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CONTRIBUTIONS OF RENORMALON SINGULARITIES

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ABSTRACT

On the basis of some simple examples we suggest a general method for discussing the contribution of renormalons to the Borel transform based on the general analytic structure of the Gell-Mann-Low β function. No simple principle value or contour integral Borel representation seems to be possible. Instead a splitting of the Borel transform at each renormalon singularity seems to be necessary making the infinite sum over renormalons possibly difficult to define.

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1. INTRODUCTION

It is gradually being recognized that "renormalon" singularities of the Borel transform¹⁾⁻⁵⁾ play a fundamental role in the construction of renormalizable field theories in four dimensions. It is well known that the usual perturbation expansion in quantum field theory is a divergent asymptotic series resulting from a saddle-point expansion of a (functional) integral:

$$F(g) = \int d[u] e^{-S[g,u]} \quad (1)$$

The natural way to compensate for the illegality of making the perturbation expansion inside the integral (and to try to reconstruct F from the perturbation series) is to go to the Borel representation

$$F(g) = \int_0^\infty db e^{-\frac{b}{g}} B(b) \quad (2)$$

where $B(b)$ is (hopefully) an analytic function for $b \sim 0$ and so in this region can be computed unambiguously from the perturbation series for F . The one-dimensional integration in (2) can be thought of as, essentially, an integral over the action S in (1). The simplest situation is that S has no stationary points for positive g . B then has no singularities on the positive real axis, the integral (2) is well-defined and the perturbation series for F is Borel summable. The next most complicated situation is that S has stationary points (instanton solutions of the classical equations of motion) which give singularities of B on the positive real axis. It is currently being argued^{1),6),7)} that this can be taken into account by defining a modified Borel representation

$$F(g) = \int \underbrace{db}_{\text{contour}} e^{-\frac{b}{g}} \tilde{B}(b) \quad (3)$$

where now the integration is a contour integral around the real axis and the discontinuities of $\tilde{B}(b)$ can be unambiguously associated with the stationary points of S .

The most complicated situation is that the measure $d[u]$ in (1) is such that the integration between adjacent stationary points of S cannot be adequately represented by a smooth integral over the action as implied by the Borel representation. This happens near $g = 0$, due to the renormalon singularities. The measure $d[u]$ is in principle defined there by the

renormalized perturbation expansion. It can happen that the divergence of this series is so bad that $B(b)$ must of necessity have singularities on the positive real axis. These are the renormalon singularities. When the theory is renormalizable but not asymptotically free the renormalons are associated with ultra-violet divergences of the perturbation expansion, while for asymptotically free theories they are associated with similar infra-red divergences. The above divergences of the perturbation series can be thought of as the relic of divergences that are present in individual diagrams, in higher dimensions for ultra-violet divergences, or lower dimensions for infra-red divergences. They can be seen^{4),5)} by going to the Schwinger-Dyson equations for the theory satisfied by the (formal) sum of the perturbation series.

The purpose of this paper is to consider whether a (further) modified Borel representation generalizing (2) may exist in the presence of renormalons and if so, what form it may take. Our analysis will be based on a study of some simple integrals in which we incorporate the essential features of the Schwinger-Dyson equations and what we show to be an essential feature of the β function, that is complex singularities (or more generally real singularities on the negative axis that can be treated as pairs of complex singularities. The case when β has a zero has to be treated separately). We find that the complex singularities of the β function, while not determining the position of the renormalon singularities, nevertheless, play an essential role in determining the form of the Borel representation. We should like to emphasize the independence of the renormalons from the Landau ghost problem^{5),8)}. The β functions we consider explicitly solve the Landau ghost problem as originally envisaged⁹⁾ but they still lead to the singularities in $B(b)$.

Before discussing our analysis further it may be useful to remark on the possible relevance of the renormalons. For non-asymptotically free theories such as $g\phi^4$ and QED it has recently been argued¹⁰⁾ (although our work will question the assumptions made) that the renormalons can produce an ultra-violet stable zero of the β function which otherwise appears to be absent. This zero is believed to be essential for the theory to exist. On the other hand, Parisi has argued^{4),5)} that the incorporation of renormalons into the theory can be seen as the addition of higher-order non-renormalizable local operators to the Lagrangian. That the ultra-violet behaviour of the theory remains tractable after this process seems hard to believe. Within the present state of knowledge therefore, the renormalons can be seen either as the saviour or the

destroyer of non-asymptotically free theories. Our conclusions will be, if anything, towards the negative point of view.

For asymptotically free gauge theories such as QCD it has been argued¹¹⁾ that the addition of instanton (or meron) contributions to the theory gives a valuable approximation (the dilute gas approximation) for discussing the formation of hadrons (or quark confinement). However, the renormalons imply that the definition of the measure $d[u]$ in (1) around any particular stationary point of S is in itself sufficiently complicated that it leads to concentration of field fluctuations at arbitrarily large values of S and so makes it meaningless to try to compute isolated contributions from instanton configurations. In fact, Parisi has shown⁵⁾ that the infra-red divergence problems found in computing instanton contributions in QCD are directly associated with the renormalon singularities. He further argued that a renormalon "gas" picture could replace the instanton gas picture and give a similar picture of confinement. Since the renormalon singularities are present in the $1/N$ expansion of $SU(N)$ Yang-Mills theory they could therefore provide the possibility of seeing confinement within this expansion which Witten¹²⁾ has argued is so desirable. Finally we note that infra-red renormalon singularities necessitates the addition to the theory of non-local operators, dual to local operators in a specific sense described by Parisi⁵⁾. They may therefore provide a possible route to understanding which non-local operators must be used to formulate the theory and why the quantum theory should display the electric-magnetic duality which Mandelstam has argued¹³⁾ is essential for confinement but which appears to be absent in the initial local Lagrangian.

The most direct analysis of the renormalons is Parisi's discussion^{4),5)} of the Borel-transformed Schwinger-Dyson equations for $g\phi^4$ starting from the lowest order solution of the renormalization group equations for the Green's functions of the theory. In the next section we briefly review Parisi's results giving additional arguments to confirm that the ultra-violet (or infra-red) behaviour of the Borel transform is controlled by the lowest-order term in the β function even when the theory is not asymptotically (or not infra-red) free. In Section 3 we consider in detail a simple integral which is essentially the lowest-order Schwinger-Dyson integral for the six-point function. We insert four-point functions obtained by solving the renormalization group equation with what we argue is a sufficiently realistic β function to enable us to make some general conclusions. These we discuss in Section 4. We find that there is no

simple contour or principle value integral representation [as assumed in Ref. 10)] which would allow the sum over all renormalon singularities to be clearly well-defined, and argue that this is further evidence of the importance of the renormalons in determining the existence of a theory which is not asymptotically free of the need to redefine the vacuum in an asymptotically free theory.

2. GENERAL FORMALISM

The renormalization group equations for the Green's functions Γ_N of $g\phi^4$ are (in Parisi's notation⁵⁾)

$$\left[-p \frac{\partial}{\partial p} + \beta(g) \frac{\partial}{\partial g} + d_N + \delta_N(g) \right] \Gamma_N(p, g) = 0 \quad (4)$$

The Borel transform of Γ_N

$$C_N(b, p) = \frac{1}{2\pi i} \oint_{\Omega} \frac{dg'}{g'^2} e^{b/g'} \Gamma_N(p, g') + \Gamma_N(p, 0) \delta(b) \quad (5)$$

(Ω is a contour integral around the origin) satisfies

$$\begin{aligned} & \left[-p \frac{\partial}{\partial p} + \beta_2 b + d_N \right] C_N(b, p) + \\ & + \int_0^\infty [\tilde{\beta}(b-b') b' + \tilde{\delta}_N(b-b')] C_N(b', p) db' = 0 \end{aligned} \quad (6)$$

where $\beta_2 > 0$ (that is the theory is not asymptotically free).

$$[\beta(g) - \beta_2 g^2]/g^2 = -g\beta_3 + O(g^2) \quad (7)$$

$$= \int_0^\infty e^{-b/g} \tilde{\beta}(b) db$$

$$\delta'_N(g) = -\delta'_N g + O(g^2) = \int_0^\infty e^{-b/g} \tilde{\delta}'_N(b) db \quad (8)$$

Keeping only β_3 and δ'_N in defining $\tilde{\beta}$ and $\tilde{\delta}'_N$ and solving (6) gives

$$c_N(b, p) \underset{p \rightarrow \infty}{\sim} r_N p^{d_N} \left(\frac{p}{\mu}\right)^{\beta_2 b} \left(\ln \frac{p}{\mu}\right)^{-\frac{\beta_3 b + \delta'_N}{\beta_2}} \quad (9)$$

where μ is a renormalization point. Thus β_3 and δ'_N lead to only a logarithmic deviation from the power behaviour obtained by dropping the integral term in (6) altogether.

We can understand why the power-behaviour of c_N is controlled by β_2 under fairly general circumstances as follows. Consider the simplest case of the momentum dependent coupling constant $\bar{g}(t)$, $t = \ln(p/\mu)$, which satisfies

$$\frac{\partial \bar{g}}{\partial t} = \beta(\bar{g}) \quad (10)$$

and write

$$\beta(\bar{g}) = \frac{\beta_2 \bar{g}^2}{1 + R(\bar{g})} \quad (11)$$

where

$$R(\bar{g}) \underset{\bar{g} \rightarrow 0}{\sim} \bar{g} \quad (12)$$

Defining

$$G = \frac{g}{\bar{g}} \quad (13)$$

we have

$$\frac{\partial}{\partial t} \left(\frac{G}{g} \right) = - \frac{\beta_2}{1 + R\left(\frac{g}{G}\right)}, \quad G(g, t=0) = 1 \quad (14)$$

which if we write $\tau = \beta_2 g t$ becomes

$$\frac{\partial G}{\partial \tau} = - \frac{1}{1 + R\left(\frac{g}{G}\right)}, \quad G(g, \tau=0) = 1 \quad (15)$$

from which we see that a singularity of $\beta(\bar{g})$ at some finite $\bar{g}$ implies a singularity in Equation (15) when $G \sim g$, and so implies that (for small g) the change in G is $O(-1)$ before the singularity is reached. Also $\partial G / \partial \tau \sim -1$ for small τ , and so we must have a change in τ of $O(1)$. The singularity of β , which must in turn be a singularity of $\bar{g}(t, g)$, therefore occurs at

$$t \sim \frac{1}{\beta_2 g} \quad (16)$$

A singularity of $\bar{g}$ at (16) will also be generated when β is regular and G vanishes (as in the case $\beta = \beta_2 g^2$). Returning to (5) with Γ_N replaced by $\bar{g}$ and considering $p \rightarrow \infty$, we see that moving the contour Ω out to $|g'| \sim 1/\beta_2 t$ gives

$$C(b, t) = \frac{1}{2\pi i} \oint \frac{dg'}{g'^2} e^{\frac{\beta}{g'}} \bar{g}(g', t) \quad (17)$$

$$\underset{t \rightarrow \infty}{\sim} e^{\beta_2 b t} * (\text{powers of } t)$$

In the next section we shall see this argument applied to a particular example. Clearly it generalizes to a general Γ_N .

The convolution theorem for the Borel transform⁴⁾ ensures the consistent reproduction of the factor $(p/\mu)^{\beta_2 b}$ in (9) when inserted into the Borel transformed Schwinger-Dyson equations. Therefore the ultra-violet region of all loop integrations in these equations involves this factor multiplied by the canonical dimension momentum powers of the Green's functions involved. As an example consider the contribution of Fig. 1 to the equation for the six-point function. Since the propagator and four-point function have canonical dimensions -2 and 0 respectively the ultra-violet behaviour of the loop-integral after Borel transform is

$$\int \frac{d^4 q}{(q^2)^3} q^{\beta_2 b} \sim \int q^{(\beta_2 b - 3)} dq \quad (18)$$

which diverges at $\beta_2 b = 2$. Similarly Fig. 2 gives a divergence at $\beta_2 b = 4$ in the eight-point function and in general there will be singularities at $\beta_2 b = 2n$, $n = \text{integer}$. The convolution theorem then implies that all such renormalon singularities are present in all the $C_N(b, p)$.

The further use of the renormalization group and also the $1/N$ expansion to study the nature of the renormalon singularities has been discussed by Parisi^{4), 5)}. Our purpose is now to study how the higher-order terms in the β function which enter $R(\bar{g})$ in (11) determine the way that $\Gamma_N(g, p)$ must be reconstructed from its $C_N(g, p)$. That is, how the form of a modified Borel representation is determined by the higher-order terms. In the next section, therefore, we consider the insertion of model four-point functions into (18).

Before leaving the general formalism we note that if β_2 is negative (or $\beta_2 = 0$ and $\beta_3 < 0$) so that the theory is asymptotically free, then the above arguments can be repeated for the infra-red region (in a massless theory). The problem of unravelling the infra-red divergences in a gauge theory is, however, much more complicated^{1),5)}.

3. SIMPLE EXAMPLES

Consider now the β function

$$\beta = \frac{\beta_2 g^2}{1 + \gamma_2 g^2} \quad (19)$$

This has the general properties:

i) for g real and positive, β is real which is required on general grounds of real analyticity for Green's functions;

ii) for $g \rightarrow \infty$, β is finite which eliminates the Landau ghost (strictly) the Landau ghost is eliminated whenever $\beta \sim g^\alpha$, with $\alpha \leq 1$;

iii) β has no real singularities or zeros. The absence of zeros is probably necessary for positive g if we wish to apply our arguments also to confining gauge theories. Singularities for negative g are allowed but can be treated similarly to the complex singularities of the above example, as we shall see.

Any β function with properties (i)-(iii) must, of course, have complex (conjugate) singularities. By considering first the effect of the simple singularities of β at $g = \pm i\gamma_2^{-1/2}$ in (19) we shall be able to see how the general singularity structure of a large (and we think realistic) class of β functions will manifest itself in a general discussion of renormalons.

With the above form for β , equation (10) can be explicitly solved for $\bar{g}$. Writing

$$g \rightarrow \frac{1}{\sqrt{\gamma_2}} g, \quad t \rightarrow \frac{\sqrt{\gamma_2}}{\beta_2} t \quad (20)$$

we obtain

$$\frac{d\bar{g}}{dt} = \frac{\bar{g}^2}{1+\bar{g}^2} \quad (21)$$

which has the solution

$$\bar{g} = \frac{1}{2g} \left\{ [(1-gt-g^2)^2 + 4g^2]^{\frac{1}{2}} - (1-gt-g^2) \right\} \quad (22)$$

Note that the singularities of $\bar{g}$ are at $\bar{g} = \pm i$ and for large t are at

$$g_s = -(t-2i), -(t+2i), \frac{1}{t+2i}, \frac{1}{t-2i} \quad (23)$$

in agreement with the general argument of the previous section. In particular, the displacement of the singularities from the real axis is determined by the imaginary parts of the corresponding singularities in $\beta(g)$. In general therefore, if the singularities of $\bar{g}$ are displaced in pairs from the real axis for finite t , it must be possible to give a corresponding $i\epsilon$ prescription to the singularities of $\beta(g)$ for negative g . [In fact, to see the effect of a negative g singularity we can take the (unrealistic) example of $\beta = 1/(1+g)$, in which case $g_s = -1 \pm i\sqrt{2t}$, as would result from taking $\beta = \frac{1}{2}\{(1+g+i\epsilon)^{-1} + (1+g-i\epsilon)^{-1}\}$].

We now apply (17) with $\bar{g}$ given by (22). Writing $h' = 1/g'$, we obtain

$$C(\ell, t) = \frac{1}{2\pi i} \int_{\mathcal{C}'} \frac{dh'}{h'} e^{\beta h'} \left\{ [(1+th'-h'^2)^2 + 4h'^2]^{\frac{1}{2}} + (1+th'-h'^2) \right\} \quad (24)$$

where the contour Ω' encloses the branch cuts as shown in Fig. 3. As anticipated the limit $t \rightarrow \infty$ is dominated by the branch points at $h_s = t \pm 2i$. To obtain a simple closed form for the asymptotic behaviour of this particular example it is actually more convenient to join the points $t + 2i$, $t - 2i$ to form one cut, in which case we obtain (after making the rescaling (20))

$$\begin{aligned} C(b, t) \underset{t \rightarrow \infty}{\sim} \bar{C}(b, t) &= e^{\beta_2 b t} \frac{2}{\pi} \int_{-1}^1 d\alpha (1-\alpha^2)^{1/2} \cos(2b\alpha\sqrt{x_2}) \\ &= e^{\beta_2 b t} \frac{1}{\sqrt{x_2} b} J_1(2b\sqrt{x_2}) \end{aligned} \quad (25)$$

[Although in the integral Eq. (25) the point $\alpha = 0$ is included, it is important that it has zero weight, since as we shall see the oscillatory behaviour for large b obtained from the parts of the integration region with $\alpha \neq 0$, plays an important role]. For large b , (25) gives

$$\bar{C}(b, t) \underset{b \rightarrow \infty}{\sim} \sqrt{\frac{2}{\pi}} e^{\beta_2 b t} (b\sqrt{x_2})^{-3/2} \sin(2b\sqrt{x_2}) \quad (26)$$

We now note several points. First we see that the limit $\gamma_2 \rightarrow 0$ cannot be taken in (26). However, (26) is manifestly a sum of two terms having oscillatory behaviour in b . This is true of the finite b result (25), if the integration region is split into the regions $-1 < \alpha < 0$, $0 < \alpha < 1$. If we had worked directly with the cuts of $\bar{g}$ drawn in Fig. 3, then a similar splitting of Eq. (24) would have been simply a consequence of the occurrence of complex conjugate pairs of singularities of $\bar{g}$ away from the real axis. The result (25) is in agreement with (9) and shows that in this case the higher order terms in β affect only the coefficient of the exponential behaviour determined by β_2 .

If we keep only the high t result of $C(b,t)$ and insert this back into the Borel representation for $\bar{g}(g,t)$ we obtain

$$\bar{g}(gt) = \frac{2}{\pi} \int_0^1 d\alpha (1-\alpha^2)^{1/2} \int_0^\infty db e^{-\frac{b}{g}} * \quad (27)$$

$$[e^{b(\beta_2 t + 2i\alpha\sqrt{\gamma_2})} + e^{b(\beta_2 t - 2i\alpha\sqrt{\gamma_2})}]$$

$$= \frac{4}{\pi} \int_0^1 d\alpha (1-\alpha^2)^{1/2} \frac{g(1-\beta_2 gt)}{(1-\beta_2 gt)^2 + 4\alpha^2 \gamma_2 g^2} \quad (28)$$

Note that the high t limit of (28), namely $-(1/\beta_2 t)$, is not the high t limit of the original $\bar{g}$ given by (22), which is $(\beta_2/\gamma_2)t$. This is a particular example of the general point emphasized by Parisi, that to obtain the high t behaviour of Green's functions it is not sufficient to know just the high t behaviour of the Borel transform. However, for locating the positions of singularities associated with high t behaviour we can freely interchange limits and integrals.

As a model for the ultra-violet region of the Schwinger-Dyson integral of Fig. 1 with only one four-point function having momentum dependence we consider

$$\Gamma(g) = \int_0^\infty dt e^{-t} \bar{g}(gt) \quad (29)$$

For our initial discussion it will be sufficient to take (28) for $\bar{g}$. Note that with either (22) or (28), $\Gamma(g)$ is well-defined and non-singular for real g . Expression (28) is actually discontinuous near $\beta_2 gt = 1$, where it is equal to $(1/\sqrt{\gamma_2}) \text{sign}[1 - \beta_2 gt]$. This discontinuity comes from the region $\alpha \sim 0$ in (25), and again if we had used (24) with the cuts drawn as in Fig. 3, this problem would have been avoided. For simplicity, therefore, we take $\alpha = \frac{1}{2}$ only, in (27). Writing $\tau_1 = i\sqrt{\gamma_2}$ we have for the first term

$$\bar{g}_1 = \frac{\frac{1}{2}g}{1 - g(\beta_2 t + z_1)} \quad (30)$$

This gives

$$\Gamma_1(g) = \int_0^\infty dt e^{-t} \bar{g}_1(gt) \quad (31)$$

which has the formal perturbation series

$$\Gamma_1(g) = \sum_{N=0}^{\infty} g^{N+1} A_N \quad (32)$$

with

$$A_N = \frac{1}{2} \int_0^\infty dt e^{-t} (\beta_2 t + z_1)^N \quad (33)$$

$$= \frac{1}{2} \beta_2^N (N!) e^{\frac{z_1}{\beta_2}} + O\left(\frac{|z_1|^{N+1} e^{\frac{|z_1|}{\beta_2}}}{\beta_2}\right) \quad (34)$$

and since there is no oscillation of sign, the series (32) will not be Borel summable if we keep only the first term in (34). Alternatively, we expect this because of the singularity of $\bar{g}$ approaching the integration contour of (31) at $t = \infty$ within the right-half g plane. (More precisely, the pole at $t_s = 1/\beta_2 g - z_1/\beta_2$ has an imaginary part which is $O(g)$ of the real part).

Summing the contribution of the leading terms in (34) to the Borel transform of Γ we obtain

$$\zeta(b) = \sum_{N=0}^{\infty} \frac{A_N g^{N+1}}{N!} \quad (35)$$

$$\sim \frac{1}{2} e^{\frac{z_1}{\beta_2}} \frac{1}{1 - \beta_2 b} \quad (36)$$

Alternatively

$$\Gamma_1(g) = \int_0^\infty dt e^{-t} \int_0^\infty db e^{-b/g} \bar{c}_1(b, t) \quad (37)$$

where $\bar{c}_1(b, t)$ is the Borel transform of $\bar{c}_1$, that is $\frac{1}{2} e^{b(\beta_2 t + z_1)}$. Interchanging the orders of integration gives

$$\frac{1}{2} \int_0^\infty db e^{-b/g} \int_0^\infty dt e^{-t} e^{b(\beta_2 t + z_1)} \quad (38)$$

$$= \int_0^\infty db e^{-b/g} \frac{e^{bz_1}}{2} \frac{1}{1 - \beta_2 b} \quad (39)$$

which again shows the pole at $b\beta_2 = 1$. These last three equations are the essence of Parisi's general analysis of the Schwinger-Dyson equations. They show that Borel transforming the integrand and freely interchanging orders of integration does quickly uncover singularities of the Borel transform of the integral which can be more laboriously determined by direct computation as in (31)-(36). Note that the "large order" terms amount to replacing the b of e^{bz_1} by the pole value $1/\beta_2$.

Of course, (39) is not well-defined, since we must determine how to integrate around the pole. For $\Gamma_1(g)$ the answer is straightforward. Writing

$$\Gamma_1(g) = \int_0^\infty db e^{-b/g} \frac{e^{bz_1}}{2} \frac{1}{1 - (b + i\epsilon)\beta_2} \quad (40)$$

defining $b' = (1/g - z_1)b$, and deforming the integration contour to lie along the line b' real, we obtain the original definition (31). Similarly defining $\Gamma_2(g)$ from the second term in (27) we obtain

$$\Gamma_2(g) = \int_0^\infty db e^{-b/g} \frac{e^{-bz_1}}{2} \frac{1}{1 - (b-i\epsilon)\beta_2} \quad (41)$$

and for the complete $\Gamma(g)$ given by (28) and (29) we obtain

$$\Gamma(g) = \frac{1}{2} \int_0^\infty db e^{-b/g} \left\{ \frac{e^{bz_1}}{1 - \beta_2(b+i\epsilon)} + \frac{e^{-bz_1}}{1 - \beta_2(b-i\epsilon)} \right\} \quad (42)$$

$$= \int_0^\infty db e^{-b/g} \left\{ (\cos \sqrt{\tau_2} b) \text{P.V.} \frac{1}{1 - \beta_2 b} - \pi \delta(1 - \beta_2 b) (\sin \sqrt{\tau_2} b) \right\} \quad (43)$$

where P.V. = principal value.

To extend this last result to the complete $\bar{g}(g, t)$ given by (22) and to generalize it to more complicated functions we first write a dispersion relation in $1/g$

$$\bar{g}(g, t) = \int du' \frac{\rho_+(u', t)}{\frac{1}{g} - u'} + \int du' \frac{\rho_-(u', t)}{\frac{1}{g} - u'} \quad (44)$$

where ρ_+ is non-zero only for $\text{Im}(u') > 0$ and ρ_- is non-zero only for $\text{Im}(u') < 0$. The (naive) Borel transform of $\bar{g}$ is

$$C(b, t) = \int du' g_+ e^{bu'} + \int du' g_- e^{bu'} \quad (45)$$

The $t \rightarrow \infty$ limit of $C(b, t)$ will be controlled by the part of the integration region $u' \sim \beta_2 t$ (on which $\bar{g}$ will be singular because of the general argument of the previous section) giving

$$C(b, t) \underset{t \rightarrow \infty}{\sim} e^{\beta_2 b t} \int du'' e^{bu''} [\tilde{f}_+(u''t) + \tilde{f}_-(u''t)] \quad (46)$$

The generalization of (42) will therefore be

$$|f(g)| = \int_0^\infty e^{-\frac{b}{2}t} [B_+(b + i\epsilon) + B_-(b - i\epsilon)] db \quad (47)$$

where

$$B_\pm(b) = \int_0^\infty dt e^{(\beta_2 b - 1)t} \int du'' e^{bu''} \tilde{f}_\pm(u''t) \quad (48)$$

and so both B_+ and B_- are singular at $b = 1/\beta_2$. As is clear from (43), equation (47) would be a principal value integral only if the u'' integration over $\tilde{p}_+$ and $\tilde{p}_-$ separately were real, which we have argued is never the case in reality.

4. INTERPRETATION AND GENERALIZATION OF RESULTS

We now describe what we consider to be the general implications of the simple examples considered in the previous section. We began with complex singularities of the β function (19) at $\bar{g} = \pm i$. These then

gave rise to complex singularities of $\bar{g}(g,t)$, that is (23), which following the general argument of Section 2 approach $g = 0$ like $1/t$ as $t \rightarrow \infty$, with the sign of the imaginary part determined by that of the imaginary part of the original singularity in β . The singularities in $\bar{g}$ give exponential increase as $t \rightarrow \infty$ of the Borel transform of $\bar{g}$. The phases of the increasing terms are governed by the same imaginary parts. Finally, the exponential increase gives a singularity of the Borel transform of the Schwinger-Dyson integral and to obtain a Borel representation of the integral we must treat separately the parts of the integrand having respectively singularities in the upper and lower-half g plane. The final representation then has the form (47). Clearly this argument goes through if we replace $\bar{g}(g,t)$ by a general Γ_N or a product of Γ_N 's in the Schwinger-Dyson integral. Since the argument is based only on the presence of complex singularities in the β function it appears rather general.

We emphasize now that (43) or (47) is not a principal value integral of the kind assumed by Khuri¹⁰⁾. Nor does it seem likely that in general (47) can be written as a real part or contour integral of any single analytic function. In fact, it appears that as b increases, in order to incorporate each new renormalon singularity in the Borel representation it will be necessary to write a separate dispersion relation for the integrand of the relevant Schwinger-Dyson integral leading to a new splitting of the Borel transform as in (47). Therefore it appears from our analysis that in the absence of systematic cancellations the infinite set of renormalons requires an infinite splitting process to define a Borel transform. The convergence of such a procedure is certainly very far from obvious. We contrast this with the situation where just instantons are involved and a representation of the form (2) is claimed to hold. Such an equation is clearly well-defined even in the presence of an infinite set of singularities along the real axis.

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FIG.1

FIG.2

FIG. 3