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ANOMALIES AND SUPERSYMMETRIC REGULARIZATION BY DIMENSIONAL REDUCTION

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ABSTRACT

We show that the superconformal anomalies of super Yang-Mills theory may be correctly calculated using the supersymmetry preserving dimensional regularization by dimensional reduction.

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1. . INTRODUCTION

Two loop calculations in the Wess-Zumino model have shown that dimensional regularization preserves supersymmetry provided one uses the rules^{1),2)}

$$\begin{aligned} \text{Tr}(\mathbb{1}) &= 4, \quad \mathbb{1} = \text{unit Dirac matrix} \\ \{\gamma_5, \gamma_\mu\} &= 0 \quad \forall \gamma_\mu \end{aligned} \quad (1.1)$$

The first of these rules ensures the equality of the number of boson and fermion degrees of freedom; the second is required for various scalar-pseudoscalar cancellations typical of supersymmetric theories. The extension of these results to gauge theories appears problematical at first because a vector W_μ has n components in n dimensions while a spinor still has four by the above rule for $\text{Tr } 1$. A solution to this problem was proposed by Siegel³⁾. In his method the $n = 4$ Lagrangian is dimensionally reduced to $n < 4$ dimensions. A vector W_μ is split into an n component vector W_μ and $\varepsilon = 4 - n$ "ε scalars", $W_{(\mu)}$, but the total number of bosons remains n independent. The effective quantum Lagrangian for super Yang-Mills theory is therefore

$$\begin{aligned} \mathcal{L}_{\text{eff.}} &= \left\{ -\frac{1}{4} (G_{\mu\nu}^a)^2 - \frac{1}{2} (\partial_\mu W^a)^2 + C^{*a} \partial_\mu (D_\mu C)^a - \frac{1}{2} \bar{\lambda} \not{\partial} \lambda + \frac{1}{2} (D^a)^2 \right\} \\ &+ \left\{ -\frac{1}{2} [(D_\mu W_{(\mu)})^a]^2 - \frac{1}{4} [W_{(\mu)}^b W_{(\nu)}^c \varepsilon^{abc}]^2 - \frac{1}{2} \varepsilon^{abc} \bar{\lambda}^a \gamma_\mu \lambda^c W_{(\mu)}^b \right\} \end{aligned} \quad (1.2)$$

The terms in the second bracket constitute the ε scalar Lagrangian. Capper, Jones and van Nieuwenhuizen⁴⁾ have shown that this modified form of dimensional regularization preserves supersymmetric Ward identities at one loop. The one loop corrections to the super Yang-Mills Ward identity have also been calculated by Majumdar, Poggio and Schnitzer⁵⁾, using a prescription in which ε scalars are ignored but $\gamma_\mu \gamma^\mu = 4$ instead of the usual $\gamma_\mu \gamma^\mu = n$ of dimensional regularization. It is not surprising that these two methods give the same result because ε scalars effectively convert n component indices to four-component indices, but this latter method is also restricted to $n < 4$. For $n < 4$ the momentum ℓ_μ in $\ell_\mu \gamma^\mu$ serves to project out the first n component of γ^μ thus providing the correct n dependence of the diagrams. For $n > 4$, however, this argument fails and the n dependence can then depend on how the calculation is done. Because of this problem one should restrict the method of Ref. 5) also to $n < 4$, and then it is clearly equivalent to dimensional reduction. We have in fact tried several other ways to extend the results of Refs 4) and 5) to $n > 4$

without success. One which is perhaps worth mentioning is the attempt to go to $n > 4$ in the dimensional reduction scheme by interpreting the ε scalars as Faddeev-Popov type ghosts which then subtract $|\varepsilon|$ degrees of freedom. But this would imply a vanishing kinetic term for the ε scalars, as well as a non-commuting Lagrangian on account of the $\lambda\gamma_{(\nu)}\lambda W_{(\nu)}$ interaction. For these reasons we restrict ourselves in this paper to dimensional reduction as applied in Ref. 4).

The problems we investigate centre on the calculation of the superconformal anomalies^{6),7)} of $\partial \cdot J^5$, $\gamma \cdot S$ and θ_μ^μ , of the axial current, J_μ^5 , supersymmetry current, S_μ , and energy momentum tensor $\theta_{\mu\nu}$. There are two central problems:

1) the calculation of $\partial \cdot J^5$ by 't Hooft and Veltman⁸⁾ requires rules differing from those of (1.1), viz.

$$\begin{aligned} \{\gamma_5, \gamma_\mu\} &= 0, \quad \mu = 1 \dots 4 \\ [\gamma_5, \gamma_\mu] &= 0, \quad \mu > 4 \dots n \end{aligned} \quad (1.3)$$

with $n > 4$. Since we can no longer use this prescription we must find another way to calculate the anomaly, consistent with the rules of (1.1).

2) The supersymmetry current is

$$S_\mu = \sigma^{\sigma\tau} G_{\sigma\tau}^a \gamma_\mu \lambda^a \quad (1.4)$$

Using standard dimensional regularization we have $\gamma_\mu \sigma_{\sigma\tau} \gamma_\mu = (n-4)\sigma_{\sigma\tau}$, which gives $\gamma \cdot S = (n-4)\sigma \cdot G^a \lambda^a$ and since $\sigma \cdot G^a \lambda^a$ is an infinite operator we find $\gamma \cdot S \neq 0$ at one loop. In dimensional reduction, however, the currents are obtained from the dimensionally reduced Lagrangian, $\mathcal{L}_n$. This means that the indices σ, τ in (1.4) are effectively four component indices because of the ε scalars, while the Noether index μ is n component because all derivatives in $\mathcal{L}_n$ are n dimensional. Therefore we cannot use $\gamma_\mu \sigma_{\sigma\tau} \gamma_\mu = (n-4)\sigma_{\sigma\tau}$. In fact the correct identities are

$$\begin{aligned} \gamma_\mu \sigma_{\sigma\tau} \gamma_\mu &= (n-4)\sigma_{\sigma\tau} \quad ; \sigma, \tau \leq n \\ \gamma_\mu \sigma_{\sigma(\tau} \gamma_\mu &= (n-2)\sigma_{\sigma\tau} \quad ; \sigma \leq n, (\tau) > n \\ \gamma_\mu \sigma_{(\sigma\tau)} \gamma_\mu &= n\sigma_{(\sigma\tau)} \quad ; (\sigma), (\tau) > n \end{aligned} \quad (1.5)$$

We see from (1.5) that the ε scalars can make a contribution to the one loop result for $\gamma \cdot S$, upsetting the calculation of ordinary dimensional regularization⁹⁾. This is unwanted because the current values for $\partial \cdot J^5$, $\gamma \cdot S$ and θ_μ^μ are such that these anomalies are related by supersymmetry.

We consider these two problems in turn in Sections 2 and 3. In Section 2 we give a derivation of the axial anomaly consistent with (1.1). Our analysis is close to that of Chanowitz, Furman and Hinchliffe¹⁰⁾, who have discussed this problem in a different context, but is different in detail. There remain, however, some questions of consistency that are particular to $n < 4$ that we have not resolved to our satisfaction. In Section 3 we calculate the $\gamma \cdot S$ anomaly and show that the contributions of ε scalars to $\gamma \cdot S$ and $\partial \cdot S$ cancel so that dimensional regularization and dimensional reduction agree at one loop.

There is no special problem with the trace anomaly. At one loop the calculation of θ_μ^μ is identical to that of ordinary dimensional regularization¹¹⁾. We wish to emphasize that the agreement between ordinary dimensional regularization and dimensional reduction is at one loop only. There is good reason to believe that the results will differ at two loops. This could be a welcome feature of dimensional reduction as it may provide a resolution of the well-known puzzle of the higher loop anomalies⁷⁾.

2. THE AXIAL ANOMALY

The diagrams contributing to the one loop axial anomaly are shown in Fig. 1. Multiplying the J_ρ^5 vertex by $(p + q)_\rho = (p + q + \ell)_\rho - \ell_\rho$ we find the usual integrals to evaluate for $\partial \cdot J^5$

$$\left\{ \frac{\int d^4 \ell \text{Tr} [\gamma_5 (\not{p} + \not{q} + \not{\ell}) \not{\ell} \gamma_\nu (\not{\ell} + \not{q}) \gamma_\mu (\not{\ell} + \not{p} + \not{q})]}{\ell^2 (\ell + q)^2 (\ell + p + q)^2} - \frac{\int d^4 \ell \text{Tr} [\gamma_5 \not{\ell} (\not{\ell} + \not{q}) \gamma_\mu (\not{\ell} + \not{p} + \not{q})]}{(\ell + q)^2 (\ell + p + q)^2} \right\} + \begin{pmatrix} \mu \leftrightarrow \nu \\ p \leftrightarrow q \end{pmatrix} \quad (2.1)$$

The question is now: is the correct non-zero result for (2.1) consistent with the standard four-dimensional Dirac algebra? This standard algebra tells us that

$$\text{Tr}(\gamma_5 \gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma) = 4 \epsilon_{\mu\nu\rho\sigma} \quad (2.2)$$

Using this result the second integral in (3.1) is seen to vanish on shifting $k \rightarrow k - q$. If in addition we now assume that

i) γ_5 is totally anticommuting

ii) $\text{Tr}(AB) = \text{Tr}(BA)$ for strings of Dirac matrices A, B ,

then we can write the first integral of (2.1) in the same form as the second and show that it too vanishes. This would give the incorrect result $\partial \cdot J^5 = 0$. In order to get the correct result we must give up one of the above two assumptions. 't Hooft and Veltman⁸⁾ gave up i) while Chanowitz, Furman and Hinchliffe¹⁰⁾ gave up ii). According to the rules of (1.1) we must follow the latter authors. As they have shown, the trace of (2.2) cannot be consistently defined away from $n = 4$ if it is also assumed to have the cyclicity property, $\text{Tr}(AB) = \text{Tr}(BA)$. However, once we have renounced use of the cyclic property of the trace we can no longer derive the inconsistency of (2.2), now can we conclude that $\partial \cdot J^5 = 0$. Fortunately it is never required for calculational purposes that we use $\text{Tr}(AB) = \text{Tr}(BA)$. Any trace can be reduced to a few elementary ones such as $\text{Tr}(\gamma_\mu \gamma_\nu)$, $\text{Tr}(1)$ and $\text{Tr}(\gamma_5 \gamma_\mu \gamma_\nu \gamma_\rho \gamma_\tau)$ by use of standard Dirac algebra, and these few are defined to be the usual quantities of four dimensions.

For example, to calculate the first integral of (2.1) without using the cyclic trace property we (anti) commute $(\not{p} + \not{q} + \not{k})$ to the end of the trace. In this way we obtain

$$\begin{aligned} & \text{Tr}[\gamma_5 (\not{p} + \not{q} + \not{k}) \not{k} \gamma_\nu (\not{k} + \not{q}) \gamma_\mu (\not{k} + \not{p} + \not{q})] = \\ & -(\not{k} + \not{p} + \not{q})^2 \text{Tr}[\gamma_5 \not{k} \gamma_\nu (\not{k} + \not{q}) \gamma_\mu] - 8(\not{p} + \not{q} + \not{k})^\rho \not{k}^\sigma (\not{k} + \not{q})^\lambda (\not{p} + \not{q} + \not{k})^\tau A_{\sigma\nu\lambda\mu\tau, \rho} \end{aligned} \quad (2.3)$$

where the A tensor is defined by

$$\begin{aligned} A_{\sigma\nu\lambda\mu\tau, \rho} \equiv & -\epsilon_{\nu\lambda\mu\tau} g_{\sigma\rho} + \epsilon_{\sigma\lambda\mu\tau} g_{\nu\rho} - \epsilon_{\sigma\nu\mu\tau} g_{\lambda\rho} \\ & + \epsilon_{\sigma\nu\lambda\tau} g_{\mu\rho} - \epsilon_{\sigma\nu\lambda\rho} g_{\mu\tau} \end{aligned} \quad (2.4)$$

As this six component tensor is antisymmetric in its first five indices it necessarily vanishes for $n = 4$. Away from $n = 4$, however, we are not allowed to assume that it still vanishes. Indeed, this assumption is equivalent to the assumption of the cyclic property of the trace in (2.2). Taking the trace of (2.4) we find

$$A_{\sigma\nu\lambda\mu\tau,\rho} g^{\sigma\rho} = (4-n) \epsilon_{\nu\lambda\mu\tau} \quad (2.5)$$

which shows that the A tensor cannot vanish away from $n = 4$. Proceeding with the calculation the first term on the right-hand side of (2.3) gives zero leaving us to evaluate (after shifting the integrals)

$$8 A_{\sigma\nu\lambda\mu\tau,\rho} \left\{ - \int d^4 l \frac{(l+p)^\rho (l-q)^\sigma l^\lambda (l+p)^\tau}{l^2 (l+p)^2 (l-q)^2} + \int d^4 l \frac{(l-q)^\rho (l+p)^\sigma l^\lambda (l-q)^\tau}{l^2 (l+p)^2 (l-q)^2} \right\} \quad (2.6)$$

Owing to the antisymmetry properties of the A tensor, the quadratic and linearly divergent pieces drop out. The convergent part cannot contribute, leaving us with the logarithmically divergent part. Only the following piece contributes

$$\begin{aligned} & -16 A_{\nu\lambda\mu\tau\sigma,\rho} p^\sigma q^\tau \int d^4 l \frac{l^\rho l^\lambda}{l^2 (l+p)^2 (l-q)^2} \\ &= \frac{16i}{n} \epsilon_{\nu\lambda\mu\tau\sigma} p^\sigma q^\tau \pi^{\frac{n}{2}} (4-n) \Gamma(2-\frac{n}{2}) + O(n-4) \\ &\xrightarrow{n \rightarrow 4} 8i\pi^2 \epsilon_{\nu\lambda\mu\tau\sigma} p^\sigma q^\tau \end{aligned} \quad (2.7)$$

which is the desired result (one gets half this number for a Majorana fermion).

There are two points that need clarification.

i) Is this derivation unambiguous? There are several ways of doing the Dirac algebra, and it is not guaranteed that these will give the same result for $n \neq 4$. They can give results which differ by a quantity antisymmetric in five indices. In fact, different calculational routes do give results which differ in just this way. But this ambiguity is equivalent to the initial ambiguity of where to start the trace on the diagram of Fig. 1. Because the trace is not

cyclic, a different choice will give a different result. All these results give an anomaly but not necessarily in $\partial \cdot J^5$. In order to place the anomaly unambiguously in $\partial \cdot J^5$ one must also impose gauge invariance.

ii) What about the failure of the trace property? It means that γ_5 and γ_μ cannot be considered as finite dimensional matrices away from $n = 4$. Moreover, even if we had an infinite dimensional representation of the γ_μ we could still not define γ_5 by $\gamma_5 = \gamma_1 \gamma_2 \gamma_3 \gamma_4$. The reason for this is that for traces containing no γ_5 's (or equivalently an even number of them) one can deduce that the cyclic property of the trace is satisfied, just as one can deduce that it is not satisfied for traces containing a single γ_5 . This follows directly from the above rules. In the usual approach to the anomaly with $n > 4$ the choice of $\gamma_5 = \gamma_1 \gamma_2 \gamma_3 \gamma_4$ does not in any case satisfy the rules of (1.1) and is, therefore, disallowed from the start. For $n < 4$, however, this choice of γ_5 is totally anticommuting and therefore allowed by the rules of (1.1), but still not allowed because of the axial anomaly. We have not found any satisfactory representation of γ_5 and γ_μ and this fact leaves open in our opinion some questions of the consistency of the method.

3. THE $\gamma \cdot S$ ANOMALY

Consider the following matrix element of $\bar{S}_\mu$

$$\left\langle \bar{\lambda}^a(x) \gamma_\mu G_{\sigma\tau}^a(x) \sigma^{\sigma\tau} \lambda_b(y) W_\rho^c(z) \right\rangle_{\text{conn.}}^{\text{phys.}} \quad (3.1)$$

The one loop diagrams contributing to this S matrix element are shown in Fig. 2. They lead to the evaluation of the following expression.

$$\begin{aligned} & \frac{4i \delta_{bc}}{p^2 q^2} \left\{ \frac{d^4 l}{l^2 (l-p)^2 (l+q)^2} \left[- (l+q)_\sigma \not{l} \gamma_\tau \not{p} \gamma_\rho (l-p)_\mu \sigma_{\sigma\tau} \right]_1 \right. \\ & + \left[2 p_\sigma l_\tau \not{l} \gamma_\rho (l+\not{q}) \gamma_\mu \sigma_{\sigma\tau} + 2 (l-p)_\sigma l_\rho \not{l} \gamma_\tau (l+\not{q}) \gamma_\mu \sigma_{\sigma\tau} \right. \\ & + \left. 2 (l-p)_\sigma \not{l} \gamma_\tau (l+\not{q}) \gamma_\mu \sigma_{\sigma\tau} - (l-p)_\sigma \not{l} \gamma_\mu \sigma_{\sigma\tau} (l+q)^2 \right]_2 \\ & + \left. \left[- (l+p)_\sigma \not{l} \gamma_\mu \sigma_{\sigma\tau} (l+q)^2 \right]_3 + \left[- \not{l} \gamma_\sigma \not{p} \gamma_\mu \sigma_{\sigma\tau} (l-p)^2 \right]_4 \right\} \end{aligned} \quad (3.2)$$

In this expression the summation over internal indices is four-dimensional because of the ε scalar diagrams. Specifically, the ε scalar contribution to (3.1) is

$$\frac{4i\delta_{bc}}{p^2 q^2} (4-n) \left(\frac{d^4 l}{l^2 (l-p)^2 (l+q)^2} \right) \left\{ \frac{1}{2} \text{tr} \left(\gamma_\mu \gamma_\nu (l-p) \gamma_\rho (l+q) \right) - l_\rho \text{tr} \left(\gamma_\mu \gamma_\nu (l-p) \right) \right\} \quad (3.3)$$

It is straightforward to check that this gives no contribution to $\gamma \cdot S, \partial \cdot S$ and does not destroy gauge invariance. Therefore, the calculation reduces to that of ordinary dimensional regularization, which has been done by Hagiwara, Pi and Tsao⁹⁾. We have redone their calculation as a check on our results and confirm that $\partial \cdot S = 0$ while $\gamma \cdot S = (3/4\pi^2) \sigma \cdot G^a_{\lambda} a$.

CONCLUSIONS

Regularization by dimensional reduction³⁾ appears to preserve the supersymmetry of supersymmetric gauge theories^{4),5)}. As we have shown it can also be used to calculate the superconformal anomalies with the usual one loop results. We have concentrated on the axial and supersymmetry anomalies because the energy momentum tensor presents no special problem. Unlike $\gamma \cdot S$ it is manifest that for θ^μ_μ the ε scalars can contribute only from two loops. Will they in fact contribute at two loops? Because the supersymmetry transformations include ε scalars it seems unavoidable that the currents given by the dimensional reduction scheme will be the appropriate superconformal currents at two loops. That is, we would expect S_μ as defined by dimensional reduction to be the current that is conserved at two loops. Since this contains ε scalars it is a different current from that of ordinary dimensional regularization and there is no reason to suppose that a calculation of $\gamma \cdot S$ or θ^μ_μ will give the same result in both regularization schemes. If supersymmetry is preserved at two loops the coefficients of the superconformal anomalies should continue to have equal coefficients. Our results lead us to conjecture that a two loop calculation using dimensional reduction will satisfy this requirement.

Regularization by dimensional reduction appears to give all the desired results of a supersymmetric regularization scheme, but despite this success its originator has recently argued¹²⁾ that the scheme is mathematically inconsistent. The inconsistency amounts to a calculation done in two different ways giving two different answers. As we have pointed out in Section 2 a calculation of the axial anomaly will give different answers depending on how the calculation is done but, provided we do not use the cyclic property of the trace, this amounts

to the ambiguity in the distribution of the anomaly between the vector and axial vector currents. This ambiguity may mean that dimensional reduction (and perhaps any other possible regularization scheme) is not mathematically respectable, but in this case we think that this is something that we can live with. Even if there is no totally unambiguous scheme we can still ask that the results of a calculation not be self-contradictory even though they may depend on the details of the calculation. We suggest that the "inconsistency" of Ref. 12) is of this type.

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Note added: After this paper was completed we received a preprint by P. Majumdar, E. Poggio and H. Schnitzer who discuss the anomaly of S_μ in the context of supersymmetric regularization. These authors obtain $\gamma \cdot S = 0$ and $\partial \cdot S \neq 0$. We defer discussion of this paper and more details of the ambiguities in the calculation of the axial current anomaly to a forthcoming paper, which will be published in the proceedings of the Erice conference on "Unification of the Fundamental interactions", March 1980.

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FIGURE CAPTIONS

Fig. 1 : Diagrams contributing to the axial anomaly.

Fig. 2 : Diagrams contributing to the matrix element of ∂_μ including ϵ scalars. Wiggly line is W_μ , straight line is λ , dotted line is ϵ scalar.

FIG. 1

FIG. 2