)}{u(r, E)} \simeq -\sqrt{-E} + o(1) \quad (115)$$

So one gets from (113)

$$\int_0^{r_2} dr \delta V(r) \frac{u(r, E) v(r, E)}{u(0, E) v(0, E)} = O(\sqrt{-E} \exp(-2\sqrt{-E} r_2)) \quad (116)$$

However, since δV , u , and v have a constant sign, the left-hand side of (116) is certainly larger than

$$\exp(-2\sqrt{-E} \alpha r_2) \int_0^{\alpha r_2} dr \delta V(r) \quad (117)$$

for any $0 < \alpha < 1$. Letting E go to $-\infty$ gives a contradiction unless $\delta V = 0$ for $0 < r < r_1$, so we conclude the left-hand side of eq. (113) is equal to zero and can repeat the argument and prove that δV vanishes in $r_1 < r < r_2$ and so on.

Remark: One can slightly weaken the assumption about δV . If $\int dr \delta V(r)$ has no accumulation points of zeros the argument goes through because the estimates on u and v are also valid for higher derivatives.

Proposition 16: If two potentials are finite for finite r and go to infinity like r^n and have an infinite set of levels E_m in common such that $\sqrt{E_{m+1}} - \sqrt{E_m} \rightarrow 0$, then their difference δV cannot have compact support.

Proof: Assume that δV has compact support $[0, R]$. Integrate this time the Schrödinger equation from the origin with boundary conditions $u(0, E) = v(0, E) = 0$, and $u'(0, E) = v'(0, E) = 1$. One gets

$$u'(R, E) v(R, E) - v'(R, E) u(R, E) = - \int_0^R dr \delta V(r) u(r, E) v(r, E) = 0 \quad (118)$$

and $\phi(E)$ is an entire function of order 1 of $k = \sqrt{E}$ and exponential type $2R$. It vanishes according to the assumption for $k = \pm i\sqrt{E_m}$, that means on a set of infinite asymptotic density on the imaginary axis. According to a known theorem [39], it is identically zero. The product $u(r, E)v(r, E)$ can now be written according to the Paley-Wiener theorem as

$$\int_{-2r}^{2r} dx e^{ikx} w(r, x) \quad (119)$$

Putting (113) into (112) we get

$$\int_0^R dr \delta V(r) w(r, x) = 0, \quad (120)$$

which can be regarded as a Volterra equation for δV , because of the support properties of w , and hence $\delta V = 0$.

From proposition 16 one deduces that changing a finite number of levels will produce a change in the potential which extends up to infinity. For that simple case the Gelfand-Levitan equation has a degenerate kernel and the explicit solution can be obtained by adapting the usual method to the case in hand.

Proposition 17: Given a potential $V_1(r)$ with energies $E_j^{(1)}$ and normalization constants $\gamma_j^{(1)}$. A potential V_2 having energy levels $E_i^{(2)}$ and constants $\gamma_i^{(2)}$ ($i = 1, \dots, M$) instead of levels $E_j^{(1)}$ and constants $\gamma_j^{(1)}$ ($j = 1, \dots, N$) is given by

$$V_2 = V_1 - 2 \left(\frac{d}{dr} \right)^2 \ln \text{Det} (1+A), \quad (121)$$

where A is an $(n+m) \times (n+m)$ -dimensional matrix with entries

$$A_{ij} = c_i \int_0^r dr' u_1(k_i, r') u_1(k_j, r') \quad (122)$$

$$c_i = \begin{cases} -\gamma_i^{(1)} & i = 1, \dots, N \\ \gamma_i^{(2)} & i = N+1, \dots, N+M \end{cases} \quad k_i^2 = \begin{cases} E_i^{(1)} & i = 1, \dots, N \\ E_i^{(2)} & i = N+1, \dots, N+M \end{cases}$$

and u_1 denotes the regular wave function for potential V_1 .

Proof: Since one uses the standard "artillery" of the inverse problem, we are very brief and sketch only the steps. From the Wiener-Boas theorem one deduces the "Povsner-Levitan" representation for the regular wave functions u_1 and u_2 (corresponding to potentials V_1 and V_2), respectively,

$$u_i(k, r) = \frac{\sin kr}{k} + \int_0^r dr' K_i(r, r') \frac{\sin kr'}{k} . \quad (123)$$

Solving one equation for $\sin(kr)/k$ and introducing it into the other one gives

$$u_2(E, r) = u_1(E, r) + \int_0^r dr' K(r, r') u_1(E, r') , \quad k^2 = E \quad (124)$$

where the new kernel K is a function of u_1 and u_2 of (123); similarly one gets a reciprocal relation

$$u_1(E, r) = u_2(E, r) + \int_0^r dr' \tilde{K}(r, r') u_2(E, r') \quad (125)$$

Multiplying now (125) first by $u_1(E, t)$, integrating with the spectral measure $d\rho_2(E)$ for potential V_2 , and using the completeness property gives

$$\int_{-\infty}^{+\infty} d\rho_2(E) u_2(E, t) u_1(E, r) = 0 \quad \text{for } t > r . \quad (126)$$

Note that the spectral measure used here corresponds to the "right" boundary conditions at the origin $u_2(E_1, 0) = 0$ and is therefore different from that used in (98). Next, multiplying (124) by $u_1(E, t)$, integrating with $d\rho_2(E)$ again, and using (126) gives

$$\int_{-\infty}^{+\infty} d\rho_2(E) u_1(E, t) u_1(E, r) + \int_0^r dr' K(r, r') \int_{-\infty}^{+\infty} d\rho_2(E) u_1(E, t) u_1(E, r') \quad (127)$$

$t < r$

Adding the identity

$$K(r, t) = \int_{-\infty}^{+\infty} d\rho_1(E) \int_0^r K(r, s) u_1(E, s) u_1(E, t) ds \quad r > t \quad (128)$$

to (127) leads us to Gelfand-Levitan equation with kernel K :

$$K(r, t) + F(r, t) + \int_0^r dr' K(r, r') F(r', t) = 0 \quad r > t. \quad (129)$$

The input $F(r, t)$ is now given in terms of the spectral measure by

$$F(r, t) = \int_0^\infty d\sigma(E) u_1(E, r) u_1(E, t) \quad (130)$$

$$\sigma(E) = \rho_2(E) - \rho_1(E)$$

Since we intend to change a finite number of constants as specified before, σ is given by the finite sum:

$$\frac{d\sigma(E)}{dE} = \sum_{i=1}^M \gamma_i^{(2)} \delta(E - E_i^{(2)}) - \sum_{j=1}^N \gamma_j^{(1)} \delta(E - E_j^{(1)}) \quad (131)$$

For the case of a degenerate kernel it is well known that the Gelfand-Levitan equation admits the solution (121), (122) in closed form.

Before turning to the inverse $\mathfrak{L}$ problem, let us remark that within the WKB approximation one gets already a unique monotonous potential given all energy levels.

Proposition 18: Assume $V(r)$ is monotonically increasing so $r(V)$ exists and all energy levels are given. Within the WKB approximation we get

a) for the S-wave case [35a]

$$\kappa(V) = 2 \int_{E_{\min}}^V \frac{dE'}{\sqrt{V - E'}} \frac{dN(E')}{dE'} \quad (132)$$

b) for the three-dimensional case

$$\kappa^3(V) = \frac{1}{6} \int_0^V \frac{d^2 N(E')}{dE'^2} \frac{dE'}{\sqrt{V - E'}} \quad (133)$$

Proof:

a) The number of bound states below some energy E is given within the semi-classical approximation by

$$N(E) = \frac{1}{\pi} \int_0^E dv \frac{dr}{dv} \sqrt{E - V} \quad (134)$$

Differentiating $N(E)$ once with respect to E , one gets an expression which can be regarded as an Abel integral equation for $r(V)$ and inverted; this gives (132).

b) To proceed similarly in the three-dimensional case, we use the well-known formula for the total number of bound states below E

$$N(E) \cong \frac{1}{6\pi^2} \int d^3x (E - V(r))^{3/2} \quad (135)$$

It has first been shown in another paper [40] that (135) becomes exact in the strong coupling limit, where one takes $\lambda V(r)$ as a potential and takes the limit $\lambda \rightarrow \infty$. To invert (135) one differentiates twice and gets again an Abel integral equation

$$\frac{d^2 N}{dE^2} = \frac{1}{6\pi} \int \frac{dV \frac{dr}{dV}(V)}{(E - V)^{1/2}} \quad (136)$$

Inverting (136) gives (133).

Remark: The obvious generalization of (135) to different "moments" of sums over energy levels [or rather quantities like $\text{Tr } \theta(-H)H^\alpha$] has been discussed in the literature: there exist also rigorous bounds on that type

$$\sum_i |E_i|^\alpha \leq C_\alpha \int d^3x |V(x)_-|^{3/2 + \alpha} \quad (137)$$

where $|V|_-$ denotes the absolute value of the attractive part of V . Such bounds lead to a whole "industry", since it was recognized that they are of great help in the problem of proving stability of matter [30].

4.3 The inverse problem for $E_1(\ell)$

We now want to discuss a non-standard inverse problem connected with confining potentials. Instead of giving ourselves the successive energy levels for a given angular momentum, we give ourselves the ground-state energies for all physical angular momenta, $E_1(\ell)$. The question is whether this fixes the potential uniquely or not. We shall not be able to give a complete answer to this question, but collect a certain number of indications towards

Conjecture 3: The confining potential is uniquely fixed by the knowledge of the ground-state energies of the system for all angular momenta.

Example: First we shall look at a particularly simple case, the case of equally spaced energy levels

$$E_l(l) = A + B l \quad (138)$$

One solution of the problem is well known. It is the harmonic oscillator

$$V(r) = \frac{B}{2} r^2 + A - \frac{3B}{2} \quad (139)$$

For this particular solution of the inverse problem, the reduced wave functions associated with $E_l(l)$ are given by (in the special case of $B = 2$)

$$u_l(r) = N_l r^{l+1} \exp(-r^2/2) \quad (140)$$

We shall content ourselves with investigating the local uniqueness of V . Suppose we change the potential by δV , infinitesimal. Then by the "Feynman-Hellmann" theorem the change of the energy levels will be

$$\delta E_l = \int_0^\infty dr \delta V |u_l(r)|^2 \quad (141)$$

An equivalent potential $V + \delta V$ will be such that all δE_l vanish. The question is then a question of completeness. We have from (140) and (141), putting $r\delta V(r) = W(x)$, $r^2 = x$

$$\int_0^\infty dx W(x) x^l e^{-x} = 0 \quad \forall l \in \mathbb{N}. \quad (142)$$

According to Szegő [41] the system $\phi_l = x^l e^{-x}$ is "closed", so that any function $f(x) = W(x)e^{-x}$, with a finite L^2 norm can be approximated by $\sum c_l \phi_l$ in such a way that $\|f(x) - \sum_l c_l \phi_l\|_2 < \varepsilon$, with ε arbitrarily small.

If δV grows less fast than any exponential, $f(x)$ is guaranteed to have a finite L^2 norm and the theorem applies. Conditions (142) guarantee therefore that $\delta V \equiv 0$. So we get the following:

Proposition 19: If the levels $E_l(l)$ are equally spaced, the harmonic oscillator potential constitutes an isolated solution to the inverse problem.

If one tries to generalize this result to other potentials one gets first the set of equations

$$\int_0^\infty dr \, r V \, |u_l(r)|^2 = 0 \quad \forall l \in \mathbb{N}.$$

however, one encounters a very difficult problem, which is to prove that the set of functions $|u_l(r)|^2$ is complete in some way; the mathematical literature is incredibly poor in this respect.

We shall try nevertheless to study these functions $|u_l(r)|^2$ and prove the following:

Proposition 20: Let $V(r)$ be a confining potential such that $V(r)/r^2 \rightarrow 0$ for $r \rightarrow \infty$. Then the reduced wave functions $u_l(r)$ associated with the levels $E_0(l)$, normalized by $\lim_{r \rightarrow 0} u_l(r)/r^{l+1} = 1$, are such that for any fixed R

$$\lim_{l \rightarrow \infty} \frac{u_l(R)}{R^{l+1}} = 1 \quad \text{and} \quad \lim_{l \rightarrow \infty} \frac{u'_l(R)}{(l+1)R^l} = 1.$$

Proof: We incorporate the normalization condition by writing the integral equation

$$u_l(r) = r^{l+1} + \frac{1}{2l+1} \int_0^r dr' u_l(r') \left[\frac{r^{l+1}}{r'^l} - \frac{r'^{l+1}}{r^l} \right] [V(r') - E_l(l)] \quad (143)$$

Define R_c to be the smallest r such that $E = V(R_c)$. If $r < R_c$ we have from eq. (143) $u_l(r) < r^{l+1}$ and by iteration

$$u_l(r) > r^{l+1} \left[1 - \frac{E_l(l)}{2l+1} \int_0^r dr' r' \left(1 - \left(\frac{r'}{r} \right)^{2l+1} \right) \right] \quad (144)$$

and hence

$$r^{l+1} > u_l(r) > r^{l+1} \left[1 - \frac{E(l)}{2l+1} \frac{r^2}{2} \right] \quad (145)$$

Since, as we have seen from the considerations of section 2.2,

$$\frac{E_1(\ell)}{2\ell+1} \rightarrow 0 \quad \text{if} \quad \frac{V(r)}{r^2} \rightarrow 0 \quad \text{for } r \rightarrow \infty,$$

the first part of proposition 20 follows. The statement about the derivative of u_ℓ follows by differentiating (143).

Remark: We believe that proposition 20 is indicative that the set of functions $u_\ell^2(r)$ is in some way complete, at least on a finite interval. According to a theorem of Müntz [42], a sequence of pure powers r^{α_n} is complete if the α_n 's have a finite density for $n \rightarrow \infty$. However, even if we succeeded in proving completeness over a finite interval, we would have the problem of letting the interval extend towards $+\infty$.

We shall, however, make another completely rigorous use of proposition 20, but restrict the generality of the possible variations of δV .

Theorem 9: Assume that the set of levels $E_1(\ell)$ is reproduced by two potentials U and V such that U/r^2 and $V/r^2 \rightarrow 0$ for $r \rightarrow \infty$, and $U(r)$ and $V(r) \rightarrow \infty$ for $r \rightarrow \infty$. If, in addition, $U - V$ has a definite sign for $r > R$, then $U = V$ for $r > R$.

Proof: Consider a given energy level $E_1(\ell)$. The wave functions, normalized as in proposition 20, associated with the potentials U and V are, respectively, u_ℓ and v_ℓ . By combination of the Schrödinger equation one gets

$$u'_\ell v_\ell - u_\ell v'_\ell \Big|_R = \int_R^\infty (V-U) u_\ell v_\ell dr = \int_0^R dr u_\ell v_\ell (U-V).$$

Assume that for $r > R$, $V - U$ has a constant sign which can be chosen to be positive.

We have

$$u'_\ell v_\ell - u_\ell v'_\ell \Big|_R > \int_{(1+E)R}^{(1+2E)R} dr u_\ell v_\ell (V-U) \quad E > 0 \quad (146)$$

For large ℓ , proposition 20 allows us to estimate both sides of (146). The left-hand side is of the order of $(2\ell + 1)R^{2\ell+1}$, while the right-hand side is larger than

$$(1+\epsilon)^{2\ell+2} R^{2\ell+2} \int_{(1+\epsilon)R}^{(1+2\epsilon)R} dr (V-U)$$

which is clearly a contradiction, unless $V = U$ for $r > R$.

Now we are working with a compact interval and we have two possibilities, one being rigorous, the other imperfect.

Corollary 1 to theorem 9: If in addition $V - U$ has a finite number of changes of sign $V \equiv U$.

Proof: First apply theorem 9, to R_n corresponding to the last change of sign of $V - U$, then to R_{n-1} , etc.

Pseudocorollary: We keep the conditions of theorem 9. We admit that we are in conditions sufficiently close to the Müntz theorem to believe that the set of the $u_\ell v_\ell$ is complete on a finite interval and conclude from $\int_0^R dr (U - V) u_\ell v_\ell = 0$ that $U \equiv V$.

This is as far as we have been able to go on a completely rigorous basis. Now we shall describe a semiclassical treatment of the problem. Assume for simplicity $(d/dr)[r^3(dV/dr)] > 0$. This guarantees that $\ell(\ell+1)/r^2 + V(r)$ has a unique minimum. It also implies that V is increasing. On the other hand, this condition is not so restrictive as it looks: all potentials satisfying condition (A) obey it.

We start by purely classical considerations, neglecting completely the radial kinetic energy. Then $E_1(\ell)$ will be the minimum of the centrifugal and the potential energy

$$E_1(\ell) = \frac{(\ell+1/2)^2}{r^2} + V(r) \quad (147)$$

with r given by

$$\frac{2(\ell+1/2)^2}{r^3} = \frac{dV}{dr} \quad (148)$$

r is a function of ℓ . We can differentiate $E_1(\ell)$ with respect to ℓ , using an interpolation in ℓ , taking into account (148) and get

$$\frac{dE_1(\ell)}{d\ell} = \frac{(2\ell+1)}{r^2} \rightarrow r^2 = \frac{2\ell+1}{dE/d\ell} \quad (149)$$

Remark: r^2 can be rewritten as $(dE_1/d[\ell(\ell+1)])^{-1}$, which according to the general concavity property contained in section 2.2, is an increasing function of ℓ .

r being given as a function of ℓ , V can also be obtained as a function of ℓ , from (147)

$$V(\ell) = E_1(\ell) - \frac{1}{2}(\ell + \frac{1}{2}) \frac{dE_1}{d\ell} \quad (150)$$

We see that at this level the inverse problem is incredibly simple. We shall try to see to what extent that kind of approach can be justified quantum-mechanically. Interpolation in ℓ is not a problem. It has been done long ago by Regge [43]. We shall return to this problem in the next section.

The quantum mechanical analogue of (147) is

$$E_1(\ell) = \int_0^\infty dr \left[\dot{u}_\ell^2 + \frac{\ell(\ell+1)}{r^2} u_\ell^2 + V(r) u_\ell^2 \right] \quad (151)$$

where, from now on we take u_ℓ^2 to be normalized by $\int_0^\infty dr u_\ell^2 = 1$. Hence we get using also inequality (47)

$$E_1(\ell) > \min \left(\frac{(\ell + \frac{1}{2})^2}{r^2} + V(r) \right) \quad (152)$$

which resembles very much (147) except for the fact that it is an inequality.

The quantum mechanical analogue of (149) is

$$\frac{dE_1}{d\ell} = (2\ell+1) \int_0^\infty dr \frac{u_\ell^2}{r^2} \quad (153)$$

and, if the wave function is sufficiently concentrated around the minimum of the effective potential, reduces to (149) because r can be replaced by r minimum in the integral. Our condition on the potential allows in fact to make a very precise statement.

Proposition 21: Define r_0 and r_1 by

$$\frac{1}{r_0^2} = \int_0^\infty dr \frac{u_\ell^2}{r^2}, \quad V(r_1) = \int_0^\infty dr u_\ell^2 V(r) \quad (154)$$

and assume

$$\frac{d}{dr} \left(r^3 \frac{dV}{dr} \right) > 0, \quad \text{then } r_1 > r_0$$

Proof: We shall be satisfied by proving $r_1 \neq r_0$. If $r_1 = r_0$

$$\int_0^\infty dr u_\ell^2 \left\{ \frac{V(r) - V(r_0)}{V'(r_0)} + \frac{r_0^3}{2} \left(\frac{1}{r^2} - \frac{1}{r_0^2} \right) \right\} = 0 \quad (155)$$

Calling the bracket $y(r)$ we see that $y(r_0) = y'(r_0) = 0$, but $(d/dr)r^3(dy/dr) > 0$, which guarantees $y(r) \geq 0$. The sign of the inequality between r_0 and r_1 is intuitively clear.

Remark: From proposition 21, we conclude as long as (149) holds

$$E(\ell) > V(r_0) + \frac{\ell+1/2}{2} \frac{dE}{d\ell}, \quad r_0 = \sqrt{\frac{2\ell+1}{dE/d\ell}}. \quad (156)$$

This can even be made more practical by using convexity and avoiding the use of non-integer ℓ . We have

$$\frac{2\ell+1}{2\ell-1} [E(\ell) - E(\ell-1)] > \frac{dE}{d\ell} > \frac{2\ell+1}{2\ell+3} [E_{\ell+1} - E_\ell] \quad (157)$$

and hence we get the definition and the inequality

$$\bar{r}_0 = \sqrt{\frac{2\ell-1}{E(\ell) - E(\ell-1)}}, \quad V(\bar{r}_0) < E(\ell) - \frac{1}{2}(\ell+1/2) \frac{\ell+1/2}{\ell+3/2} [E(\ell+1) - E(\ell)]. \quad (158)$$

This is a strict and probably useful inequality, but it does not tell us how close we are to the potential.

Now we shall abandon rigour and try to approximate the effective potential $\ell(\ell+1)/r^2 + V(r)$ by a harmonic potential and calculate the zero-point energy. To make this estimate, we use the potential determined without quantum effects and compute the curvature at the minimum. This gives finally

$$\left\{ \begin{array}{l} V(r) = E(\ell) - \frac{\ell(\ell+1)}{2\ell+1} \frac{dE}{d\ell} + \sqrt{\frac{V''}{2} + 3 \frac{\ell(\ell+1)}{r^4}} \\ r^2 = \frac{2\ell+1}{\frac{dE}{d\ell}} \\ V'' = -\frac{1}{2} \left(\frac{dE}{d\ell} \right)^2 + \frac{\frac{d}{d\ell} [(2\ell+1)(dE/d\ell)]}{\frac{d}{d\ell} [(2\ell+1)(dE/d\ell)]} \end{array} \right. \quad (159)$$

It seems unfortunately difficult to find a systematic iterative procedure to improve this formula. As it is it gives rather good results for large r .

Example: Take $E_0 = 2\ell + 3$ ($V = r^2$). Equations (156) give $V = r^2 + 1$, while eqs. (159) give

$$V(r) = 2r^2 + 2 - \frac{r^{4/4}}{r^2} - \sqrt{1 + \frac{3(r^{4/4})}{r^4}} \simeq r^2 + \frac{1}{4r^2} + \dots \quad (160)$$

and therefore no constant term.

Example: Let

$$E = -\frac{1}{4(\ell+1)^2} \quad (V = -\frac{1}{r})$$

Equations (159) give

$$V = -\frac{1}{r} + O\left(\frac{1}{r^2}\right)$$

while (156) contains, after the leading term a contribution in $r^{-3/2}$.

Our conclusion is that experimentally our improved semiclassical procedure is excellent to get the large distance behaviour of the potential, and can certainly be used as a starting point in the search for the exact potential.

On the other hand, there is a very well posed, but difficult, mathematical problem left.

5. REGGE TRAJECTORIES FOR CONFINING POTENTIALS

In section 4 we have already been led to introduce an interpretation of the ground-state energies for non-integrals. In section 2 we have also mentioned convexity properties in ℓ . Confining potentials are in a way simpler than ordinary potentials in what concerns analyticity properties in the angular momentum

complex plane, because of the absence of the continuous spectrum. Following Regge [43] we can try to solve the Schrödinger equation for complex ℓ , with $\text{Re } \ell > -1/2$:

$$\left(-\left(\frac{d}{dr}\right)^2 + \frac{\lambda}{r^2} + V(r)\right)u(r) = E(\ell)u(r), \quad \lambda = \ell(\ell+1) \quad (161)$$

and we get immediately by multiplication with u^* and integration

$$\int_0^\infty dr \left\{ |u'|^2 + \frac{\ell(\ell+1)}{r^2} |u|^2 + V(r) |u|^2 \right\} = E(\ell) \int_0^\infty dr |u|^2 \quad (162)$$

which implies that normalizable solutions of the Schrödinger equation are such that

$$\text{Im } \lambda \int_0^\infty dr \frac{|u|^2}{r^2} = \text{Im } E \int_0^\infty dr |u|^2, \quad (163)$$

and hence for real angular momentum E is real. In non-confining potentials, solutions of the Schrödinger equation for ℓ real, large enough are not any more normalizable and the reality argument breaks down and, as everybody knows, the energies become complex, or, conversely, real positive energies correspond to complex momenta.

We think it is worth studying Regge trajectories in confining potentials, because there are practically no examples of Regge trajectories known, except in the oscillator and Coulomb cases, and except for numerical calculations in the Yukawa potential.

The problem is that $E(\ell)$, defined implicitly by (161), is analytic whenever the derivative

$$\frac{dE}{d\ell} = (2\ell+1) \frac{\int_0^\infty dr \frac{u_\ell^2}{r^2}}{\int_0^\infty dr u_\ell^2} \quad (164)$$

exists, but except for ℓ real nobody knows when this derivative exists because

$\int_0^\infty dr u_\ell^2$ may vanish. What is known is that if $E(\ell)$ [or $\ell(E)$] has isolated singularities in the complex ℓ plane these singularities cannot be poles or essential singularities because from (162) $\text{Im } E / \text{Im } \lambda > 0$, or since $\text{Re } \ell > -1/2$,

$\text{Im } E(\ell) / \text{Im } \ell > 0$.

From now on we shall restrict ourselves to pure power potentials

$$V(r) = r^\alpha \quad (165)$$

This will allow us to use the complex r plane. In fact, most of what we shall say now is an adaptation of the work of J.J. Loeffel and one of us (A.M.) [44,45] on the anharmonic oscillator.

For a potential of the type (165) the solution of the Schrödinger equation behaves at large distances like the WKB solution. One can use the WKB solution as a starting point to establish an asymptotic expansion of the solution:

$$u_0 = r^{-\alpha/4} \exp\left\{-\frac{r^{\alpha/2+1}}{\frac{\alpha}{2}+1}\right\} \left(1 + O(r^{1-\frac{\alpha}{2}})\right) \text{ for } \alpha > 2, \quad (166)$$

$$u_0 = r^{l+1} \exp\left(-\frac{r^2}{2}\right) \text{ for } \alpha = 2, \quad (167)$$

$$u_0 = r^{-\alpha/4} \exp\left\{-\frac{r^{\alpha/2+1}}{\frac{\alpha}{2}+1} + \frac{E}{2} \frac{r^{1-\frac{\alpha}{2}}}{1-\frac{\alpha}{2}} + O(r^{1-\frac{3\alpha}{2}})\right\} \text{ for } \alpha < 2. \quad (168)$$

In the case $\alpha < 2$ more and more terms are needed in the expansion as α approaches zero, if one wants u_0 to be such that $\lim_{r \rightarrow \infty} (u/u_0) = 1$.

Because the potential is an analytic function of $z = r$ except at the origin, the solution can be analytically continued in the complex r planes for fixed l (real or complex). It can be shown that the solution is decreasing in the sector

$$-\frac{\pi}{2+\alpha} < \text{Arg } z < \frac{\pi}{2+\alpha},$$

where it behaves like the analytic continuation of (166), (167), and (168), i.e. it is dominated by

$$\exp\left\{-\frac{z^{\alpha/2+1}}{\alpha/2+1}\right\}$$

Remark (a): If $\alpha > 2$, the two independent solutions behave like

$$z^{-\alpha/2} \exp\left\{\pm \frac{z^{\alpha/2+1}}{\alpha/2+1}\right\}$$

and hence $|u|^2$ is still square integrable along the rays $\arg z = \pm\pi/(2 + \alpha)$.

We shall need to continue further the solution in

$$-\frac{3\pi}{\alpha+2} < \arg z < \frac{3\pi}{\alpha+2}.$$

In the sector

$$\frac{\pi}{\alpha+2} < \arg z < \frac{3\pi}{\alpha+2}$$

u is, *a priori*, a superposition of solutions asymptotically behaving like

$$u_0 = z^{-\alpha/4} \exp(P(z))$$

$$v_0 = z^{-\alpha/4} \exp(-P(z))$$

where

$$P(z) = -\frac{z^{\alpha/2+1}}{\alpha/2+1} + \frac{\pi}{2} \frac{z^{1-\alpha/2}}{1-\alpha/2} + \dots$$

contains only positive powers of z . But $u_0/v_0 \rightarrow \infty$ for

$$\frac{\pi}{\alpha+2} < \arg z < \frac{3\pi}{\alpha+2}$$

and hence $|u/u_0| < \text{const.}$ We get the following situation

$$u/v_0 \rightarrow 1 \quad \text{for} \quad |\arg z| < \frac{\pi}{2+\alpha}$$

$$|u/v_0| < \text{const} \quad \text{for} \quad |\arg z| < \min\left\{\frac{3\pi}{2+\alpha}, \pi\right\}$$

According to Montel's theorem we conclude

$$u/v_0 \rightarrow 1 \quad \text{for} \quad |\arg z| < \min\left\{\frac{3\pi}{2+\alpha}, \pi\right\}. \quad (169)$$

Let us now describe our strategy to establish the analyticity properties of the energy levels $E_n(\ell)$ with respect to ℓ or λ . We said already that the only singularities we must worry about are branch points (natural boundaries can be discarded after a painful argument which we shall not reproduce here). To exclude branch points first find a characterization of the successive energy levels

for complex λ . For real $\lambda > -1/2$ the levels are labelled by the number of zeros of $n(r)$ on the real positive axis. The first step will be to show that in the angle

$$|\operatorname{Arg} z| < \min \left\{ \pi, \frac{3\pi}{2+\alpha} \right\},$$

there are no other zeros for real λ . We can show that it is still possible to characterize these energy levels for complex λ by the number of zeros either in $|\arg z| < \pi/(2 + \alpha)$ in the case where $\alpha > 2$ or in

$$-\frac{\pi}{2+\alpha} < \operatorname{Arg} z < \frac{2\pi}{2+\alpha} \quad \text{for } \alpha < 2 \quad \operatorname{Im} \lambda > 0$$

and

$$-\frac{2\pi}{2+\alpha} < \operatorname{Arg} z < \frac{\pi}{2+\alpha} \quad \text{for } \alpha < 2 \quad \operatorname{Im} \lambda < 0$$

We shall prove that the number of zeros in these regions does not vary along any path in the complex λ plane starting from and returning to the real axis, with $\operatorname{Re} \lambda > -1/2$. This will demonstrate the absence of branch points.

Step 1: λ real $> -1/2$. The Schrödinger equation written along a complex ray $z = te^{i\phi}$ becomes

$$\left(-\left(\frac{d}{dt}\right)^2 + \frac{\lambda}{t^2} + t^\alpha e^{i(2+\alpha)\phi} \right) u = e^{2i\phi} E u \quad (170)$$

By multiplying by u^* and integrating, we get

$$\operatorname{Re} u' u^* = \int_0^t dt' \left\{ |u'|^2 + |u|^2 \left(\frac{\lambda}{t'^2} + t'^\alpha \cos(2+\alpha)\phi - E \cos 2\phi \right) \right\}, \quad (171)$$

$$\operatorname{Im} u' u^* = \int_0^t dt' \left\{ t'^\alpha \sin(2+\alpha)\phi - E \sin 2\phi \right\} |u|^2. \quad (172)$$

If $|\phi| < \pi/(2 + \alpha)$, $u \rightarrow 0$ at infinity and the integration sign $\int_0^t$ in (171) and (172) can be replaced by $-\int_t^\infty$. In that case we see that since the integrand in (172) is monotonous, we can always choose the limits in such a way that the integrand has a constant sign; therefore we have

Property 1: For ℓ real $> -1/2$, u and u' have no zero in

$$0 < |\phi| < \frac{\pi}{2+\alpha}.$$

We try now to investigate $\pi/(2+\alpha) \leq |\arg \phi|$. One considers the combination

$$\operatorname{Re}(u'u^*) \cos \beta - \sin \beta \operatorname{Im}(u'u^*)$$

It will be different from zero if one can find β such that

$$\cos \beta > 0, \quad \cos((2+\alpha)\phi + \beta) > 0, \quad \sin(2\phi + \beta) < 0$$

This is possible, if simultaneously

$$|\phi| < \frac{3\pi}{\alpha+2}, \quad |\phi| < \frac{2\pi}{\alpha}$$

Hence we get

Property 2: For ℓ real $> -1/2$ and $\alpha \leq 4$, u and u' have no zeros in

$$0 < |\phi| < \min\left(\frac{3\pi}{\alpha+2}, \pi\right)$$

Step 2: We make λ complex, for instance, $\operatorname{Im} \lambda > 0$. Equation (172) becomes then

$$\operatorname{Im} u'u^* = \int_0^t dt' |u|^2 \left\{ \frac{\operatorname{Im} \lambda}{t'^2} + t'^{\alpha} \sin(2+\alpha)\phi - |\epsilon| \sin(2\phi + \arg \epsilon) \right\} \quad (173)$$

If $0 < \phi < \pi/(2+\alpha)$ the integral tends to zero if $t \rightarrow \infty$. However, $\sin(2+\alpha)\phi > 0$; so we conclude $\sin(2\phi + \arg \epsilon) > 0$. Hence, using continuity to choose the good determination

$$0 < \arg \epsilon < \frac{\alpha\pi}{2+\alpha}, \quad \operatorname{Im} \lambda > 0 \quad (174)$$

Next we prove a number of properties of zeros.

Property 3: For $\operatorname{Im} \lambda > 0$, u has no zeros for $-\pi/(2+\alpha) < \phi \leq 0$.

Proof: For ϕ in that interval the integrand in (173) is monotonous and one can choose the integration limits as $(0, t)$ or (∞, t) .

Property 4: For $\text{Im } \lambda > 0$, $\alpha > 2$, u has no zero for

$$\phi = \frac{\pi}{2+\alpha}$$

Proof: According to remark (a), $|u|$ is square integrable and we get

$$\text{Im } u' u^* = \int_0^t dt' |u|^2 \left\{ \frac{\text{Im } \lambda}{t'^2} - |E| \sin\left(\frac{2\pi}{\alpha+2} + \text{Arg } E\right) \right\} = - \int_t^\infty dt' |u|^2 \left\{ \dots \right\} \quad (175)$$

Again the integrand being monotonous $\text{Im } u' u^*$ cannot vanish.

Property 5: For $\text{Im } \lambda > 0$, $\alpha < 2$, u has no zeros for

$$\phi = \frac{2\pi}{2+\alpha}$$

Proof: There we can only use the limits $(0, t)$:

$$\text{Im}(u^* u') = \int_0^t dt' |u|^2 \left\{ \frac{\text{Im } \lambda}{|t'|^2} - |E| \sin\left(\frac{4\pi}{2+\alpha} + \text{Arg } E\right) \right\}. \quad (176)$$

According to (174) and condition $\alpha < 2$, we have

$$\pi < \frac{4\pi}{2+\alpha} < \frac{4\pi}{2+\alpha} + \text{Arg } E < \frac{(4+\alpha)\pi}{2+\alpha} < 2\pi$$

and the integrand in the above expression is positive. Finally we notice

Property 6: u has no zeros for $|t|$ large enough, and

$$|\phi| < \min \left\{ \pi, \frac{3\pi}{2+\alpha} \right\}$$

Proof: This follows from the remark that $u/u_0 \rightarrow 1$ in this sector and that u_0 has no zeros.

Step 3: Continuation in λ : We vary λ continuously starting from real $\lambda > -1/4$ and taking for instance the case of $\text{Im } \lambda > 0$. Initially from property 6 we know that in the angle $|\phi| < \min [3\pi/(2+\alpha), \pi]$ if $\alpha < 4$, $|\phi| < \pi/(2+\alpha)$ if $\alpha > 2$, $u(r)$ has only zeros on the positive real axis, $\phi = 0$, the number of zeros (not including $z = 0$) being equal to $u - 1$ for the n^{th} level.

Now we distinguish two cases:

i) $\alpha > 2$. If we make λ complex there cannot be any zero on the lines $\phi = \pm\pi/(\alpha+2)$ from properties 3 and 4 and no zero can come from infinity in the sector $|\phi| < \pi/(\alpha+2)$, from property 6. Since the zeros are continuous functions of λ ,

their number inside the sector $|\phi| < \pi/(\alpha + 2)$ is equal to what it was initially, for real λ . If we return to real λ , we get back to the same energy level, since the number of zeros has not changed. Therefore there are no branch points in the λ complex plane cut from $-\infty$ to $-\frac{1}{4}$.

ii) $\alpha < 2$. Take for instance $\text{Im } \lambda > 0$. There are no zeros in $-\pi/(\alpha + 2) < \phi \leq 0$, no zeros on the line $\phi = 2\pi/(2 + \alpha)$, no zeros at infinity in the sector $0 < \phi < \min \{3\pi/(2 + \alpha), \pi\}$. Therefore again the number of zeros in the sector $-\pi/(\alpha + 2) < \phi < 2\pi/(\alpha + 2)$ remains the same as for λ real and continues to characterize the energy levels. Again we conclude that $E(\lambda)$ has no branch points in the λ complex plane, cut from $-\infty$ to $-\frac{1}{4}$.

We conclude with:

Theorem 10: The energy levels $E_n(\lambda)$ in a potential $V = r^\alpha$ can be continued for complex λ . The functions $E_n(\lambda)$ are analytic in $\text{Re } \lambda > -\frac{1}{2}$.

Remarks: We have also the property

$$\frac{\partial_{\text{Im}} E_n(\lambda = \ell(\ell+1))}{\partial_{\text{Im}} \lambda} > 0$$

which indicates that $E_n(\lambda)$ is a Herglotz function, which can be written as

$$E_n(\lambda) = A + B\lambda + \frac{2}{\pi} \int_{-\infty}^{-\frac{1}{4}} \frac{d\lambda' \partial_{\text{Im}} E_n(\lambda')}{\lambda'(\lambda' - \lambda)} \quad (177)$$

with $\text{Im } E_n > 0$. It is easy to check that this fits with the concavity of E_n with respect to $\ell(\ell + 1)$ mentioned in section 2.2

However, we have

$$|\text{Arg } E| < \frac{\alpha \pi}{2 + \alpha}$$

If $\alpha < 2$, this means $|\text{arg } E| < \pi/2$. We conclude that $[E_n(\lambda)]^2$ is also a Herglotz function, and therefore $E_n(\lambda)$ increases at most like $\lambda^{1/2}$ if $\alpha \leq 2$. Since $(\lambda + \frac{1}{4})^{1/2} = \ell + \frac{1}{2}$, it is tempting to speculate that for $\alpha \leq 2$, $E_n(\ell)$ is itself a Herglotz function. However, we would need to establish the analyticity of $E_n(\ell)$ for $\text{Re } \ell < -\frac{1}{2}$ and this is difficult. The Herglotz property of $E_n(\ell)$ for $\alpha < 2$ fits with our proposition 2 on the concavity of $E_n(\ell)$ with respect to ℓ [46].

Conclusion

It would be very desirable to extend these results to superpositions of powers, at least with positive weights. However, it is a weakness of our approach that it cannot handle these superpositions, because the various critical lines in the z complex plane change with α .

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Table 1

Minimum of the integrand in eq. (67)

n	r (posn. of minimum)	I _n	
		Numerical	Asymptotic estimate
7	1.52	0.00387	0.00421
8	1.44	0.00287	0.00309
9	1.37	0.00222	0.00237
10	1.31	0.00177	0.00187
20	1.01	0.00041	0.00042

Table 2

Ratio of $c_{n,\ell}^2$ [see eq. (75)] to their
Thomas-Fermi approximation values for the
potential $V(r) = \omega^4 r^2/4$

	n = 1	n = 2	n = 3	n = 4
$\ell = 0$	1.023	1.005	1.002	1.001
$\ell = 1$	0.951	0.985	0.993	0.996
$\ell = 2$	0.820	0.927	0.961	0.976

Figure captions

- Fig. 1 : Present experimental situation of the J/ψ family and most likely quantum number attributions.
- Fig. 2 : The T family with data from the Fermilab and DESY experiments.
- Fig. 3 : Example of evidence for the T as seen in e^+e^- collisions.
- Fig. 4 : Evidence of the T' in e^+e^- collisions.
- Fig. 5 : Level ordering for Coulomb and harmonic oscillator potentials.
- Fig. 6 : Sign of $E_{2S} - E_{1P}$ for $V = -(1/r) + \lambda r^\delta$.
- Fig. 7 : Sign of $E_{2S} - E_D$ for $V = -(1/r) + \lambda r^\delta$.
- Fig. 8 : $E_1(\ell)$ for $V = r^\alpha/4$, $0.5 < \alpha < 2$. One notices the concavity of the curves.
- Fig. 9 : $|\psi_n(0)|$ $n = 1, 2, 3, 4$ for $V(r) = -(\alpha/r) + (1 - \alpha)r$. Notice the absence of crossings.
- Fig. 10 : $\ln |\psi_3(0)/\psi_2(0)| / \ln |\psi_2(0)/\psi_1(0)|$ for $V = r^\alpha$, $-1 < \alpha \leq 3$.

Fig. 2

Fig. 3

Fig. 4

$$V(r) = -1/r$$

$$V(r) \propto r^2$$

Fig. 5

Sign of $E_{2S} - E_P$ for $V = -1/r + \lambda r^\delta$

Fig. 6

Sign of $E_{2S} - E_D$ for $V = -1/r + \lambda r^\delta$

Fig. 7

Fig. 8

Fig. 9

Fig. 10