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PERTURBATION THEORY FOR TWO-DIMENSIONAL GAUGE MODELS

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ABSTRACT

Euclidean symmetric integration, previously proposed for the $SU(N)$ Yang-Mills theory in the limit of large N , is used to study the propagators in the Schwinger model and the massive vector meson model in two dimensions. The result of summing the perturbation series agrees with the exact solution in each case. Therefore, with Euclidean symmetric integration, perturbation theory is capable of dealing with non-analytic behaviour in the coupling constant.

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1. INTRODUCTION

Recent developments in high-energy physics suggest that (1) Yang-Mills theory¹⁾ is correct with a suitable choice of the underlying group, and (2) non-perturbative effects²⁾ are important. It must be emphasized that the importance of non-perturbative effects does not mean that perturbation theory cannot be used; it only means that it is not sufficient to carry perturbation theory to any finite order. There are numerous examples in physics where a summation of the perturbation series gives useful results: one of the most famous ones is, perhaps, the treatment of the hard-sphere boson gas by Lee and Yang³⁾. More recently, the prediction of the increasing total cross-section⁴⁾ was also obtained by summing the perturbation series of quantum field theory.

In connection with the SU(N) Yang-Mills theory, several years ago 't Hooft⁴⁾ studied the most interesting limit of $N \rightarrow \infty$. More precisely, if g_0 is the Yang-Mills coupling constant, then the limit is $N \rightarrow \infty$ with $g_0^2 N$ fixed (say g^2). In this limit, only the planar diagrams need be kept. In the special case of one space and one time dimension, the use of an axial gauge removes all the Yang-Mills self-coupling, and the above $N \rightarrow \infty$ can be treated exactly. In particular, in the light-cone gauge the fermion self-energy $\Sigma(p)$ satisfies the integral equation:

$$\Sigma(p) = \frac{ig^2}{\pi^2} \int \frac{d^2 k}{k_-^2} \frac{k_- + p_-}{(k+p)^2 + m_0^2 - (k_- + p_-) \Sigma(k+p) - i0} \quad (1.1)$$

where $p_{\pm} = p_1 \pm p_0$ and m_0 is the bare fermion mass.

As it stands, Eq. (1.1) has no meaning, because of the appearance of the factor k_-^2 in the denominator. Two essentially different interpretations of Eq. (1.1) have been proposed:

1) Contour integration over k_+ (corresponding to an "instantaneous x^+ interaction"⁵⁾) followed by cut-off in k_- ⁵⁾ or principal value definition for $1/k_-^2$ ⁶⁾. The physical (gauge invariant) results are the same within these two procedures. However, in the first one the gauge invariant quantities (in particular the fermion propagator) are cut-off dependent, therefore we shall mainly refer to the second one, also called the "regular cut-off"⁶⁾. It leads to the self-energy⁶⁾

$$\Sigma(p) = \frac{g^2}{\pi^2} \frac{1}{p_-} \quad (1.2)$$

and thus to the dressed propagator

$$S'(p) = \frac{-i p_-}{p^2 + m_0^2 - g^2/\pi - i0} \quad (1.3)$$

2) Euclidean symmetric integration⁷⁾ defined via the Wick rotation:

$$k_0 = -i k_2 \quad ; \quad d^2 k = -i dk_1 dk_2 \quad (1.4)$$

leads⁶⁾ to

$$\Sigma(p) = \frac{2g^2}{\pi} \frac{p_+}{p^2 + m_0^2 + \sqrt{(p^2 + m_0^2)^2 - (4g^2/\pi) p^2}} \quad (1.5)$$

and

$$S'(p) = \frac{-2i p_-}{p^2 + m_0^2 + \sqrt{(p^2 + m_0^2)^2 - (4g^2/\pi) p^2}} \quad (1.6)$$

While the $1/N$ approximation sharply restricted the class of contributing diagrams, the first regularization procedure leads to an even more drastic cut: in fact, the self-energy Eq. (1.2) is given by the second order diagram alone which is the only one to survive out of the rainbow graphs.

These higher-order rainbow graphs are no longer zero under the "Euclidean symmetric integration", which correspondingly provides a more complicated structure for the propagator. For instance, for $m_0 = 0$, $S'(p)$ of Eq. (1.6) shows a finite cut at space-like momenta, while $S'(p)$ of Eq. (1.3) [which coincides with S' of Eq. (1.6) in second order] has a pole in this region at $p^2 = g^2/\pi$.

In view of the qualitative differences between these two procedures, it is desirable to apply them to simpler two-dimensional cases. In this paper, we apply the Euclidean symmetric integration⁷⁾ to the Schwinger model^{8),9)} and the massive vector meson model¹⁰⁾. We shall see that the propagators obtained in this way indeed agree with the exact answers. In this process, we shall also learn the capability of perturbation theory to deal with non-analytic behaviour in the

coupling constant. We note that there is a great deal of similarity between these models and the $SU(N)$ Yang-Mills theory in the axial gauge. Indeed, the Schwinger model is one of the earliest models that show confinement. Perturbation series for the Schwinger model has previously been considered¹¹⁾, but in these earlier works an unconventional gauge is used for the purpose of avoiding singular integrals being studied here and no explicit comparison with the exact solution is attempted.

After we briefly review the exact solution of the Schwinger model in Section 2, we calculate in Section 3 the lower-order diagrams and compare them with the expansion of the Green functions. In Section 4 we show that the perturbation series to all orders sums up back to the exact solution. In Section 5, it is shown how, by modifying slightly the previous analysis, we can obtain the solution of the vector meson mode. In Section 6 we consider the principal value prescription in the Schwinger model. Finally, in Section 7 we comment on some more general aspects of the infra-red behaviour of two-dimensional gauge theories. Some of the calculations are gathered in the Appendices.

We use the metric $g^{11} = -g^{00} = 1$ and the representation of the γ matrices with diagonal γ_5 .

2. THE OPERATOR SOLUTION OF THE SCHWINGER MODEL

The Schwinger model is described by the Lagrangian

$$\mathcal{L} = \frac{i}{2} \bar{\psi} \not{\partial} \psi + \frac{1}{2} \partial_\mu A^\nu \partial^\mu A_\nu + g \bar{\psi} A \psi \quad (2.1)$$

In approaching this model perturbatively we will be led to that solution which exhibits a fermion and a photon. This is therefore the so-called covariant solution, which in the Lorentz gauge, was discussed in Ref. 12):

$$\psi = : e^{i \gamma_5 \sqrt{\pi} (\tilde{\eta} + \tilde{\Sigma})} \psi_f : \quad (2.2)$$

$$A_\mu = -\frac{1}{m} \epsilon_{\mu\nu} \partial^\nu (\tilde{\eta} + \tilde{\Sigma}) \quad (2.3)$$

It is written in the canonical free fields $\psi_f, \tilde{\eta}, \tilde{\Sigma}$ (a fermion and two pseudo-scalars):

$$i \not{\partial} \psi_f = 0 ; \quad \square \tilde{\eta} = 0 ; \quad (-\square + m^2) \tilde{Z} = 0 ; \quad m^2 = \frac{g^2}{\pi} \quad (2.4)$$

$\tilde{\eta}$ being quantized with indefinite metric¹²⁾. A_μ is transverse and has the degrees of freedom of a photon. Only in two dimensions is such a construction possible.

This solution is valid on a subspace of the Hilbert space, where

$$\left\langle \frac{1}{\sqrt{\pi}} : \bar{\psi}_f \gamma^\mu \psi_f : - \partial^\mu \tilde{\eta} \right\rangle = 0 \quad (2.5)$$

which thus defines the physical states. It shows ultra-violet freedom and non-trivial infra-red behaviour.

The light cone formulation of the theory was given in Ref. 13). The solution reached through a local gauge transformation from Eqs. (2.2), (2.3) is:

$$\psi = : e^{i(1+\gamma_5)\sqrt{\pi}(\tilde{\eta} + \tilde{Z})} \psi_f : \quad (2.6)$$

$$A_- = 0 ; \quad A_+ = -\frac{4}{m} \partial_+ (\tilde{\eta} + \tilde{Z})$$

and promotes one of the components of the fermion as the only dynamical variable.

The fermion two-point function is

$$S'(x) = S(x) e^{2 \mathcal{D}(x)} \quad (2.7)$$

and the photon propagator is given by the "++" component of the tensor

$$D'_{\mu\nu}(x) = \left(-g_{\mu\nu} + \frac{\partial_\mu \partial_\nu}{\square^2} \right) \mathcal{D}(x) \quad (2.8)$$

with

$$\mathcal{D}(x) = K_0(m\sqrt{x^2+i0}) + \ln\left(\frac{m}{2}\sqrt{x^2+i0}\right) - \gamma \quad (2.9)$$

where K_0 is the modified Bessel function, $\gamma \approx 0.5772$ is the Euler constant and $S(x)$ is the free fermion propagator in the light cone gauge.

The ultra-violet free and the infra-red singular behaviour is evident from the properties of the propagator

$$S'(x) \simeq \begin{cases} S(x) & \text{for } x^2 \rightarrow 0 \\ \frac{m^2}{4} e^{-2\gamma} x^2 S(x) & \text{for } x^2 \rightarrow \infty \end{cases} \quad (2.10)$$

In momentum space we have

$$S'(p) = S(p) \sqrt{p^2 - i0} \int_0^\infty dg \, \gamma_2(g \sqrt{p^2 - i0}) e^{2D(g)} \quad (2.11)$$

with

$$D(g) = K_0(mg) + \ln\left(\frac{1}{2} mg\right) - \gamma \quad (2.12)$$

and

$$D'_{\mu\nu}(p) = \frac{-g_{\mu\nu} + \frac{p_\mu p_\nu}{p^2}}{p^2 + m^2} \quad (2.13)$$

As it is clear also from Eq. (2.10), $S'(p)$ must still be defined in the sense of distribution at $p^2 = 0$. In what follows we shall always understand $p^2 \neq 0$.

For our purposes we need $S'(p)$ up to fourth order. $D(p)$ of Eq. (2.12) goes like $(mp)^2 \ln(mp)$ for small mp (remember $m^2 = g^2/\pi$). By expanding the exponential in Eq. (2.11) we get from the first three terms (see Appendix A)

$$S'(p) \simeq S(p) \left\{ 1 + \ln\left(1 + \frac{m^2}{p^2}\right) + \frac{1}{2} \left[\left(\ln \frac{\sqrt{1 + 4m^2/p^2} + 1}{\sqrt{1 + 4m^2/p^2} - 1} \right)^2 + \left(\ln \frac{m^2}{p^2} \right)^2 - 2 \left(\ln\left(1 + \frac{p^2}{m^2}\right) \right)^2 \right] \right\} \quad (2.14)$$

By further expanding the logarithms in m^2/p^2 we get from Eq. (2.14)

$$S'(p) \simeq S(p) \left\{ 1 + \frac{m^2}{p^2} + \frac{m^4}{p^4} \left(\frac{1}{2} + 2 \ln \frac{m^2}{p^2} \right) + \mathcal{O}(m^6 \ln^2 m) \right\} \quad (2.15)$$

We remark that the Lorentz gauge propagator is to be obtained from Eq. (2.7) by replacing the 2 in the exponent by $\frac{1}{2}$ and using the Lorentz gauge $S(p)$. It is clear that the expansion Eq. (2.15) is meaningful only for $p^2 \neq 0$ and it breaks down for $p^2 \rightarrow 0$.

3. LOWER ORDER PERTURBATION CALCULATIONS

The Feynman rules for a gauge theory in the light cone gauge are simplified due to the γ matrices' algebra. They are essentially the same for two-dimensional quantum electrodynamics and for the two-dimensional $SU(N)$ Yang-Mills theory in the $1/N$ approximation (see Fig. 1):

$$\begin{aligned} S(p) &= \frac{1}{(2\pi)^2} \frac{-i p_-}{p^2 + M^2 - i0} \\ D(p) &= \frac{4}{(2\pi)^2} \frac{-i}{p_-^2} \\ \text{vertex} &: (2\pi)^2 i g \end{aligned} \tag{3.1}$$

where $p_{\pm} = p_1 \pm p_0$ and a factor -1 has to be taken for each fermionic loop. In the Schwinger model the fermion is massless ($M = 0$) while the $SU(N)$ model was studied for the general mass case. In what follows we have to take $M = 0$ in the Feynman rules Eq. (3.1). We shall discuss the relevance of a fermionic mass in Section 7.

The integrals with photon propagator are defined in the Euclidean symmetric integration as

$$\int \frac{d^2 k}{k_-^2} \dots = \frac{1}{i} \int \frac{dk_1 dk_2}{(k_1 + i k_2)^2} \dots \tag{3.2}$$

where symmetric integration is used for the variables k_1 and k_2 . We then get (see Fig. 2 and Appendix B):

$$(a) = -4g^2 \int \frac{d^2 k}{(k_- - p_-)^2} \frac{1}{k_+ + i0} = -4g^2 \frac{1}{i} \int_0^\infty dk_1 \int_0^{2\pi} d\theta \frac{e^{i\theta}}{(ke^{i\theta} - pe^{i\alpha})^2} = \frac{g^2}{\pi} \frac{S(p)}{p^2}$$

$$\begin{aligned}
 (b) &= -g^2 \int \frac{d^2 k}{k_+ (k_+ + p_+)} = -\frac{g^2}{\pi} \frac{D^{-1}(p)}{p^2} \\
 (c) &= 2 \frac{g^3}{\pi} i \frac{1}{q_+} \left(\frac{1}{p_- + q_-} - \frac{1}{p_-} \right) \\
 (d) &= \frac{4g^4}{\pi^2} i \int \frac{d^2 k d^2 q}{k_-^4} \frac{1}{(p_+ - k_+) q_+ (k_+ + q_+)} = -\frac{g^4}{2\pi^2} \frac{S^{-1}(p)}{p^4} \\
 (e) &= -\frac{4g^4}{\pi^2} i \int \frac{d^2 k d^2 q}{k_-^2 q_-^2} \frac{1}{(p_+ + k_+)^2 (p_+ + k_+ + q_+)} = \frac{g^4}{\pi^2} \frac{S^{-1}(p)}{p^4} \\
 (f) &= -\frac{4g^4}{\pi^2} i \int \frac{d^2 k d^2 q}{k_-^2 q_-^2} \frac{1}{(p_+ + k_+)(p_+ + k_+ + q_+)(p_+ + q_+)} \text{ is divergent.}
 \end{aligned} \tag{3.3}$$

As perturbation theory with the Feynman rules Eq. (3.1) cannot reproduce the $\ln(m^2/p^2)$ of Eq. (2.15), consistency requires a divergence to appear in fourth order, and this comes in the graph (f) in Eq. (3.3) and Fig. 2. To get control of this divergence, we must introduce a regulator mass. However, such a mass naturally appears in the theory. As can be seen from (b) in Eq. (3.3), the vacuum polarization is singular. On the other side, as explicitly shown by Georgi and Rowles¹⁴⁾, the higher radiative corrections cancel and thus the full photon propagator is obtained through iteration of the second order graph (b) in Eq. (3.3) and Fig. 2.

With

$$m^2 = g^2/\pi \tag{3.4}$$

we then have

$$D'_{\mu\nu}(p) \cong \frac{-g_{\mu\nu} + \frac{p_\mu p_\nu}{p^2}}{p^2 + m^2} \quad (3.5)$$

(the symbol $\cong$ means that we skip some constant factors). This corresponds to the exact Green function Eq. (2.13). In the light cone gauge this reads

$$D'(p) = \frac{4}{(2\pi)^2} \frac{-i p_+}{p_- (p^2 + m^2)} \quad (3.6)$$

We then modify the Feynman rules Eq. (3.1) and Fig. 1 by replacing D by D' of Eq. (3.6) and obtain for the diagram (f) in Eq. (3.3) and Fig. 2 to lowest order in m the finite expression

$$(f) = \frac{g^4}{\pi^2} \frac{S^{-1}(p)}{p^4} \left(-1 + 2 \ln \frac{m^2}{p^2} \right) \quad (3.7)$$

Adding together the diagrams we get for the fermion propagator in fourth order

$$S^{(4)}(p) = S(p) \left\{ 1 + \frac{g^2}{\pi} \frac{1}{p^2} + \frac{g^4}{\pi^2} \frac{1}{p^4} \left(\frac{1}{2} + 2 \ln \frac{m^2}{p^2} \right) \right\} \quad (3.8)$$

which by Eq. (3.4) is seen to agree explicitly with the expansion (2.15) of the exact Green function. We see that the divergence of the graph (f) in Eq. (3.3) and Fig. 2 calculated with $D(p)$ was indeed related to the logarithmic non-analyticity for $m \rightarrow 0$ in Eqs. (2.15) and (3.8).

4. PERTURBATIVE SOLUTION OF THE SCHWINGER MODEL

Encouraged by the success in fourth order of perturbation theory with Euclidean symmetric integration, we proceed here to investigate the higher order terms. We shall see in this section that the exact solution of the Schwinger model can be obtained in this way.

It is convenient to introduce the notation

$$\begin{aligned} p_+ &= p_1 + p_0 = p_1 - i p_2 = \bar{z} ; \\ p_- &= p_1 - p_0 = p_1 + i p_2 = \bar{\bar{z}} ; \quad p^2 = |\bar{z}|^2 = r^2 \end{aligned} \quad (4.1)$$

We first investigate the radiative corrections. The general graph of Fig. 4 corresponds to the loop integral

$$R_n(z_1, \dots, z_n) = \int d^2z \frac{1}{z+z_1} \dots \frac{1}{z+z_1+\dots+z_n} \quad (4.2)$$

with

$$z_1 + z_2 + \dots + z_n = 0$$

For any diagram containing the part R_n (see Fig. 3) one has to add all permutations of the vertices $1, 2, \dots, n$ modulo cyclic permutations. It has been shown by Georgi and Rawls¹⁴⁾, that for $n > 2$:

$$\sum_{P(1, \dots, n)} R_n(z_1, \dots, z_n) = 0 \quad (4.3)$$

[for $n = 2$ there is no sum and we get the known result (b) in Eq. (3.3)].

Now, for any class of diagrams we can add any number of second-order loops on the photon lines. Thereby 1) all the fermionic loops are taken into account and 2) nothing else can be put on the photon lines.

This leads to replacing anywhere the bare propagator D with the dressed one Eq. (3.6):

$$D(p) \rightarrow D'(p) \cong \frac{z/\bar{z}}{p^2 + m^2} \quad (4.4)$$

which thus reproduces the exact photon Green function of Eq. (2.13).

As the second step we consider an arbitrary $2n^{\text{th}}$ order diagram of the fermion propagator. Owing to the previous remark this is of the type shown in Fig. 4, where the photon lines correspond to $D'(p)$ of Eq. (4.4), and where we singled out the photon line of momentum z_n connecting the vertex $2n$ with the vertex k . We then have for this graph

$$\begin{aligned}
 S_n^k(z) &\cong \int d^2 z_1 \dots d^2 z_{n-1} d^2 z_n \frac{z_1 / \bar{z}_1}{r_1^2 + m^2} \dots \frac{z_{n-1} / \bar{z}_{n-1}}{r_{n-1}^2 + m^2} \frac{z_n / \bar{z}_n}{r_n^2 + m^2} \\
 &\times \frac{1}{z + Z_1} \dots \frac{1}{z + Z_{k-1}} \frac{1}{z + Z_{k-1} + z_n} \dots \\
 &= \int d^2 z_1 \dots d^2 z_{n-1} d^2 z_n \frac{z_1 / \bar{z}_1}{r_1^2 + m^2} \dots \frac{z_{n-1} / \bar{z}_{n-1}}{r_{n-1}^2 + m^2} \frac{1 / \bar{z}_n}{r_n^2 + m^2} \\
 &\times \frac{1}{z + Z_1} \dots \left(\frac{1}{z + Z_{k-1}} - \frac{1}{z + Z_{k-1} + z_n} \right) \dots
 \end{aligned} \tag{4.5}$$

where Z_j means some combination of the momenta not including z_n .

Taking out the vertices k and $2n$ and summing over all other distributions we get the $2(n-1)^{st}$ order S_{n-1} . Adding back the vertices k and $2n$ and summing over the positions of the vertex k we get the $2n^{th}$ order S_n . Thus

$$S_n = \sum_{(\text{all other distributions})} \cdot \sum_{(\text{positions of } k: 1, \dots, 2n-1)} \cdot S_n^k \tag{4.6}$$

But looking at the last line of the expression Eq. (4.5) for S_n^k , we see that the terms in the brackets compensate between neighbouring members of the inner sum in Eq. (4.6). In this way we are left with only two terms which are parts of the graphs with $k = 1$ and $k = 2n-1$. The last one is, however, zero, due to symmetric integration, and after completing with the outer free propagators we obtain the iteration equation:

$$z_0 S_n(z_0) = -i \frac{g^2}{\pi^2} \int \frac{d^2 z}{|z - z_0|^2 + m^2} \frac{1}{z - \bar{z}_0} S_{n-1}(z) \tag{4.7}$$

Written as an integral equation, with Minkowski momenta, this reads

$$p_+ S'(p) = p_+ S(p) - i \frac{g^2}{\pi^2} \int \frac{d^2 k}{(k-p)^2 + m^2} \frac{1}{k_- - p_-} S'(k) \quad (4.8)$$

which for the Fourier transform

$$S'(x) = \frac{1}{2\pi} \int d^2 p e^{ipx} S'(p) \quad (4.9)$$

leads to ($m = g/\sqrt{\pi}$):

$$x_+ \frac{\partial}{\partial x_+} S'(x) = x_+ \frac{\partial}{\partial x_+} S(x) + (1 - m\sqrt{x^2 + i0} K_1(m\sqrt{x^2 + i0})) S'(x) \quad (4.10)$$

It is easy to check that the exact Green's function, as given by (2.7) and (2.9), follows from this differential equation.

We have thus obtained a perturbative solution for the propagators of the Schwinger model in the light cone gauge which coincides with the (so-called covariant) operator solution. Note that the analysis in this section is essentially independent of gauge, due to the simple algebra of the massless fermion in two dimensions. With the corresponding modifications we can obtain in the same way the solution in the Lorentz gauge.

5. THE VECTOR MESON THEORY

If in the Lagrangian Eq. (2.1) we introduce a mass term

$$\frac{1}{2} \mu^2 A_\mu A^\mu \quad (5.1)$$

we shall be dealing with the vector meson model¹⁰⁾. The usual vector meson bare propagator is

$$\frac{-g^{\mu\nu} + \frac{k^\mu k^\nu}{\mu^2}}{k^2 + \mu^2} = \frac{-g^{\mu\nu} + \frac{k^\mu k^\nu}{k^2}}{k^2 + \mu^2} - \frac{\frac{k^\mu k^\nu}{\mu^2}}{k^2} \quad (5.2)$$

Due to the current conservation, the self-energy is transverse¹⁵⁾ and is given by the second-order graph (b) in Eq. (3.3), because the same argument concerning the higher radiative corrections as in Section 4 applies. Therefore only the transverse part in Eq. (5.2) gets modified by the interaction, and we find the dressed propagator

$$\frac{-g^{\mu\nu} + \frac{k^\mu k^\nu}{k^2}}{k^2 + \mu^2 + g^2/\pi} - \frac{k^\mu k^\nu}{\mu^2 k^2} \quad (5.3)$$

From Eq. (5.3) one can already see two features of the solution in this gauge:

- 1) it is no longer expressible in terms of only one massive vector field and
- 2) the limit $\mu \rightarrow 0$ can be taken only after rearranging with the help of an indefinite metric massless scalar, both of them known from the exact solutions [see Refs. 10), 12) and 16)]. In what follows we shall stick to the Feynman gauge, in order to directly compare with the results of Ref. 10).

With

$$D_{\mu\nu}(k) = \frac{i}{(2\pi)^2} \frac{g_{\mu\nu}}{k^2 + \mu^2 - i0} \quad (5.4)$$

$$S(p) = \frac{i}{(2\pi)^2} \frac{\not{p}}{-p^2 + i0}$$

and the vertex $(2\pi)^2 i g \gamma^\mu$, we get, corresponding to the previous discussion

$$D'_{\mu\nu}(k) = \frac{i}{(2\pi)^2} \left(\frac{-g_{\mu\nu} + \frac{k_\mu k_\nu}{k^2}}{k^2 + \mu^2 + g^2/\pi} - \frac{k_\mu k_\nu}{k^2 + \mu^2} \right) \quad (5.5)$$

which coincides with the exact photon Green function presented in Ref. 10).

As the two components of the fermion field are totally separated, the γ^+ and γ^- components of the fermion self-energy each bear the same discussion as in Section 4, with the photon denominators replaced as follows:

$$\frac{1}{r^2 + m^2} \longrightarrow \frac{1}{r^2 + \tilde{m}^2} - \frac{1}{r^2 + \mu^2} \quad (5.6)$$

where

$$\tilde{m}^2 = \mu^2 + g^2/\pi = \mu^2 + m^2 \quad (5.7)$$

Correspondingly the integral equation becomes:

$$\not{p} S'(p) = \not{p} S(p) - i \frac{g^2}{4\pi^2} \int \frac{d^2 k}{k_- - p_-} \left(\frac{1}{(k-p)^2 + \tilde{m}^2} - \frac{1}{(k-p)^2 + \mu^2} \right) S'(k) \quad (5.8)$$

leading to

$$\begin{aligned} x^\mu \partial_\mu S'(x) = x^\mu \partial_\mu S(x) + & \left[\frac{g^2/\pi}{4\tilde{m}^2} \left(1 - \tilde{m} \sqrt{x^2 + i0} K_1(\tilde{m} \sqrt{x^2 + i0}) \right) \right. \\ & \left. - \frac{g^2/\pi}{4\mu^2} \left(1 - \mu \sqrt{x^2 + i0} K_1(\mu \sqrt{x^2 + i0}) \right) \right] S'(x) \end{aligned} \quad (5.9)$$

Thus, we obtain

$$S'(x) = S(x) \exp \left\{ \frac{g^2}{2\pi} \right. \quad (5.10)$$

$$\left. x \left[\frac{1}{\tilde{m}^2} \left(K_0(\tilde{m} \sqrt{x^2}) + \ln \frac{\tilde{m}}{2} \sqrt{x^2} - \gamma \right) - \frac{1}{\mu^2} \left(K_0(\mu \sqrt{x^2}) + \ln \frac{\mu}{2} \sqrt{x^2} - \gamma \right) \right] \right\}$$

which again coincides with the exact solution given in Ref. 10).

6. SCHWINGER MODEL WITH PRINCIPAL-VALUE INTEGRATION

We discuss briefly the corresponding problem using principal-value integration^{5),6)}, a problem that we have not solved. As emphasized previously^{7),17)}, there are at least two interpretations of the principal-value integral. Principal value in the mathematical sense leads to integrals that do not commute, a consequence of the Poincaré-Bertrand theorem¹⁸⁾. The alternative is to replace $1/p_-^2$ by

$$\frac{1}{2} \left[\frac{1}{(p_- + i\varepsilon)^2} + \frac{1}{(p_- - i\varepsilon)^2} \right] \quad (6.1)$$

with the same ε for all momentum variables. Here the corresponding p_+ integrations are carried out first. With (6.1), the calculation up to fourth order leads, for the graphs (a)-(d) in Fig. 2, to the same results as above [see Eq. (3.3)], the graph (f) is again divergent, while the graph (e) is now zero.

The procedure of Section 4 can be applied with little modification. However, because it seems necessary to carry ε along explicitly, the evaluation is much more complicated. Taking the limit $\varepsilon \rightarrow 0$ will eventually lead to a fermion propagator different from that of (2.7). The meaning of this solution on the basis of (6.1) is not clear to us, and the relation with (2.7) remains an important open problem that deserves further investigation.

7. DISCUSSION

We have shown that, for two gauge models in one space dimension, perturbation theory is powerful enough to give the exact propagators, even though the fermion propagator is not analytic in the coupling constant. In both cases, Euclidean symmetric integration is used.

Consider the Schwinger model in the limit^{*)} of small m^2 . In fourth order we have seen explicitly that in carrying out the integrations this mass appears logarithmically. On the basis of the integral equation or the exact solution, in $2n^{\text{th}}$ order there is a term of the form $(\ln m^2)^{n-1}$, but no inverse powers of m^2 appear. However, in the light cone gauge in sixth order, there are three fermion self-energy diagrams that give $1/m^2$ terms. These three terms cancel each other when added together (see Appendix C). In the eighth order, there are both m^{-4} and m^{-2} terms; they again cancel when added together. This phenomenon holds to all orders.

With this knowledge, we speculate about the possible behaviour in other models.

1) If a non-zero mass M is introduced for fermion, then this cancellation can no longer hold. For example, in the sixth order, in addition to terms of the order $[\ln(p^2/m^2)]^2$, there may be a term of the order $(p^2/m^2)(M^2/m^2)$, where M is assumed to be much less than m . Therefore, for sufficiently high momenta, the fermion mass is important and can change the structure of the theory¹⁹⁾.

*) To clarify the discussions of this section, we add the following remarks. First, in writing down the orders of magnitude for the various terms in $2n^{\text{th}}$ order perturbation theory, we do not include the factor g^{2n} . Secondly, the m used here is the mass that appears in the Feynman rule for the propagator of the gauge particle. It does not matter whether m is the mass of this particle, or merely a cut-off used to avoid infra-red divergence.

2) If the $U(1)$ gauge of the Schwinger model is replaced by a $U(N)$ or $SU(N)$ group, the integrals for each diagram are not changed, but the weight depends on N and is different. Even though the above-mentioned cancellation still obtains in the sixth order, the cancellation is much less extensive. In the eighth order, the m^{-4} terms are still expected to cancel but the m^{-2} terms are not. Then the structure of the theory is again quite different. Whether this conjecture is correct or not is probably best settled by an explicit calculation in the eighth order.

3) This $U(N)$ or $SU(N)$ case differs further from the Schwinger model in that the self-energy of the gauge particle is no longer given by the second-order diagram only. The important diagrams for the fermion propagator are rather different in the two limits of $N \rightarrow \infty$ and $m \rightarrow 0$. We speculate that these two limits and the third limit of $M \rightarrow 0$ are all different and do not commute. Thus this is a very rich subject that remains to be understood.

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APPENDIX A

We list here some integrals which we could not find in the tables [for the rest see Ref. 20)]:

$$p \int_0^\infty ds \, J_1(ps) K_0(ms) K_0(\mu s) = \frac{1}{4} \left(\ln \frac{m^2 + \mu^2 + p^2 + \sqrt{(m^2 + \mu^2 + p^2)^2 - 4m^2\mu^2}}{2\mu^2} \right) \left(\ln \frac{m^2 + \mu^2 + p^2 + \sqrt{(m^2 + \mu^2 + p^2)^2 - 4m^2\mu^2}}{2m^2} \right) \quad (\text{A.1})$$

$$p \int_0^\infty ds \, J_1(ps) K_0(ms) \ln \frac{ms}{2} = -\frac{1}{2} \ln \left(1 + \frac{p^2}{m^2} \right) \left(\gamma + \frac{1}{2} \ln \left(1 + \frac{p^2}{m^2} \right) \right) \quad (\text{A.2})$$

$$p \int_0^\infty ds \, J_1(ps) \left(\ln \frac{ms}{2} \right)^2 = \left(\gamma + \frac{1}{2} \ln \frac{p^2}{m^2} \right)^2 \quad (\text{A.3})$$

APPENDIX B

To illustrate the Euclidean integration we calculate here the fermion self-energy in second order [graph (a) in Fig. 2]

$$\begin{aligned}
 (a) &= 4ig^2 \int dk_1 dk_2 \frac{1}{(k_1 - i k_2 + p_1 - i p_2)(k_1 + i k_2)^2} \\
 &= 4ig^2 \int_0^\infty k dk \int_0^{2\pi} d\theta \frac{1}{(ke^{-i\theta} + pe^{-i\alpha}) k^2 e^{2i\theta}} \\
 &= 4g^2 \int_0^\infty \frac{dk}{k} \oint_{|z|=1} \frac{z dz}{kz + pe^{-i\alpha}} \quad (B.1) \\
 &= 8\pi ig^2 \int_0^\infty \frac{dk}{k} \theta(k-p) \left(-\frac{p}{k} e^{-i\alpha}\right) = \frac{m^2}{p^2} \frac{1}{S(p)}
 \end{aligned}$$

(see Eq. (3.3)).

APPENDIX C

In sixth-order perturbation theory, the leading terms as $m \rightarrow 0$ of the five diagrams shown in Fig. 5 cancel. These diagrams are characterized by having a photon line connecting point 1 with point 5 (or 2 and 6). Without such a line, a dimensional argument shows that the leading contribution is of a lower order of magnitude. In general, for any order of perturbation theory, the corresponding requirement is to have a photon line connecting the first point at either end with the next-to-last point at the other end.

As $m \rightarrow 0$ the leading contributions of these five diagrams are proportional respectively to

$$I_i = \lim_{m \rightarrow 0} \frac{1}{m^2} \int dx_1 dy_1 dx_2 dy_2 dx_3 dy_3 \times \frac{\delta_1/\bar{\delta}_1}{r_1^2+m^2} \frac{\delta_2/\bar{\delta}_2}{r_2^2+m^2} \frac{\delta_3/\bar{\delta}_3}{r_3^2+m^2} \frac{1}{\beta_1+1} \frac{1}{\beta_1+\beta_2+1} F_i \quad (C.1)$$

where we have normalized p to be 1, and

$$F_a = F_e = \frac{1}{\beta_1+1} \frac{1}{\beta_1+\beta_3+1} \frac{1}{\beta_3+1} \quad (C.2)$$

$$F_b = F_d = \frac{1}{\beta_1+\beta_2+\beta_3+1} \frac{1}{\beta_1+\beta_3+1} \frac{1}{\beta_3+1} \quad (C.3)$$

and

$$F_c = \frac{1}{\beta_1+\beta_2+\beta_3+1} \frac{1}{\beta_1+\beta_2+1} \frac{1}{\beta_2+1} \quad (C.4)$$

We want to show by explicit calculation that

$$2 I_a + 2 I_b + I_c = 0 \quad (C.5)$$

For each I , the important region of contribution is

$$\beta_1 \sim -1, \quad \beta_2 \sim 0 \quad \text{and} \quad \beta_3 \sim 0 \quad (C.6)$$

For I_C there is an additional region

$$\beta_1 \sim 0, \quad \beta_2 \sim -1 \quad \text{and} \quad \beta_3 \sim 0 \quad (\text{C.7})$$

For I_C the contributions from the regions (C.6) and (C.7) are equal. From (C.6) and (C.7) we see the necessity of having a photon line connecting one end point with the next-to-last point at the other end.

The limit $m \rightarrow 0$ can thus be taken explicitly. With a change of variable $z_1 + 1$ to a new z_1 , we obtain

$$I_i = \int dx_1 dy_1 dx_2 dy_2 dx_3 dy_3 \times \frac{\beta_2/\bar{\beta}_2}{r_2^2 + 1} \frac{\beta_3/\bar{\beta}_3}{r_3^2 + 1} \frac{1}{\beta_1} \frac{1}{\beta_1 + \beta_2} F_i^{(0)} \quad (\text{C.8})$$

where

$$F_a^{(0)} = F_e^{(0)} = \frac{1}{\beta_1} \frac{1}{\beta_1 + \beta_3} \quad (\text{C.9})$$

$$F_b^{(0)} = F_d^{(0)} = \frac{1}{\beta_1 + \beta_2 + \beta_3} \frac{1}{\beta_1 + \beta_3} \quad (\text{C.10})$$

and

$$F_c^{(0)} = 2 \frac{1}{\beta_1 + \beta_2 + \beta_3} \frac{1}{\beta_1 + \beta_2} \quad (\text{C.11})$$

If we define $G(z)$ by

$$G(\beta) = \int dx' dy' \frac{\beta'/\bar{\beta}'}{r'^2 + 1} \frac{1}{\beta + \beta'} \quad (\text{C.12})$$

then

$$I_a = \int dx dy \beta^{-2} [G(\beta)]^2 \quad (\text{C.13})$$

and

$$I_c = -2 \int dx dy \beta^{-2} G(\beta) G(-\beta) \quad (\text{C.14})$$

The corresponding formula for I_b is slightly more complicated. The partial fraction

$$\frac{1}{\delta_1 + \delta_2 + \delta_3} \frac{1}{\delta_1 + \delta_3} = \frac{1}{\delta_2} \left(\frac{1}{\delta_1 + \delta_3} - \frac{1}{\delta_1 + \delta_2 + \delta_3} \right) \quad (C.15)$$

shows that

$$I_b = I'_b + I''_b \quad (C.16)$$

where

$$F_b^{(0)'} = \frac{1}{\delta_2} \frac{1}{\delta_1 + \delta_3} \quad (C.16)$$

and

$$F_b^{(0)''} = - \frac{1}{\delta_2} \frac{1}{\delta_1 + \delta_2 + \delta_3} \quad (C.18)$$

In I''_b , if z_1 is first replaced by $z_1 - z_2$ and z_2 is then replaced by $-z_2$ the result is

$$I''_b = I'_b \quad (C.19)$$

Therefore

$$I_b = 2 \int dx dy \delta^{-1} G(\delta) \tilde{G}(\delta) \quad (C.20)$$

where

$$\tilde{G}(\delta) = \int dx' dy' \frac{\delta' / \bar{\delta}'}{r'^2 + 1} \frac{1}{\delta'} \frac{1}{\delta + \delta'} \quad (C.21)$$

Since it follows from (C.12) and (C.21) that

$$G(\delta) = -G(-\delta) \quad (C.22)$$

and

$$\delta \tilde{G}(\delta) + G(\delta) = \int dx' dy' \frac{1}{\bar{\delta}' (r'^2 + 1)} = 0 \quad (C.23)$$

we get from (C.13), (C.14), and (C.20) that

$$2 I_a = I_c = -I_b \quad (C.24)$$

which immediately implies (C.5).

Straightforward calculation, not needed for the present purpose, actually gives

$$\widetilde{G}(z) = \pi \ln \frac{r^2+1}{r^2} \quad (\text{C.25})$$

and

$$I_a = \frac{1}{3} \pi^5 \quad (\text{C.26})$$

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FIGURE CAPTIONS

- Fig. 1 : Feynman rules for QED in the light-cone gauge.
- Fig. 2 : Lowest order QED graphs.
- Fig. 3 : General fermion loop.
- Fig. 4 : $2n^{\text{th}}$ order diagram of the fermion propagator.
- Fig. 5 : Sixth-order diagrams that give the leading contributions as $m \rightarrow 0$.

FIG.1

FIG.2

FIG.3

FIG.4

(a)

(b)

(c)

(d)

(e)

FIG.5