Ref.TH.2835-CERN

ON ABELIAN SELF-DUAL MULTIMONOPLES

W. Nahm
CERN - Geneva

A B S T R A C T

Abelian self-dual monopoles are Dirac monopoles which also act as the source of a scalar field equal in strength to the potential of the magnetic field. The static field generated by several such monopoles with arbitrary positions can be described by adapting the ADHM formalism developed for instantons. In particular one obtains an exact expression for the Green function of the covariant Laplace operator.

Ref.TH.2835-CERN
20 March 1980

In a preceding paper¹⁾ we have adapted the ADHM formalism²⁾ for self-dual gauge fields to the BPS monopole^{3),4)}. This monopole is constructed from an SU(2) gauge field. But asymptotically only the U(1) part contributes. There one obtains a Dirac monopole coupled in addition to a scalar field which is equal in strength to the potential of the magnetic field. Let us consider such Abelian self-dual monopoles which can easily be superposed⁵⁾. Here we show how one can adapt the ADHM formalism to these multimonopoles. On the one hand, this formulation allows the construction of zero modes and Green functions for the Dirac operator and related operators^{6),7)}. On the other hand, it should be a step toward the construction of non-Abelian multimonopoles.

The ADHM formalism yields potentials of the form

$$A_\mu = -i (v_1, \dots, v_n)^+ \partial_\mu (v_1, \dots, v_n), \quad (1)$$

where the v_i form a complete orthonormal set of solutions of a system of linear equations

$$\Delta^+ v = 0. \quad (2)$$

Different choices of the v_i correspond to different gauges. For instantons, Δ is a matrix acting on a finite dimensional vector space. For monopoles it will be a matrix of differential operators in some Hilbert space.

If we write positions with the help of sigma matrices, e.g.,

$$X = X^0 - i X^k \sigma_k, \quad (3)$$

we have for a BPS monopole at position $\vec{\alpha}$

$$\Delta = i \frac{\partial}{\partial z} + X - \alpha. \quad (4)$$

Here x^0 is just a dummy co-ordinate, and A_0 is the scalar field. Correspondingly, α^0 is fixed but arbitrary and will be put to zero.

The v_i belong to the Hilbert space $L^2(I) \otimes C^2$, with

$$I = \left\{ z \mid -\frac{1}{2} \leq z \leq \frac{1}{2} \right\}. \quad (5)$$

The C^2 factor is acted upon by the sigma matrices. Thus it serves as a two-dimensional representation of $SU(2)$. Equation (2) now has two independent solutions. Thus Eq. (1) yields the required $SU(2)$ potential.

The differential operator of Eq. (4) may be treated naively, but it is useful to specify domain and range in such a way that Δ and its adjoint become bounded operators. More precisely, let Δ_k , $k = 1, 2$, denote the two columns of Δ . Then we have a complex

$$W^1(I) \xrightarrow{\delta_k} L^2(I) \otimes C^2 \xrightarrow{\delta_k} W^{-1}(I), \quad (6)$$

where

$$\delta_k = \begin{cases} \Delta_k & \text{on } W^1 \\ \varepsilon_{kl} \Delta_l^+ & \text{on } L^2 \otimes C^2, \end{cases} \quad (7)$$

and

$$\varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (8)$$

The two Sobolev spaces

$$W^{\pm 1}(I) = \left\{ \sum_{n=1}^{\infty} a_n \sin \pi n(z + \frac{1}{2}) \mid \sum_{n=1}^{\infty} n^{\pm 2} |a_n|^2 < \infty \right\} \quad (9)$$

are dual to each other. Note that the elements of $W^1(I)$ are continuous functions which vanish at the boundary of I . Thus we need not worry about boundary terms by passing from Δ to its adjoint Δ^+ .

The ADHM formalism requires

$$\delta_k(x) \delta_l(x) + \delta_l(x) \delta_k(x) = 0, \quad (10)$$

which is easily seen to be fulfilled.

Surprisingly, the Abelian self-dual monopole is described by a Δ which still has the form of Eq. (4) but with a changed domain. We now have

$$W^1(R_+) \xrightarrow{\delta_k} \mathcal{L}^2(R_+) \otimes \mathbb{C}^2 \xrightarrow{\delta_k} W^{-1}(R_+) \quad (11)$$

with

$$R_+ = \{z \mid 0 \leq z < \infty\} \quad (12)$$

and

$$W^{\pm 1}(R_+) = \left\{ \int_0^\infty f(\xi) \sin(z\xi) d\xi \mid \int_0^\infty |f(\xi)|^2 (1+\xi^2)^{\pm 1} d\xi < \infty \right\}. \quad (13)$$

The remarks made for $W^1(I)$ also apply to $W^1(R_+)$.

As only square integrable solutions are allowed, Eq. (2) now only has one independent solution

$$V(x, z) = \sqrt{2r} \exp(ix^0 z - rz) w(x), \quad (14)$$

with

$$r^2 = \sum_{i=1}^3 (x^i - \alpha^i)^2, \quad (15)$$

$$\frac{1}{2} (1 + r^{-1} (x^i - \alpha^i) \sigma_i) w(x) = w(x). \quad (16)$$

The last equation together with the normalization condition

$$W^+ W = 1 \quad (17)$$

determines $w(x)$ up to a phase factor. Equation (1) yields the required $U(1)$ potential. The singularity at the monopole position is reflected by the fact that Eq. (2) has no solution at this point. Moreover, the solutions of Eq. (2) form a non-trivial fibre bundle, as one cannot choose solutions $w(x)$ of Eqs (16) - (17) such that the dependence on x is continuous. But it is evident in our formalism that the resulting Dirac strings are just gauge artifacts, as it should be.

To go over to multimonopoles, we use the fact that the tensor product of $U(n)$ and $U(m)$ potentials yields a $U(nm)$ potential. For $n = m = 1$ one again obtains a $U(1)$ potential, namely just the linear superposition. Thus for the special case of Abelian monopoles, multimonopoles are just tensor products of single monopoles, which may even have coinciding positions. As tensor products can be treated in the ADHM formalism, our problem is solved in principle.

The tensor product formalism has been developed by Corrigan, Goddard and Templeton⁸⁾ and independently by Drinfeld and Manin⁹⁾. The first paper contains more explicit results, the second one gives a clearer mathematical foundation and improves some parts of the formalism. It is advantageous to combine both approaches^{10),11)}.

Let us treat the tensor product of two Abelian monopoles in detail. We take $\vec{\alpha} = (a, 0, 0)$ and $\vec{\alpha}' = -\vec{\alpha}$ as their positions. The tensor product of two complexes of type (11) yields

$$\begin{aligned} W^1 \otimes W^{1'} &\xrightarrow{\delta_k} W^1 \otimes \mathcal{L}^{2'} \otimes C^{2'} + \mathcal{L}^2 \otimes C^2 \otimes W^{1'} \xrightarrow{\delta_k} \\ W^1 \otimes W^{-1'} + \mathcal{L}^2 \otimes \mathcal{L}^{2'} \otimes C^2 \otimes C^{2'} + W^{-1} \otimes W^{1'} &\xrightarrow{\delta_k} \\ \mathcal{L}^2 \otimes C^2 \otimes W^{-1'} + W^{-1} \otimes \mathcal{L}^{2'} \otimes C^{2'} &\xrightarrow{\delta_k} W^{-1} \otimes W^{-1'}. \end{aligned} \quad (18)$$

To preserve Eq. (10) one puts

$$\delta_k (X \otimes Y') = (\delta_k X) \otimes Y' \pm X \otimes (\delta_k' Y') \quad (19)$$

where the plus sign applies for $X = \mathbb{L}^2 \otimes \mathbb{C}^2$, the minus sign for $X = W^1, W^{-1}$. Operations which lead out of the factor complexes yield zero.

The complex (18) is too long for the ADHM formalism and has to be projected down to a complex of type (11). Here we need to do this only for the middle part, because our new v will be an element of this space. We write

$$V(z, z') = V^-(z, z') \oplus V^0(z, z') \oplus V^+(z, z'), \quad (20)$$

with

$$V^\pm \in W^{\pm 1} \otimes W^{\mp 1}, \quad (21)$$

$$V_{IJ}^0 = \left((u^0 - iu^k \sigma_k) \varepsilon \right)_{IJ} \in \mathcal{L}^2 \otimes \mathcal{L}^{2'}. \quad (22)$$

The projection goes to the subspace defined by

$$\delta_k(x) \delta_l(x) v = 0 \quad \text{for all } k, l, x. \quad (23)$$

This is equivalent to

$$\begin{aligned} \text{tr} \left[(i\partial + (x-\alpha)^+)(i\partial + x - \alpha) v^+ + (i\partial' + (x+\alpha)^+)(i\partial' + x + \alpha) v^- \right. \\ \left. + 2(i\partial' + x + \alpha)(i\partial + (x-\alpha)^+)(u^0 - iu^k \sigma_k) \right] = 0 \end{aligned} \quad (24)$$

for all x ,

where the trace is over a 2×2 matrix.

If one takes

$$z_\pm = z \pm z', \quad (25)$$

$$\partial_\pm = \frac{1}{2}(\partial \pm \partial'), \quad (26)$$

all derivatives with respect to z_+ cancel, whereas the dependence on z_- is fixed by

$$(\partial_-^2 - a^2)v = 0, \quad (27)$$

and by the boundary conditions coming from the definition of $W^1, W^{1'}$. Altogether Eq. (24) yields

$$v^\pm = -u^0 \mp ia^{-1} \partial_- u^1, \quad (28)$$

$$u^0 = ik(z_+) \tanh az_+ \cosh az_- + l(z_+) \coth az_+ \sinh az_-, \quad (29)$$

$$u^1 = k(z_+) \cosh az_- - il(z_+) \sinh az_-, \quad (30)$$

$$u^2 = -n(z_+) \cosh az_- - im(z_+) \sinh az_-, \quad (31)$$

$$u^3 = m(z_+) \cosh az_- - in(z_+) \sinh az_-. \quad (32)$$

The norm on this subspace is given by

$$\begin{aligned} \|v\|^2 &= \iint_0^\infty (\bar{v}_{IJ}^0 v_{IJ}^0 - \bar{v}^+ v^- - \bar{v}^- v^+) dz dz' = \\ &= \int_0^\infty (|k|^2 + |l|^2 + |m|^2 + |n|^2)(z_+) a^{-1} \sinh(2az_+) dz_+. \end{aligned} \quad (33)$$

Equation (2) takes the form

$$\delta_k(x) v = 0, \quad (34)$$

i.e., in terms of (k, l, m, n)

$$\left(i\partial_+ + 2ia \begin{pmatrix} \tanh az_+ & & & \\ & \coth az_+ & & \\ & & 0 & \\ & & & 0 \end{pmatrix} + \begin{pmatrix} 0 & -x^1 & -x^2 & -x^3 \\ x^1 & 0 & x^3 & -x^2 \\ x^2 & -x^3 & 0 & x^1 \\ x^3 & x^2 & -x^1 & 0 \end{pmatrix} \right) \begin{pmatrix} k \\ l \\ m \\ n \end{pmatrix} = 0. \quad (35)$$

This equation is hard to solve directly. One can, however, take the tensor product $v \otimes v'$ of the solutions of Eq. (2) for single monopoles and project it down to the subspace given by Eqs (29) - (32). The relevant part of the projection operator is denoted in Refs 8) and 10) by $1 - \mathcal{M}$. It is obtained from an operator M , which in our case is defined by

$$(-(\partial_y - \partial_{y'})^2 + 4a^2) M(z, z', y, y') = \delta(y - z) \delta(y' - z'), \quad (36)$$

with the boundary conditions

$$M(z, z', 0, y') = M(z, z', y, 0) = 0 \quad (37)$$

[compare Ref. 1) for the case $a = 0$]. One finds

$$M(z, z', y, y') = -\frac{1}{4a} \delta(y + y' - z - z') \times \\ \times \left[\sinh a|y - y' - z + z'| + \sinh a(z - z') \coth a(z + z') \sinh a(y - y') \right. \\ \left. - \cosh a(z - z') \tanh a(z + z') \cosh a(y - y') \right]. \quad (38)$$

This projection is given by

$$V_{IJ}^0(z, z') = V_I(z) V_J'(z') + \\ + \left[(i\partial + x - \alpha)(i\partial' + (x + \alpha)^*) \varepsilon \right]_{IJ} \int M(z, z', y, y') V_K(y) \varepsilon_{KL} V_L'(y') dy dy'. \quad (39)$$

This yields

$$k(z_+) = -iy \left[\frac{r_+ r_- \sinh r_- z_+}{\sinh a z_+} - \frac{a}{\cosh a z_+} (r_+ \cosh r_- z_+ - r_- \sinh r_- z_+ + a \tanh a z_+ \cosh r_- z_+) \right] \quad (40)$$

$$l(z_+) = -y \left[\frac{r_+ r_- \cosh r_- z_+}{\cosh a z_+} - \frac{a}{\sinh a z_+} (r_+ \sinh r_- z_+ - r_- \cosh r_- z_+ + a \coth a z_+ \sinh r_- z_+) \right] \quad (41)$$

$$m(z_+) = ay \left(x^2 \frac{\cosh r_- z_+}{\cosh a z_+} + ix^3 \frac{\sinh r_- z_+}{\sinh a z_+} \right) \quad (42)$$

$$n(z_+) = -ia y \left(x^2 \frac{\sinh r_- z_+}{\sinh a z_+} + ix^3 \frac{\cosh r_- z_+}{\cosh a z_+} \right) \quad (43)$$

Here

$$r_{\pm} = \frac{r}{2} (r \pm r'), \quad (44)$$

$$y = \frac{\sqrt{rr'}}{r^2 - a^2} \exp(-r_+ z_+ + ix^0 z_+) w_I \epsilon_{IJ} w'_J, \quad (45)$$

with w, w' given by Eqs (16) - (17).

Now the construction is finished, and one may obtain Green's functions, etc., as for instantons. For example, the Green's functions $G(x, y)$ of the covariant Laplacian is essentially given by the scalar product of $(k, l, m, n)(x)$ with $(k, l, m, n)(y)$. Of course, the scalar product is the one given by the norm (33).

The tensor product of N monopoles can be constructed similarly, but instead of four functions k, l, m, n one obtains 2^N functions. In fact, the x dependent

part of the basic operator δ will act according to the N -fold tensor product of two-dimensional representations of $SU(2)$. For $N = 1, 2$ this can be read off from Eqs (4) and (35).

For $N = 1$ the only essential difference between self-dual $U(1)$ and $SU(2)$ monopoles is the shape of the z interval, which is bounded for $SU(2)$ and unbounded for $U(1)$. One is tempted to pass to a bounded interval also for multi-monopoles. In that case, however, one obtains 2^N different solutions for v and an $SU(2^N)$ potential. In fact, one obtains the same result by taking directly the N -fold tensor product of $SU(2)$ monopoles.

REFERENCES

- 1) W. Nahm, "A simple formalism for the BPS monopole", CERN TH. 2796, to be published in Phys. Lett. B.
- 2) M.F. Atiyah, N.J. Hitchin, V.G. Drinfeld and Yu.I. Manin, Phys. Lett. 65A (1978) 185.
- 3) M.K. Prasad and C.M. Sommerfield, Phys. Rev. Lett. 35 (1975) 760.
- 4) E.B. Bogomolny, Sov. J. Nucl. Phys. 24 (1976) 861.
- 5) N.S. Manton, Nucl. Phys. B126 (1977) 525.
- 6) L.S. Brown, R.D. Carlitz, D.B. Creamer and C. Lee, Phys. Lett. 70B (1977) 180; 71B (1977) 103; Phys. Rev. D17 (1978) 1583.
- 7) E.F. Corrigan, D.B. Fairlie, P. Goddard and S. Templeton, Nucl. Phys. B140 (1978) 31.
- 8) E.F. Corrigan, P. Goddard and S. Templeton, Nucl. Phys. B151 (1979) 93.
- 9) V.G. Drinfeld and Yu.I. Manin, Yad. Fiz. 29 (1979) 1646. For a German translation see DESY-L-Ueb-233.
- 10) E.F. Corrigan and P. Goddard, "Some aspects of instantons", DAMTP 79-18.
- 11) P. Goddard, private communication.

12