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THE INFRA-RED BEHAVIOUR OF THE RUNNING
COUPLING CONSTANT IN YANG-MILLS THEORIES

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1. INTRODUCTION

In Quantum Electrodynamics (QED) the infra-red (IR) structure of the theory is well understood. In Quantum Chromodynamics (QCD) it is not, though considerable effort has been spent in recent years in studying it. One reason for this interest is the hope that (because of the apparently stronger IR singularities in QCD over those in QED) confinement can be understood through the IR properties of QCD. Indeed, it has been argued¹⁾ that an "effective potential" in momentum space between quarks is proportional to $g^2(q^2)/q^2$ where $g(q^2)$ is singular in the IR (that is, as $q^2 \rightarrow 0$) then the "effective potential" produces confinement. In particular, if $g^2(q^2) \sim 1/q^2$ as $q^2 \rightarrow 0$, this can be crudely translated into a potential growing linearly with distance for large distances. There are clearly difficulties and uncertainties with these ideas. First the identification of $g^2(q^2)/q^2$ as an "effective potential" has been made only in a leading log study of the IR properties of QCD, and an investigation of non-leading logs casts doubt on it²⁾. Second, $g^2(q^2)$ is not really gauge invariant, and it is clearly uncomfortable to attribute a physical result, such as confinement, to a non-gauge invariant quantity.

We do not, here, propose to involve ourselves in the discussion of whether or not the above speculations are correct, but we do wish to study what can be said about $g^2(q^2)$ in the IR limit, and the remarks in the above paragraph can be taken as our motivation.

A convenient way to identify $g^2(q^2)$ is to study the gluon propagator in axial gauge, for in this gauge the gluon wave function renormalization constant determines $g^2(q^2)$ directly:

$$g^2(q^2) = Z(q^2/\Lambda^2) g_0^2. \quad (1.1)$$

Within the class of axial gauges, Z , and hence g , is gauge invariant.

In QED we recall that a convenient method of obtaining the IR behaviour of the electron propagator is to use the Dyson equation combined with the Ward identity to generate an integral equation whose solution defines a function having the same IR behaviour as the true propagator³⁾. We may hope that an analogous procedure can be used in QCD to obtain the IR behaviour of the glue propagator.

For the sake of completeness, we give a brief outline of the QED case. The Dyson equation reads (for convenience we take a spinless electron)

$$S^{-1}(p) = p^2 - m_0^2 + e_0^2 \int d^4k \, D(k) S(p-k) \\ \cdot (2p-k)_\mu (\delta_{\mu\nu} - \xi \frac{k_\mu k_\nu}{k^2}) \Gamma_\nu(p, p-k; k). \quad (1.2)$$

Here S is the electron propagator,

$$D_{\mu\nu}(k) = D(k) (\delta_{\mu\nu} - \xi k_\mu k_\nu / k^2)$$

is the photon propagator in a gauge specified by the parameter ξ , and Γ_ν is the proper vertex function in which a photon of momentum $p-k$ connects to one of momentum k .

In general, the vertex can be written

$$\begin{aligned} \Gamma_\nu(p, p-k; k) &= (2p-k)_\nu \Gamma_1 \\ &+ \left(k_\nu - \frac{(2p-k)_\nu k^2}{(2p-k) \cdot k} \right) \Gamma_2 \end{aligned} \quad (1.3)$$

where Γ_1 and Γ_2 are scalar functions of p^2 , $(p-k)^2$ and k^2 .

Now the Ward identity fixes Γ_1 in terms of the electron propagator:

$$\Gamma_1 = \frac{S^{-1}(p) - S^{-1}(p-k)}{p^2 - (p-k)^2} \quad (1.4)$$

Γ_2 is, however, not determined. Thus in general we cannot use (1.2) to evaluate S .

In the IR limit, $p^2 \rightarrow m^2$ (or small k), however, we know that $D(k^2) \rightarrow Z_3/k^2$ (where the physical charge $e = Z_3^{1/2} e_0$) and the Γ_2 term in Γ does not matter. In this limit, therefore,

(1.2) does become a closed integral equation defining a function having the same IR behaviour as the true propagator. This equation is

$$S(p) = \frac{1}{p^2 - m^2} \left\{ 1 + 4e^2 \int \frac{d^4 k}{k^2} \left(p^2 - \xi \frac{(p \cdot k)^2}{k^2} \right) \left[\frac{S(p-k) - S(p)}{(p-k)^2 - p^2} \right] \right\} \quad (1.5)$$

The equation is linear and may be solved by a Laplace transform. In the IR limit, the solution is

$$S(p) = \int_0^\infty ds e^{is(p^2 - m^2) + G(s)}$$

where

$$G(s) = 4e^2 \int \frac{d^4 k}{k^2} \left(p^2 - \xi \frac{(p \cdot k)^2}{k^2} \right) \int \frac{e^{is(k^2 - 2p \cdot k)} - 1}{k^2 - 2p \cdot k} \Big|_{p^2 = m^2} \quad (1.6)$$

The IR limit ($p^2 \rightarrow m^2$) is controlled by $s \rightarrow \infty$. As $s \rightarrow \infty$,

$$G(s) \rightarrow \frac{\alpha}{\pi} (1 + \xi/2) \log s. \quad (1.7)$$

Thus, as $p^2 \rightarrow m^2$,

$$S(p) \rightarrow \left(\frac{1}{p^2 - m^2} \right)^{1 + \frac{\alpha}{\pi} (1 + \xi/2)} \quad (1.8)$$

In QED this is, of course, a well-known result⁴⁾. At issue here is whether or not we can use a similar technique to obtain the glue propagator in QCD.

First of all we shall limit ourselves to a pure Yang-Mills theory and ignore quarks. The reason for this is our belief that the dominant IR singularities in QCD come from the pure glue sector of the theory: quarks feel the effect of confinement but play no role in generating it. We are thus dealing with a massless theory, in which the only mass scale is provided by the ultra-violet (UV) cut-off Λ . (Upon renormalization, this of course is translated into a renormalization point M or a renormalization group invariant -- and thereby physical -- mass μ .) In a massless theory, leading log UV and IR approximations coincide. Since QCD is UV free, we know the leading log UV value of g :

$$g^2(q^2) \xrightarrow{q^2 \rightarrow +\infty} \frac{g^2(M^2)}{1 - b g^2(M^2) \log q^2/M^2}. \quad (1.9)$$

Formally, this is also the leading log IR result, but since in QCD $b < 0$, there is a singularity as $q^2 \rightarrow 0$ and $\log q^2/\Lambda^2 \rightarrow -\infty$. Hence the result is not valid in the IR: we must go beyond a leading log calculation. Fortunately, the Dyson equation/Ward identity approach will permit us to do this.

The Dyson equation for the glue propagator is shown in Fig. 1. (We will write it down in detail in Section 2.) Our procedure will be to eliminate the triple and quadruple glue vertex functions appearing in it using the Ward identities satisfied by these functions, hoping that, as in QED, the undetermined parts of the vertices are irrelevant in the IR regime.

What basis do we have for this hope? In QED the "moment form factor" (essentially Γ_2) is known to be free of IR singularities, and, furthermore, the spinological coefficient of it in the vertex vanishes with the photon momentum. Hence it can be neglected. In pure Yang-Mills theory we have not been able to find a proof that the undetermined parts of the vertex are irrelevant in the IR. As we shall argue in the following section, we can show that they are irrelevant in the regime where any one of the three momenta in the three glue vertex is small relative to the other two. But, unfortunately, we also need to know the vertex in the regime where all three are comparable. So, at present, our use of the Ward identities to eliminate the vertex functions in the Dyson equation must be looked on as an assumption.

This assumption can also be looked at as a conjecture that the UV and IR singular parts of functions in massless pure Yang-Mills theories are intimately connected. Suppose we look for simplicity at a function of a single momentum q in a massless theory. The unrenormalized function $A(q/\Lambda, g_0)$ renormalizes according to

$$A(q/\Lambda, g_0) = Z(M/\Lambda) A_{ren}(q/M, g(M)) \quad (1.10)$$

where the factor Z is present or absent according as A is UV infinite or finite. If we choose $M = q$, (1.10) reads

$$A(q/\Lambda, g_0) = Z(q/\Lambda) A_{\text{ren}}(1, g(q)). \quad (1.11)$$

Now let $q \rightarrow 0$. If A_{ren} has a finite limit (which it automatically does if $g(q)$ has, as in massless QED, for example) then the only IR singularity comes from Z , so only their functions which are UV singular are IR singular. We know from the UV renormalization program that all functions in the vertex which are UV singular are fixed in terms of the propagator by the Ward identity. Hence if all IR singularities are in $Z(q/\Lambda)$, all IR singular parts of the vertex are also fixed by the Ward identity. In QED this is true, because $A_{\text{ren}}(1, g(q))$ in (1.11) is indeed finite. In QCD (especially since $g(q)$ may be singular as $q \rightarrow 0$) we do not know if this is true: it must be taken as an assumption. (We remark that a one loop calculation, A_{ren} is finite in QCD, so the assumption is trivially true at this level.)

Once the vertex functions are eliminated, we obtain a (non-linear) integral equation for a function whose IR behaviour is then supposed to coincide with that of the true gluon propagator. It is this equation which we propose to obtain and to study, and from which we hope to extract the IR behaviour of $g(q^2)$.

$$\begin{aligned}
 \Pi_{\mu\nu} = & \quad (\text{wavy line})^{-1} \\
 & + \frac{1}{2} \quad \text{[Diagram: A loop with two shaded vertices and two external wavy lines]} \\
 & + \frac{1}{6} \quad \text{[Diagram: A loop with three shaded vertices and two external wavy lines]} \\
 & + \quad \text{[Diagram: A tadpole with one shaded vertex and one external wavy line]}
 \end{aligned}$$

Fig. 1

2. THE PROPAGATOR AND VERTEX FUNCTION

In axial gauge the gluon propagator $\Delta_{\mu\nu}^{ab}(q)$ describing the propagation of a gluon with four-momentum q satisfies

$$n_\mu \Delta_{\mu\nu}^{ab} = \Delta_{\mu\nu}^{ab} n_\nu = 0 \quad (2.1)$$

where n is the direction of the gauge. It can be written in the form

$$\Delta_{\mu\nu}^{ab} = - \frac{\delta_{ab}}{q^2} \left\{ B \left(\delta_{\mu\nu} - \frac{n_\mu n_\nu}{n^2} + \frac{N_\mu N_\nu}{(1-\gamma) n^2} \right) - A \frac{N_\mu N_\nu}{(1-\gamma) n^2} \right\} \quad (2.2)$$

in terms of two scalar functions A and B . These depend on the dimensionless variables q^2/Λ^2 *) and

$$\gamma \equiv q^2 n^2 / (q \cdot n)^2. \quad (2.3)$$

We define

$$N_\mu = n_\mu - \frac{q_\mu n^2}{q \cdot n}, \quad (2.4)$$

so that $n \cdot N = 0$.

*) The UV cut-off Λ^2 provides the only mass scale in the theory, so q^2 is referred to this. Upon renormalization, the mass scale will be translated into a renormalization point M^2 , or equivalently, a renormalization group invariant mass μ^2 which has physical meaning.

When $B = 1$ and $A = \gamma$, $\Delta_{\mu\nu}^{ab}$ becomes the free gluon propagator $\Delta_{\mu\nu}^{(0)ab}$. We shall henceforth suppress explicit colour indices, in the interest of simplicity.

We define an "inverse gluon propagator" $\Pi_{\mu\nu}$ through

$$\Delta_{\mu\lambda} \Pi_{\lambda\nu} = \delta_{\mu\nu} - \frac{q_\mu q_\nu}{q \cdot n}. \quad (2.5)$$

The extra term in (2.5) results from the constraint (2.1)*).

Since $q_\nu \Pi_{\mu\nu} = \Pi_{\mu\nu} q_\nu = 0$ we can write

$$\begin{aligned} \Pi_{\mu\nu} = & - \left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) g \\ & + \left(\delta_{\mu\nu} - \gamma \frac{q_\mu q_\nu}{q^2} - (1-\gamma) \frac{n_\mu n_\nu}{n^2} + \frac{N N_\nu}{n^2} \right) f. \end{aligned} \quad (2.6)$$

Evidently the scalar functions f and g satisfy

$$\gamma f - g = \frac{n_\mu \Pi_{\mu\nu} n_\nu}{n^2} \frac{1}{1-\gamma} \quad (2.6a)$$

and

$$(1+\gamma)f - 3g = \Pi_{\mu\mu} \quad (2.6b)$$

*) We could avoid this term by adding $\xi q_\mu q_\nu / (q \cdot n)^2$ to $\Delta_{\mu\nu}$ and taking the limit $\xi \rightarrow 0$ at the end of the calculations.

and are related to A and B by

$$B = \frac{q^2}{q-f}, \quad A = \frac{q^2}{q/4-f}. \quad (2.7)$$

The inverse propagator satisfies the Dyson equation

$$\begin{aligned} \Pi_{\mu\nu} = & - (q^2 \delta_{\mu\nu} - q_\mu q_\nu) \\ & + \frac{1}{2} \int dk \, i \Gamma_{\mu\lambda\lambda'}^{(0)}(q, -k, -k') \Delta_{\lambda\sigma}(k) \\ & \cdot \Delta_{\lambda'\sigma'}(k') i T_{\sigma\sigma'\nu}(k, k', -q) \quad (2.8) \\ & + \text{quadruple glue terms} \end{aligned}$$

which is represented graphically in Fig. 1. In Eq. (2.8), $k' = q-k$ and dk stands for

$$i g_0^2 C_A \frac{d^4 k}{(2\pi)^4} \quad (2.9)$$

where C_A is the Casimir eigenvalue of the adjoint representation of the colour group. $\Gamma^{(0)}$ is the bare vertex function: explicitly

$$\begin{aligned} i \Gamma_{\lambda_1 \lambda_2 \lambda_3}^{(0)}(q_1, q_2, q_3) = & (q_1 - q_2)_{\lambda_3} \delta_{\lambda_1 \lambda_2} \\ & + \text{cyclic permutations} \end{aligned} \quad (2.10)$$

Equation (2.8) is plagued with overlapping divergences which makes it inconvenient for determining the UV behaviour of $\Pi_{\mu\nu}$. Nevertheless we may expect it to be useful for studying the infra-red behaviour of $\Pi_{\mu\nu}$ since in this case the relevant values of the magnitudes of the momenta k, k' and q should be much less than the UV cut-off Λ .

The proper vertex Γ appearing in (2.8) satisfies the Ward identity

$$i \Gamma_{\sigma_1 \sigma_2 \sigma_3}(q_1, q_2, q_3)(q_3)_{\sigma_3} = \Pi_{\sigma_1 \sigma_2}(q_1) - \Pi_{\sigma_1 \sigma_2}(q_2). \quad (2.11)$$

In the kinematic regime where all momenta are small (compared to Λ) we have shown that (2.11) completely determines the vertex function Γ . If the corresponding Ward identity for the four-gluon vertex also determines it in the same kinematic regime, then the Dyson equation (2.8) becomes a closed set of non-linear integral equations determining the infra-red behaviour of the gluon propagator.

We have not yet completed our analysis of the four-gluon vertex. Nevertheless, we feel that the results obtained by ignoring the four gluon term in (2.8) (even though this violates the explicit gauge invariance of the theory) and using the results of our study of the three gluon vertex are of sufficient interest to justify an interim report at this time. We do not anticipate that our basic conclusion will be altered when we do include the four-gluon term.

Even without the four-gluon term, (2.8), together with our solution for Γ , yield a non-linear integral equation for the infra-red behaviour of the propagator. We shall search for solutions of this equation in which the propagator is an infra-red singular factor, independent of the gauge choice η_μ , times the spinology of the free propagator. We do this for several reasons: first,

because this is the structure of the IR behaviour of the electron propagator in spin 0, spin $\frac{1}{2}$, and spin 1 QED. Second, because it is evidently true to lowest order in perturbation theory. Third, because it yields a great simplification of the equation.

Thus we look for solutions to the propagator in which $f = 0$ and

$$g(q^2/\Lambda^2, \gamma) = g(q^2/\Lambda^2)$$

only; that is, solutions of the form

$$\Pi_{\mu\nu}(q) = -q^2 Z^{-1}(q^2/\Lambda^2) (\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}) \quad (2.12)$$

with

$$g(q^2)/q^2 = Z^{-1}.$$

Such a solution is equivalent to a propagator in which

$$A = \gamma B = \gamma Z(q^2/\Lambda^2)$$

so that

$$\begin{aligned} \Delta_{\mu\nu}(q) &= - \frac{Z(q^2/\Lambda^2)}{q^2} \left(\delta_{\mu\nu} - \frac{\eta_\mu \eta_\nu}{n^2} + \frac{N_\mu N_\nu}{n^2} \right) \\ &\equiv - \frac{Z(q^2/\Lambda^2)}{q^2} \Sigma_{\mu\nu}(q). \end{aligned} \quad (2.13)$$

With the above form for $\Pi_{\mu\nu}$, we have shown that the most general expression for Γ consistent with the Ward identity, Bose symmetry, and free of kinematic singularities is

$$\begin{aligned}
 & i \Gamma_{\sigma_1 \sigma_2 \sigma_3}^{abc}(q_1, q_2, q_3) \\
 &= f^{abc} \delta_{\sigma_1 \sigma_2} \left[(q_1)_{\sigma_3} z^{-1}(q_1) - (q_2)_{\sigma_3} z^{-1}(q_2) \right] \\
 &+ f^{abc} \frac{z^{-1}(q_1) - z^{-1}(q_2)}{q_1^2 - q_2^2} \left[q_1 \cdot q_2 \delta_{\sigma_1 \sigma_2} - (q_1)_{\sigma_2} (q_2)_{\sigma_1} \right] \\
 &\quad \cdot (q_2 - q_1)_{\sigma_3} \\
 &\quad (2.14) \\
 &+ F^{abc}(q_1^2, q_2^2, q_3^2) \left[q_1 \cdot q_2 \delta_{\sigma_1 \sigma_2} - (q_1)_{\sigma_2} (q_2)_{\sigma_1} \right] \\
 &\quad \cdot \left[(q_1 \cdot q_3) (q_1)_{\sigma_3} - (q_2 \cdot q_3) (q_2)_{\sigma_3} \right] \\
 &+ G^{abc}(q_1^2, q_2^2, q_3^2) \\
 &\quad \cdot \left[\delta_{\sigma_1 \sigma_2} (q_2 \cdot q_3) (q_1)_{\sigma_3} - q_1 \cdot q_3 (q_2)_{\sigma_3} \right] \\
 &\quad + \text{cyclic permutations}
 \end{aligned}$$

We have here temporarily re-instated the colour indices a, b, c ; thus f^{abc} denotes the structure constants of the colour group. The functions F and G are free of kinematic singularities; F is symmetric under an interchange $a \leftrightarrow b, q_1^2 \leftrightarrow q_2^2$ while

G is antisymmetric. Aside from this, F and G remain undetermined, since the tensors multiplying them satisfy the homogeneous Ward identity.

We note that the first line in Eq. (2.14) reduces to the bare vertex if Z is replaced by unity. The second line has a structure similar to that of the vertex function in QED. The undetermined parts involving F and G are all proportional to a product $(q_1)_{\lambda_1}(q_2)_{\lambda_2}(q_3)_{\lambda_3}$ and hence vanish when any one of the three momenta vanishes. If indeed this is the relevant regime for the infra-red, then we are led to drop the F and G terms in (2.14). Thus (2.8) and (2.14) together yield a closed non-linear integral equation for the function Z .

Let us recapitulate the assumptions needed to obtain this equation:

- i) The infra-red structure of $\Pi_{\mu\nu}$ is a gauge invariant scalar function $Z^{-1}(q^2/\Lambda^2)$ times the free $\Pi_{\mu\nu}$.
- ii) The contribution of the four-gluon vertex in (2.8) can be ignored.
- iii) The F and G terms in the expression (2.14) of the vertex do not contribute in the IR regime.

Of these, only (iii) is really essential, and its validity must be studied further.

With regard to (i) and (ii) we make the following comments: First, we have carried out an analysis of the Ward identity for Γ without assumption (i); that is, including both the functions f and g in Π . We have obtained a general expression for Γ , free of kinematic singularities, analogous to and a generalization of (2.14). The use of this form in (2.8) then yields a pair of coupled integral equations for f and g . This more general situation will be discussed in a separate paper.

Second, we are now engaged in a study of the Ward identity for the four-gluon vertex which will fix it in terms of f and g . Thus we will be able to include its effect in (2.8) as well. This will also be reported elsewhere.

3. THE INTEGRAL EQUATION AND ITS "SOLUTION"

Given the ansatz (2.12), and the resulting vertex (2.14), the Dyson equation (2.8) becomes an integral equation for $Z(q^2/\Lambda^2)$.

$$\begin{aligned} \Pi_{\mu\nu}(q) - \Pi_{\mu\nu}^{(0)}(q) &= \left(1 - \frac{1}{Z(q)}\right) (q^2 \delta_{\mu\nu} - q_\mu q_\nu) \\ &+ \frac{1}{2} \int \frac{dk}{k^2 k'^2} Z(k) Z(k') i \Gamma_{\lambda\lambda'}^{(0)}(q, -k, -k') \\ &\quad \Sigma_{\lambda\sigma}(k) \Sigma_{\lambda'\sigma'}(k') i \Gamma_{\sigma\sigma'}(k, k', -q) \\ &+ \text{quadruple glue terms,} \end{aligned} \tag{3.1}$$

where $i\Gamma^{(0)}$ is given in (2.10) and $i\Gamma$ is given in (2.14) with the F and G terms left out. (Σ , we recall, is defined in (2.13).)

It is important to keep in mind that (3.1) defines a function Z which is not the true gluon propagator, but only has the same IR limit. Thus, while the $Z(q)$ which in the solution of (3.1) is defined for all q , only the $q^2 \rightarrow 0$ limit has physical significance. Therefore, the mass scale, which for the true propagator comes in because of the need for a UV cut-off, can appear quite differently here. Here, in fact, we have two choices. We can insert no cut-off and look for solutions Z which are well behaved as $q^2 \rightarrow \infty$. In this case the solutions must scale, so that $Z = Z(q^2/M^2)$ where M is arbitrary. Alternatively, we can put in a cut-off Λ^2 ,

and allow badly behaved Z 's as $q^2 \rightarrow \infty$; then Λ^2 becomes the mass scale and $Z = Z(q^2/\Lambda^2)$. This, we emphasize, makes a different equation from the first choice. In either case, however, we care only about the $q \rightarrow 0$ behaviour.

First of all, on grounds of gauge invariance, we expect $\Pi_{\mu\nu}(q)|_{q=0} = 0$. The right-hand side of (3.1) is readily evaluated at $q = 0$, noting that

$$\frac{Z(k) - Z(k')}{k^2 - k'^2} \rightarrow \frac{\partial Z(k)}{\partial k^2} \quad (3.2)$$

in this limit. We find after a tedious but straightforward calculation, that the triple gluon terms contribute

$$\begin{aligned} \Pi_{\mu\nu}(q) / q = 0 &= \\ &= (\delta_{\mu\nu} + 2 \frac{\eta_{\mu\nu}}{n^2}) \int \frac{dk}{k^2} [Z(k) + k^2 \frac{\partial}{\partial k^2} Z(k)] \end{aligned} \quad (3.3)$$

To guarantee $\Pi_{\mu\nu}(0) = 0$, then, we must require the integral to vanish. This means we must have

$$k^2 Z(k) \Big|_{k^2=\infty} = k^2 Z(k) \Big|_{k^2=0} \quad (3.4)$$

Equation (3.4) should be viewed as a constraint on the function $Z(k)$. The constraint is evidently satisfied if $Z(k) = \text{const}/k^2$, but can of course also be satisfied in other ways.

We now wish to look for solutions $Z(k)$ to the integral equation (3.1), subject to the constraint (3.4). To simplify the spinology, let us multiply (3.1) by $n_\mu n_\nu / n^2$. This removes the quadruple-gluon term from the equation, because the bare quadruple-gluon vertex transfers the gauge direction n_μ to one of the internal propagators to which it is orthogonal.

The quadruple-gluon tadpole diagram (see Fig. 1) is also eliminated, because $\int dk \Delta_{\lambda\lambda}(k) = 0$.

Thus we obtain the integral equation*)

$$\begin{aligned} \frac{1}{Z(q)} (1 - \gamma) &= (1 - \gamma) \\ + \frac{1}{q^2 Z(q)} \cdot \frac{1}{2} \int \frac{dk}{k^2 k'^2} \frac{n \cdot (k - k')}{n^2} Z_{\lambda\sigma}(k) Z_{\lambda\sigma'}(k') \\ \cdot \left\{ Z(k') n \cdot k \frac{Z(k) - Z(q)}{k^2 - q^2} q_\sigma (q + k)_\sigma \right. & \quad (3.5) \\ - Z(k) n \cdot k' \frac{Z(k') - Z(q)}{k'^2 - q^2} (q + k')_\sigma q_{\sigma'} \left. \right\} \end{aligned}$$

*) We can obtain another independent scalar equation from (3.1) by taking the trace -- that is, by multiplying by $\delta_{\mu\nu}$. If our ansatz (2.12) is correct, this equation will lead to nothing new. If it is not, then it, together with the equation obtained from $n_\mu n_\nu / n^2 \cdot \Pi_{\mu\nu}$ provide two coupled equations for the two functions A and B appearing in the propagator, as outlined in Section 2.

$$\begin{aligned}
 &= \frac{1}{2q^2} \int \frac{dk}{k^2 k'^2} \frac{n \cdot (k - k')}{n^2} \Sigma_{\lambda\sigma}(k) \Sigma_{\lambda\sigma'}(k') \\
 &\quad \left\{ \delta_{\sigma\sigma'} [Z(k') n \cdot k - Z(k) n \cdot k'] \right. \\
 &\quad \left. + n \cdot (k - k') \frac{Z(k) - Z(k')}{k^2 - k'^2} [k \cdot k' \delta_{\sigma\sigma'} - k'_\sigma k_{\sigma'}] \right\}
 \end{aligned}
 \tag{3.5}$$

cont'd)

Equation (3.5) is now a scalar non-linear integral equation for the function $Z(q)$. What we should like to do is to study this equation, to see whether or not it has solutions. If it does we should like to study their uniqueness, and finally to identify the IR behaviour of those solutions. As a matter of practice, however, we are unable to carry out this program. Instead we content ourselves with the following.

We assume there exists a solution which has a power singularity $Z(q) \rightarrow (M^2/q^2)^\alpha$ as $q^2 \rightarrow 0$. Equation (3.5) then provides a constraint on what this power can be. To see how this constraint comes about, we rewrite (3.5) in the form

$$\frac{1}{Z(q)} = \frac{1 + \frac{\gamma}{\gamma-1} \int \frac{dk}{k^2 k'^2} Z(k) K}{1 + \frac{\gamma}{\gamma-1} \int \frac{dk}{k^2 k'^2} Z(k) Z(k') L}
 \tag{3.6}$$

where the dimensionless kernels K and L can be read off from (3.5). Now suppose $Z(q) \rightarrow (M^2/q^2)^\alpha$. Then (3.6) will, as $q^2 \rightarrow 0$, take the form

$$\left(\frac{q^2}{M^2}\right)^\alpha = \frac{1 + \left(\frac{M^2}{q^2}\right)^\alpha F(\alpha, \gamma)}{1 + \left(\frac{M^2}{q^2}\right)^{2\alpha} G(\alpha, \gamma)} \quad (3.7)$$

where the functions F and G are evaluated by carrying out the integrals in the numerator and denominator of (3.6). Then, provided $\alpha > 0$, the two sides of (3.7) will coincide if there is a value of the power α such that

$$1 = \frac{F(\alpha, \gamma)}{G(\alpha, \gamma)} \quad (3.8)$$

for all γ .

We have not yet carried out this procedure for an arbitrary power α , because we have not been able to, in general, evaluate the functions $F(\alpha, \gamma)$ and $G(\alpha, \gamma)$. We can, however, note the following. As $\alpha \rightarrow 1$, both F and G become singular like $(1 - \alpha)^{-1}$. (This is easily seen from an inspection of the kernels K and L defined through (3.6).) The coefficients of these singularities are determined by the region $k \ll q$ in the integrals in (3.6). In this region, the integrals can be evaluated easily, so that the coefficients of the singularities can be calculated. They turn out to be equal: that is, as $\alpha \rightarrow 1$,

$$F(\alpha, \gamma) \rightarrow \frac{C(\gamma)}{1 - \alpha} + \dots$$

and

$$G(\alpha, \gamma) \rightarrow \frac{C(\gamma)}{1-\alpha} + \dots$$

with the same coefficient $C(\gamma)$. This again suggests that the power $\alpha = 1$ characterizes the infra-red singularity of Z , so that as $q^2 \rightarrow 0$,

$$Z(q) \rightarrow (M^2/q^2) \quad (3.9)$$

It is important to emphasize that the appearance of $\alpha = 1$ is not a coincidence, but has its roots in the form of the Ward identity, and in particular in the fact that the vertex involves the structure

$$\frac{Z(k) - Z(q)}{k^2 - q^2}, \quad \frac{Z(k) - Z(k')}{k^2 - k'^2}.$$

As we already saw in (3.3), other powers than $\alpha = 1$ appear not to work.

Conversely, it is even more important to bear in mind that we have not shown that (3.9) is true; we have only shown that if there exists a solution with a power singularity, then there is a suggestion that the power is one. More work is needed to confirm this suggestion.

Nevertheless, the $1/q^2$ behaviour is very encouraging. A naive interpretation of its meaning suggests a confining effective potential $V(r) \sim r$ for large r . Also note that the power we obtain is independent both of the coupling constant (and therefore our "solution" is a non-perturbative one) and of the colour gauge group. Thus any non-Abelian gauge group produces the same IR behaviour, and hence the same confining effective potential.

We have, unfortunately, no deep physical understanding of why the precise power $\alpha = 1$ should arise. Artificial models analogous to the equation discussed here can give singularities with different powers⁵⁾. Other arguments for confinement, also giving rise to linear potentials, are often based on the existence of classical solutions to the Yang-Mills theory such as instantons or merons. The existence of such solutions does not affect the form of the Dyson equation or Ward identity used here. Thus any physics which these solutions represent should be included in our approach, and presumably manifests itself in the existence of unusual non-perturbative solutions. Perhaps it is one of these which we have found, though any connection of our result to instantons is, to say the least, completely obscure at present.

In any event, we feel that the results obtained here are sufficiently encouraging that the Dyson equation-Ward identity approach is worth pursuing further. In particular, we need to go beyond the small momentum expansion, to study the existence and uniqueness of solutions. We also need to fully include the quadruple glue term, to assure ourselves that it does not alter anything. And we need to incorporate quarks and to clarify the connection between a $(1/q^2)^2$ singularity and confinement.

The work described here is being carried out in collaboration with James Ball and with Marshall Baker. As is evident from the preceding, the work is not yet finished. We hope we will have further results to report in the near future.

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