Ref.TH.2779-CERN

FUNDAMENTAL MONOPOLES AND MULTIMONOPOLE SOLUTIONS
FOR ARBITRARY SIMPLE GAUGE GROUPS

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A B S T R A C T

Magnetic monopole solutions for an arbitrary compact simple gauge group are considered in the Prasad-Sommerfield limit. For each group and choice of symmetry breaking there is a set of fundamental monopoles with minimal topological charges and possessing no internal degrees of freedom; the number of these is less than or equal to the rank of the gauge group. It is shown that if the unbroken gauge group is non-Abelian, all solutions with higher topological charges belong to p -parameter families, where p is the number of position and group orientation parameters needed to describe a set of non-interacting fundamental monopoles with the given topological charge. It is argued that these solutions, some examples of which are given, should therefore be interpreted as multimonopole configurations. An extension of these results to the case of a non-Abelian unbroken gauge symmetry is conjectured and shown to be valid for a number of examples.

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1. INTRODUCTION

The existence of classical magnetic monopole solutions to spontaneously broken gauge theories was first demonstrated by 't Hooft and Polyakov¹⁾ in the context of an $SU(2)$ theory with a triplet of Higgs scalars. Although no additional $SU(2)$ solutions have been found, various spherically symmetric ansatzes have been used to obtain a great number of solutions for larger gauge groups²⁾. It is natural to ask how these solutions should be viewed: should they be interpreted as being all on an equal footing, or can they be understood in terms of a small number of fundamental monopoles. Also, it is not known whether there are any solutions which are not spherically symmetric. In this paper we address both these questions.

For the most part we work in the Prasad-Sommerfield³⁾ limit of vanishing scalar field potential. In this limit the scalar field becomes massless and mediates a long-range force, which has been shown to just cancel the long-range magnetic repulsion between monopoles⁴⁾. This absence of long-range interaction between monopoles suggests the possibility of static multimonopole solutions. Indeed, it has been shown for the $SU(2)$ theory that an arbitrary Prasad-Sommerfield solution with n units of magnetic charge will have n times the mass of a unit monopole⁵⁾ and will depend on precisely the number of parameters needed to describe n non-interacting monopoles⁶⁾. While the existence of solutions with $n > 1$ has not been shown, this result already implies that the unit monopole is unique in not having any internal degrees of freedom.

In this paper this analysis is extended to an arbitrary simple compact gauge group. We are led to a picture in which there are not one but several types of fundamental monopoles; the number of these is at most equal to the rank of the gauge group. Each is spherically symmetric and has no internal degrees of freedom. Their masses can in general be quite different. For all other solutions the parameter count and mass formulas are just those expected for a multimonopole system containing an appropriate number of fundamental monopoles. In particular, most of the known solutions are of the latter type. These thus belong to larger families of solutions which in general are not spherically symmetric. We argue

that these should all be interpreted as multimonopole solutions; the ones which are presently known correspond to the case where the component monopoles happen to be superimposed at the same point.

The remainder of this paper is arranged as follows: Section 2 reviews some properties of monopoles and of the Prasad-Sommerfield limit. It contains a discussion of some properties of the root vectors which play an important role in our analysis. In Section 3 we count the zero modes about a Prasad-Sommerfield solution. We then determine the number of parameters and suggest an interpretation of the result. The methods of this section are valid only when the unbroken subgroup of the gauge group is Abelian. The extension of our results to the general case is the subject of Section 4. Section 5 contains some comments on our results. There are three appendices.

2. REVIEW OF MONOPOLES

We consider a Yang-Mills theory based on a compact simple gauge group G , of rank ℓ and dimension d , with a Higgs scalar field in the adjoint representation. The Lagrangian is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F^{\mu\nu a} + \frac{1}{2} (D_\mu \phi)^a (D^\mu \phi)^a - V(\phi) \quad (2.1)$$

where

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f_{abc} A_\mu^b A_\nu^c$$

$$(D_\mu \phi)^a = \partial_\mu \phi^a + f_{abc} A_\mu^b \phi^c \quad (2.2)$$

Here the scale of the gauge field has been chosen so that the gauge coupling constant is unity, while the f_{abc} are the structure constants of the group, defined by the commutation relations of the generators:

$$[T_a, T_b] = i f_{abc} T_c \quad (2.3)$$

We will not give the potential V explicitly, but only specify that it give the scalar field a non-vanishing vacuum expectation value. The nature of the vacuum expectation value determines the subgroup of G which remains unbroken. This subgroup may be as small as a product of ℓ $U(1)$ factors; we shall refer to this case as that of maximal symmetry breaking. For special choices of the vacuum expectation value, however, the unbroken gauge group is larger and thus non-Abelian. In either case the scalar field picks out a special $U(1)$ subgroup for which we may define magnetic and electric charges by

$$\begin{aligned} Q_M &= V^{-1} \int dS_i \phi^a B_i^a \\ Q_E &= V^{-1} \int dS_i \phi^a E_i^a \end{aligned} \quad (2.4)$$

where

$$\begin{aligned} E_i &= F_{0i} \\ B_i &= \frac{1}{2} \epsilon_{ijk} F_{jk} \\ V &= \lim_{r \rightarrow \infty} (\phi^a \phi^a)^{1/2} \end{aligned} \quad (2.5)$$

and the integration is over a surface at spatial infinity.

In the $SU(2)$ theory, topological considerations require that Q_M be of the form $4\pi n$. The analogous results in theories with larger gauge groups are somewhat more complicated; before describing them, we first review some properties of the Lie algebra. We begin by choosing an ℓ -dimensional Abelian subalgebra H , with a basis given by ℓ generators H_i . Next we construct raising and lowering operators E_α , obeying

$$[H_i, E_\alpha] = \alpha_i E_\alpha \quad (2.6)$$

These may be normalized so that

$$[E_\alpha, E_{-\alpha}] = \sum_{i=1}^l \alpha_i H_i \quad (2.7)$$

Our form for the Lagrangian assumes a set of generators which are orthonormal with respect to the Cartan inner product. Such a set is given by the H_i together with certain linear combinations of the E_α , namely

$$\begin{aligned} L_\alpha &= \frac{1}{\sqrt{2}} (E_\alpha + E_{-\alpha}) \\ L_{\bar{\alpha}} &= -\frac{i}{\sqrt{2}} (E_\alpha - E_{-\alpha}) \end{aligned} \quad (2.8)$$

The roots $\vec{\alpha}$ may be viewed as vectors forming a lattice in an l -dimensional Euclidean space⁷⁾. A basis for the lattice can be chosen to be l simple roots $\vec{\beta}^{(a)}$, with the property that any other root is a linear combination of simple roots with integral coefficients, all of the same sign. Roots are termed positive or negative according to the sign of the coefficients. It will be convenient to specify a particular set of simple roots as follows. Let ϕ_0 be the asymptotic value of the Higgs field along the positive z -axis. (The asymptotic value in any other direction can differ from this only by a gauge transformation.) The subalgebra H can be chosen to contain any given element of the Lie algebra; in particular we choose it so that

$$\phi_0 = \sum_{a=1}^d \phi_0^a T^a = v \sum_{i=1}^l h_i H_i = v \vec{h} \cdot \vec{H} \quad (2.9)$$

Here v is defined by Eq. (2.5) and we have adopted vector notation for the inner product in the Euclidean space spanned by the roots. If the ℓ -component vector $\vec{h}$ is orthogonal to none of the roots, then we require that the simple roots all have positive inner products with $\vec{h}$; there is a unique set satisfying this condition. If $\vec{h}$ is orthogonal to some of the roots, then we can only require that the $\vec{h} \cdot \vec{\beta}^{(a)}$ be non-negative; this leaves some ambiguity in the choice of the simple roots, which we discuss further in Section 4. The physical difference between these two possibilities can be seen by considering the vector field mass matrix. The vector fields which remain massless are those corresponding to the H_1 and to the L_{α} and $L_{-\alpha}$ for which $\vec{\alpha} \cdot \vec{h} = 0$. Thus maximal symmetry breaking occurs if none of the roots are orthogonal to $\vec{h}$; otherwise, the unbroken gauge group is non-Abelian. In Appendix A we describe in more detail the choice of simple roots for the classical groups.

Having completed these preliminaries, we now turn to the topological charges. For any finite energy solution the asymptotic form of B_i will be

$$B_i = \frac{r_i}{4\pi r^3} G(\Omega) \quad (2.10)$$

where G^a is covariantly constant; let G_0^a be its value along the positive z -axis.

The asymptotic vanishing of $D_i \phi$ implies that $(D_i D_j - D_j D_i) \phi$ and hence $f_{abc} F_{ij}^b \phi^c$ also vanish. It follows that H can always be chosen so that not only Eq. (2.9)

but also

$$G_0 = \sum_{a=1}^d G_0^a T^a = \sum_{i=1}^d g_i H_i = \vec{g} \cdot \vec{H} \quad (2.11)$$

hold.

Topological arguments then lead to the quantization condition^{8,9)}

$$\exp(i \vec{g} \cdot \vec{H}) = I \quad (2.12)$$

The general solution to this condition is

$$\vec{g} = 4\pi \sum_{a=1}^l n_a \vec{\beta}^{(a)*} \quad (2.13)$$

where the n_a are integers and the

$$\vec{\beta}^{(a)*} = \frac{\vec{\beta}^{(a)}}{\vec{\beta}^{(a)2}} \quad (2.14)$$

are the duals of the simple roots. When there is maximal symmetry breaking, the n_a are the topologically conserved charges corresponding to the homotopy class of the scalar field configuration at spatial infinity. If, on the other hand, the unbroken gauge group is non-Abelian, only some of the n_a are topological charges; in fact, not all the n_a will be gauge-invariant. We will discuss this case further in Section 4. Finally, we note that the magnetic charge defined in Eq. (2.4) is given by

$$Q_M = \vec{g} \cdot \vec{h} \quad (2.15)$$

Thus it is quantized, but not as simply as in the SU(2) theory.

The Prasad-Sommerfield limit is obtained by letting the coupling constants in the scalar potential vanish, but retaining as a boundary condition that there be a non-vanishing scalar vacuum expectation value. In this limit the energy obeys⁵⁾

$$E \geq v (Q_M^2 + Q_E^2)^{1/2} \quad (2.16)$$

The lower bound is achieved if

$$\begin{aligned} B_i &= \cos \alpha \quad D_i \phi \\ E_i &= \sin \alpha \quad D_i \phi \\ D_0 \phi &= 0 \end{aligned} \quad (2.17)$$

with

$$\alpha = \tan^{-1} (Q_E / Q_M)$$

Fields satisfying these conditions will be solutions of the equations of motion. There may of course be other solutions to the second-order field equations which do not satisfy these first-order equations; these will not be considered in this paper. Furthermore, the dyon solutions, with both electric and magnetic charges, can all be obtained from purely magnetic solutions by a simple rescaling of lengths and fields⁶⁾. We will therefore concentrate on those solutions with $\alpha = 0$, obeying

$$B_i = \pm D_i \phi \quad (2.18)$$

The upper or lower sign is to be used according to whether Q_M is positive or negative.

A number of Prasad-Sommerfield solutions can be obtained by simple imbeddings of the SU(2) monopole solution¹⁰⁾; these provide some insight into the physical significance of the n_a . Let $\phi^s(\vec{r}; \lambda)$ and $A_i^s(\vec{r}; \lambda)$ ($s = 1, 2, 3$) be the SU(2) monopole solution corresponding to a scalar vacuum expectation value λ , and let $\vec{\beta}^{(a)}$ be any simple root with $\vec{h} \cdot \vec{\beta}^{(a)} \neq 0$. If

$$\begin{aligned} t^1 &= (2 \vec{\beta}^{(a)2})^{-1/2} (E_{\vec{\beta}^{(a)}} + E_{-\vec{\beta}^{(a)}}) \\ t^2 &= -i (2 \vec{\beta}^{(a)2})^{-1/2} (E_{\vec{\beta}^{(a)}} - E_{-\vec{\beta}^{(a)}}) \\ t^3 &= (\vec{\beta}^{(a)2})^{-1} \vec{\beta}^{(a)} \cdot \vec{H} \end{aligned} \quad (2.19)$$

then

$$\phi(\vec{r}) = \sum_{s=1}^3 \phi^s(\vec{r}; \lambda) t^s + v \left(\vec{h} - \frac{\vec{h} \cdot \vec{\beta}^{(a)}}{\vec{\beta}^{(a)2}} \vec{\beta}^{(a)} \right) \cdot \vec{H}$$

$$A_i(\vec{r}) = \sum_{s=1}^3 A_i^s(\vec{r}; \lambda) t^s$$

$$\lambda = v \vec{h} \cdot \vec{\beta}^{(a)}$$
(2.20)

gives a monopole solution satisfying

$$B_i = D_i \phi$$

$$n_b = \delta_{ab}$$
(2.21)

and with mass

$$m_a = 4\pi v \vec{h} \cdot \vec{\beta}^{(a)*}$$
(2.22)

Imbedding the SU(2) antimonopole solution in a similar manner gives a solution with the same mass, but with

$$B_i = -D_i \phi$$

$$n_b = -\delta_{ab}$$
(2.23)

By taking superpositions of widely separated monopoles and antimonopoles corresponding to various simple roots, one can obtain configurations with arbitrary integer values for the n_a . Whether there are non-trivial exact solutions of this sort is not known, but it is clear that Eq. (2.18) can be satisfied by such configurations only if the components are all monopoles or all antimonopoles. Using Eqs. (2.15), (2.16), and (2.22), we see that the energy of such a solution would be just the sum of the masses of the components, i.e. (assuming monopoles for definiteness)

$$E = \sum_{a=1}^L n_a m_a$$
(2.24)

Notice that in obtaining the solution (2.19) the fact that $\vec{\beta}^{(a)}$ was simple was never used; any positive root $\vec{\alpha}$ with $\vec{h} \cdot \vec{\alpha} \neq 0$ could equally well have been used to imbed the SU(2) solution. The topological charges for the resulting solution would be the coefficients in the expansion

$$\vec{\alpha}^* = \sum_{a=1}^L n_a \vec{\beta}^{(a)*} \quad (2.25)$$

while the mass would be given by Eq. (2.24). This result for the mass suggests that such a solution might be interpreted as a multimonopole solution in which a number of monopoles, each corresponding to a simple root, were superimposed at the same point. However, such an interpretation is faced with the objection that the SU(2) subgroup used in the imbedding is not very different from that corresponding to a simple root. In SU(N), for example, any root, whether simple or not, gives an imbedding of SU(2) as 2×2 submatrices of the $N \times N$ matrices of the fundamental representation; the subgroups corresponding to simple roots are only distinguished by making reference to ϕ_0 . Thus before adopting this interpretation we should seek some evidence that the solutions based on simple roots are qualitatively different from all others. We will find just such evidence in the next section.

3. PARAMETER COUNTING FOR MAXIMAL SYMMETRY BREAKING

We now want to determine the number of degrees of freedom for an arbitrary Prasad-Sommerfield solution. We do this by counting the physical zero modes about such a solution; i.e. the number of infinitesimal perturbations which leave the energy unchanged. As with the SU(2) theory, our analysis is based on an index theorem of Callias¹¹⁾. Since much of the analysis is similar to that of Ref. 6, we will place most emphasis on those points in which the case of a general gauge group differs from that of SU(2).

In view of the remarks preceding Eq. (2.18), it is sufficient to consider solutions with $Q_E = 0$ and obeying

$$B_i = D_i \phi \quad (3.1)$$

Expanding about such a solution and keeping terms linear in the perturbation, we obtain

$$0 = D_i^{ac} \delta \phi^c - \Phi^{ac} \delta A_i^c - \epsilon_{ijk} D_j^{ac} \delta A_k^c \quad (3.2)$$

where

$$\begin{aligned} D_i^{ac} &= \delta^{ac} D_i + f_{abc} A_i^b \\ \Phi^{ac} &= f_{abc} \phi^b \end{aligned} \quad (3.3)$$

In order to exclude those modes which simply correspond to gauge transformations, we impose the background gauge condition

$$0 = D_i \delta A_i + \Phi \delta \phi \quad (3.4)$$

Our problem then is to find the number of linearly independent solutions of Eqs. (3.2) and (3.4).

The algebraic manipulations are considerably simplified if these equations are replaced by an equivalent Dirac equation¹²⁾. If

$$\psi = I \delta \phi + i \sigma_j \delta A_j \quad (3.5)$$

then Eqs. (3.2) and (3.4) are equivalent to

$$\begin{aligned} 0 &= (-i \sigma_j D_j + \Phi) \psi \\ &\equiv \mathcal{D} \psi \end{aligned} \quad (3.6)$$

Note, however, that the δA_i and $\delta\phi$ corresponding to a given ψ are linearly independent of those corresponding to $i\psi$; the desired number of zero modes is thus twice the number of normalizable zero eigenvalues of $\mathcal{D}$.

Let us define

$$\mathcal{I} = \lim_{M^2 \rightarrow 0} \mathcal{I}(M^2) \quad (3.7)$$

where

$$\mathcal{I}(M^2) = \text{Tr} \left(\frac{M^2}{\mathcal{D}^* \mathcal{D} + M^2} \right) - \text{Tr} \left(\frac{M^2}{\mathcal{D} \mathcal{D}^* + M^2} \right) \quad (3.8)$$

Here

$$\mathcal{D}^* = -i \sigma_j D_j - \bar{\Phi} \quad (3.9)$$

is the adjoint of $\mathcal{D}$. It follows from Eq. (3.1) that

$$\begin{aligned} \mathcal{D}^* \mathcal{D} &= -D_j^2 - \bar{\Phi}^2 - 2i \sigma_j \hat{B}_j \\ \mathcal{D} \mathcal{D}^* &= -D_j^2 - \bar{\Phi}^2 \end{aligned} \quad (3.10)$$

where

$$\hat{B}_i^{ac} = f_{abc} B_i^b \quad (3.11)$$

Clearly $\mathcal{I}$ counts the zero eigenvalues of these two operators. The normalizable zero modes of $\mathcal{D}^* \mathcal{D}$ (which are the same as those of $\mathcal{D}$) each contribute 1. There would be a contribution of -1 from each normalizable zero mode of $\mathcal{D} \mathcal{D}^*$, but Eq. (3.10) shows that $\mathcal{D} \mathcal{D}^*$ is positive and has no such modes. (Note that since Φ is anti-symmetric the eigenvalues of Φ^2 are negative semidefinite). Finally, since the

continuum portions of the spectra extend down to zero, we must consider the possibility of a contribution from this source associated with singular behaviour of the spectral function at zero. For the SU(2) theory it was argued that the continuum spectra are like those for a non-relativistic particle in an exponentially decreasing potential and that no such singular behaviour occurs. In Appendix B we show that a similar argument applies for arbitrary gauge groups, but only for the case of maximal symmetry breaking. For this case there will therefore be no continuum contribution and, taking into account the factor of 2 introduced by converting to a Dirac equation, we conclude that there are 29 normalizable zero modes. For the remainder of this section we shall assume maximal symmetry breaking, leaving the discussion of the alternative case to the next section.

Our next task is to obtain an expression for η in terms of the topological charges. We begin by defining a set of Euclidean Dirac matrices

$$\gamma_k = \begin{pmatrix} 0 & -i\sigma_k \\ i\sigma_k & 0 \end{pmatrix} \quad \gamma_4 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \quad (3.12)$$

$$\gamma_5 = \gamma_1 \gamma_2 \gamma_3 \gamma_4 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

obeying

$$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu} \quad (3.13)$$

Although we are working in three dimensions, it is convenient to define a fourth component of the covariant derivative by

$$D_4 = \Phi \quad (3.14)$$

In this notation

$$\gamma \cdot D \equiv \gamma_\mu D_\mu = \begin{pmatrix} 0 & \sigma \\ -\sigma^* & 0 \end{pmatrix} \quad (3.15)$$

and

$$\downarrow(M^2) = - \int d^3x \operatorname{tr} \langle x | \gamma_5 \frac{M^2}{-(\gamma \cdot D)^2 + M^2} | x \rangle \quad (3.16)$$

where tr indicates a trace over only group and spinor indices. The integrand in this expression is in fact the divergence of a current

$$J_i = \frac{1}{2} \operatorname{tr} \langle x | \gamma_5 \gamma_i (\gamma \cdot D) \frac{1}{-(\gamma \cdot D)^2 + M^2} | x \rangle \quad (3.17)$$

(Note that, in contrast to the situation in four dimensions, the divergence of this current is non-anomalous.) We may therefore write

$$\begin{aligned} \downarrow(M^2) &= \int d^3x \partial_i J_i \\ &= \lim_{R^2 \rightarrow \infty} \int dS_i J_i \end{aligned} \quad (3.18)$$

where the surface of integration is a sphere of radius R .

To evaluate this expression we note that

$$-(\gamma \cdot D)^2 + M^2 = -D_j^2 - \Phi^2 + M^2 - \frac{1}{4} [\gamma_\mu, \gamma_\nu] G_{\mu\nu} \quad (3.19)$$

where $G_{\mu\nu}$ is defined by

$$G_{i4}^{ac} = \frac{1}{2} \epsilon_{ijk} G_{jk}^{ac} = f_{abc} B_i^b \quad (3.20)$$

We now insert into Eq. (3.17) the expansion

$$\begin{aligned} \frac{1}{-(\gamma \cdot D)^2 + M^2} &= \frac{1}{-D_j^2 - \Phi^2 + M^2} \\ &+ \frac{1}{-D_j^2 - \Phi^2 + M^2} \left(\frac{1}{4} [\gamma_\mu, \gamma_\nu] G_{\mu\nu} \right) \frac{1}{-D_j^2 - \Phi^2 + M^2} \\ &+ \dots \end{aligned} \quad (3.21)$$

Since $G_{\mu\nu}$ falls like $1/x^2$ at spatial infinity, only the first two terms in this expansion can give non-zero contributions to the surface integral in Eq. (3.18) when $R \rightarrow \infty$. Taking the trace over the Dirac indices, we find that the first term in J_i vanishes, while the second leads to

$$\begin{aligned} \hat{X}_i J_i &= -\hat{X}_i \epsilon_{i\lambda\mu\nu} \text{tr} \langle x | D_\lambda \frac{1}{-D_j^2 - \Phi^2 + M^2} G_{\mu\nu} \frac{1}{-D_j^2 - \Phi^2 + M^2} | x \rangle \\ &+ \mathcal{O}\left(\frac{1}{|x|^3}\right) \end{aligned} \quad (3.22)$$

where tr now indicates a trace over only group indices. From the asymptotic expression (2.10) we see that only terms with $\lambda = 4$ can give a contribution of order $1/x^2$ to Eq. (3.22). Furthermore, the $1/x^2$ terms are spherically symmetric, and so may be evaluated along the positive z-axis, giving

$$\begin{aligned} \hat{X}_i J_i &= -\frac{1}{2\pi x^2} \text{tr} \langle x | \hat{\Phi}_0 \frac{1}{-\nabla^2 - \hat{\Phi}_0^2 + M^2} \hat{G}_0 \frac{1}{-\nabla^2 - \hat{\Phi}_0^2 + M^2} | x \rangle \\ &+ \mathcal{O}\left(\frac{1}{|x|^3}\right) \end{aligned} \quad (3.23)$$

where

$$\begin{aligned}\Phi_0^{ac} &= f_{abc} \phi_0^b = v \sum_{i=1}^l f_{aic} h_i \\ \hat{G}_0^{ac} &= f_{abc} G_0^b = \sum_{i=1}^l f_{aic} g_i\end{aligned}\quad (3.24)$$

The expansions (2.9) and (2.11) have been used in this last expression.

To proceed further, we must evaluate the structure constants which appear in Eq. (3.24). Our generators are the H_i together with the L_α and $L_{\bar{\alpha}}$ defined in Eq. (2.8), with α running only over the positive roots. Equations (2.3) and (2.7) lead to

$$\begin{aligned}f_{\alpha j \bar{\beta}} &= -f_{\bar{\alpha} j \beta} = \delta_{\alpha \beta} \alpha_j \\ f_{ijk} &= f_{\alpha j k} = f_{\bar{\alpha} j k} = f_{\alpha j \beta} = f_{\bar{\alpha} j \bar{\beta}} = 0\end{aligned}\quad (3.25)$$

where Latin indices refer to the H_i , Greek indices to the L_α , and barred Greek indices to the $L_{\bar{\alpha}}$. Combining these results with Eqs. (3.23) and (3.24) gives

$$\begin{aligned}\hat{x}_i J_i &= \frac{v}{\pi x^2} \sum_{\alpha} ' (\vec{\alpha} \cdot \vec{h})(\vec{\alpha} \cdot \vec{g}) \langle x | \frac{1}{[-\nabla^2 + v^2(\vec{\alpha} \cdot \vec{h})^2 + M^2]^2} | x \rangle \\ &= \frac{v}{8\pi^2 x^2} \sum_{\alpha} ' \frac{(\vec{\alpha} \cdot \vec{h})(\vec{\alpha} \cdot \vec{g})}{[(\vec{\alpha} \cdot \vec{h})^2 v^2 + M^2]^{1/2}}\end{aligned}\quad (3.26)$$

where the prime indicates that the summation is only over positive roots. Therefore

$$\downarrow(M^2) = \frac{v}{2\pi} \sum_{\alpha} ' \frac{(\vec{\alpha} \cdot \vec{h})(\vec{\alpha} \cdot \vec{g})}{[(\vec{\alpha} \cdot \vec{h})^2 v^2 + M^2]^{1/2}}\quad (3.27)$$

Keeping in mind that $\vec{\alpha} \cdot \vec{h} > 0$ for any positive root $\vec{\alpha}$, we obtain for the number of zero modes

$$2 \downarrow = \frac{1}{\pi} \sum'_{\alpha} \vec{\alpha} \cdot \vec{g} \quad (3.28)$$

Recalling Eqs. (2.13) and (2.14) we rewrite this as

$$2 \downarrow = 4 \sum_{a=1}^L n_a \sum'_{\alpha} \frac{\vec{\alpha} \cdot \vec{\beta}^{(a)}}{\vec{\beta}^{(a)2}} \quad (3.29)$$

where the $\vec{\beta}^{(a)}$ are simple roots. We now note that if $\vec{\beta}$ is a simple root and $\vec{\alpha} \neq \vec{\beta}$ is any positive root, then reflection in the hyperplane orthogonal to $\vec{\beta}$ gives another positive root

$$\vec{\sigma}_{\beta}(\vec{\alpha}) = \vec{\alpha} - \frac{2 \vec{\beta} \cdot \vec{\alpha}}{\vec{\beta}^2} \vec{\beta} \quad (3.30)$$

But

$$\vec{\beta} \cdot \vec{\sigma}_{\beta}(\vec{\alpha}) = - \vec{\beta} \cdot \vec{\alpha} \quad (3.31)$$

so by pairwise cancellation of terms we obtain

$$\sum'_{\alpha} \frac{\vec{\alpha} \cdot \vec{\beta}^{(a)}}{\vec{\beta}^{(a)2}} = 1 \quad (3.32)$$

and find the number of normalizable zero modes to be

$$\begin{aligned} 2 \downarrow &= 4 \sum_{a=1}^L n_a \\ &= 4N \end{aligned} \quad (3.33)$$

For $L = 1$ this reduces to the SU(2) result.

Not all of these modes are physically significant; some will correspond to gauge transformations which are not eliminated by the background gauge condition.

In general there will be one such mode for each $U(1)$ factor of the unbroken gauge group, for a total of ℓ . If, however, the unperturbed solution is invariant under some of these $U(1)$ factors, the corresponding gauge modes will vanish. This will certainly happen whenever a solution is obtained by imbedding a solution corresponding to a gauge group of rank $\ell' < \ell$ (e.g. the $SU(2)$ imbedding of Eq. (2.20)). Furthermore, it is easy to show that any solution invariant under a $U(1)$ factor can be obtained by such an imbedding. Note that for these solutions $\ell - \ell'$ of the n_a will vanish. We conjecture, but have not been able to prove, that these are the only solutions for which any of the n_a vanish. There are thus

$$p = 4 \sum_{a=1}^{\ell} n_a - \ell + k \quad (3.34)$$

non-gauge zero modes, where k is less than or equal to the number of vanishing n_a .

Although the existence of a zero mode only require that the energy be stationary to first order in the perturbation, the positivity of $\mathcal{D}\mathcal{D}^*$ allows this result to be extended to all orders⁶⁾. We can therefore conclude that every solution with topological charges n_a belongs to a p -parameter family of solutions, with p given by Eq. (3.34).

Every solution must have at least the three parameters corresponding to the translational degrees of freedom. Examination of Eq. (3.34) shows that this minimum number occurs only if $\ell - 1$ of the topological charges vanish with the remaining one equal to unity and if furthermore the solution is invariant under $\ell - 1$ of the $U(1)$ factors. The only such solutions are the imbeddings of the $SU(2)$ monopole based on simple roots. Since these are distinguished from all other solutions (including the $SU(2)$ imbeddings based on non-simple roots) in having no internal degrees of freedom, we shall refer to them as fundamental monopoles.

As has already been pointed out, the energy of an arbitrary solution is just that expected for a multimonopole solution containing N non-interacting fundamental monopoles, with the n_a specifying the number of each type. The number of parameters is also that expected for such a solution. For each monopole there should

be three position parameters and, since $\ell - 1$ of the $U(1)$ factors leave the monopole invariant, one group orientation factor, for a total of $4N$. We have overcounted, however, since only relative group orientations are physically significant; subtracting the parameters corresponding to over-all gauge transformations gives precisely Eq. (3.34). These results suggest that all Prasad-Sommerfield solutions with non-minimal topological charges, at least for the case of maximal symmetry breaking, should be interpreted as multimonopole solutions.

4. NON-ABELIAN UNBROKEN SYMMETRY

We have seen that when there is maximal symmetry breaking, the number of parameters and the energy of an arbitrary Prasad-Sommerfield solution are given by rather simple formulas which suggest that all such solutions can be understood in terms of a few fundamental ones. While the methods of the previous section cannot be applied if the unbroken gauge group is non-Abelian, it is plausible that equally simple formulas should hold for this case also. We explore this possibility in this section.

The unbroken subgroup G' of the gauge group G will be non-Abelian whenever the scalar vacuum expectation value is such that some roots are orthogonal to $\vec{h}$; these will be the roots of G' . We can therefore no longer require that the simple roots satisfy $\vec{\beta}^{(a)} \cdot \vec{h} > 0$, but only that these inner products be non-negative; as we shall see, this is not sufficient to uniquely determine the simple roots. Let us denote by $\vec{\lambda}^a$ ($a = 1, 2, \dots, \tilde{\ell}$) those simple roots which have positive inner products with $\vec{h}$ and denote the remainder by $\vec{\gamma}^a$ ($a = 1, 2, \dots, \ell - \tilde{\ell}$). The $\vec{\gamma}^a$ will be a set of simple roots for the root diagram of the unbroken subgroup. Consequently, if G' is a product of a semi-simple group G'' and several $U(1)$ factors, the number of $\vec{\gamma}^a$ will equal the rank of G'' , while the number of $\vec{\lambda}^a$ will be the same as the number of $U(1)$'s.

Any two sets of simple roots are related by the Weyl group, generated by reflections of the form (3.30) with $\vec{\beta}$ a simple root. Those sets which satisfy our

criterion are related by the subgroup generated by those reflections for which $\vec{\beta}$ is one of the $\vec{\gamma}^{(a)}$; this is just the Weyl group of G' . Thus any two acceptable sets of simple roots are related by equations of the form

$$\begin{aligned}\vec{\lambda}'^a &= \vec{\lambda}^a + \sum_{b=1}^{l-\tilde{l}} c_{ab} \vec{\gamma}^b \\ \vec{\gamma}'^a &= \sum_{b=1}^{l-\tilde{l}} d_{ab} \vec{\gamma}^b\end{aligned}\tag{4.1}$$

In terms of a given set of simple roots the quantization condition (2.13) may be written as

$$\vec{g} = 4\pi \left(\sum_{a=1}^{\tilde{l}} \tilde{n}_a \vec{\lambda}^{a*} + \sum_{b=1}^{l-\tilde{l}} q_b \vec{\gamma}^{b*} \right)\tag{4.2}$$

The $\tilde{n}_a$, which are invariant under transformations of the form of Eq. (4.1), are the topological charges; note that they are fewer in number than for the case of maximal symmetry breaking. The q_a are not invariant under such transformations, and depend on the choice of simple roots.

There is an alternative way of viewing the effects of these transformations which is perhaps more physical. The vector $\vec{g}$ was defined by the requirement (2.11) that G_0 lie in the subalgebra H . Even for a fixed choice of H , $\vec{g}$ will not be uniquely determined if there are gauge transformations which leave ϕ_0 invariant but take G_0 from one element of H to another. The effect of such transformations on the $\tilde{n}_a$ and q_a is the same as that induced by transformations of the simple roots under the action of the Weyl group of G' .

As before, solutions can be obtained by imbedding the $SU(2)$ monopole via Eq. (2.20). This construction gives a non-trivial solution only for roots not orthogonal to $\vec{h}$; there are therefore no solutions corresponding to the $\vec{\gamma}^a$. Using a simple root $\vec{\lambda}^a$ gives a solution with

$$\begin{aligned}\tilde{n}_b &= \delta_{ab} \\ g_b &= 0\end{aligned}\tag{4.3}$$

and with mass

$$m_a = 4\pi v \vec{h} \cdot \vec{\lambda}^a\tag{4.4}$$

We expect such a solution to be a fundamental monopole in the language of the previous section. But the $\vec{\lambda}^a$ are not uniquely determined; Weyl group transformations of the form of Eq. (4.1) transform them into other roots $\vec{\lambda}'^a$ which could also have been chosen to be simple. Imbeddings based on the $\vec{\lambda}'^a$ give solutions which must be considered equally fundamental; indeed, they are just gauge transformations of the previous solutions. We are thus led to a picture in which there are $\tilde{\ell}$ fundamental monopoles, each of which may occur in one of several discrete gauge orientations^{*)}. The $\tilde{n}_a$ will be independent of the orientation, while the q_a will not. For one-monopole solutions there is no physical reason to distinguish between these orientations. They become of significance, however, when configurations with several monopoles are constructed, as they make possible several gauge-inequivalent ways of orienting two widely separated fundamental monopoles⁹⁾.

As an example, consider the SU(3) theory. If the gauge group is broken down to $U(1) \times U(1)$, then the simple roots are uniquely determined, as indicated in Fig. 1a. On the other hand, if G' is $U(1) \times SU(2)$, there are two choices of simple roots, as indicated in Fig. 1b; the two choices are related by an SU(2) rotation.

Under such a transformation,

$$\begin{aligned}\vec{\lambda} &\rightarrow \vec{\lambda}' = \vec{\lambda} + \vec{\delta} \\ \vec{\delta} &\rightarrow \vec{\delta}' = -\vec{\delta}\end{aligned}\tag{4.5}$$

*) While these are discrete as solutions of Eq. (2.11), they are connected by a continuous family of gauge transformations.

so if we write

$$\begin{aligned}\vec{g} &= \tilde{n} \vec{\lambda}^* + g \vec{\gamma}^* \\ &= \tilde{n}' \vec{\lambda}'^* + g' \vec{\gamma}'^*\end{aligned}\tag{4.6}$$

then

$$\begin{aligned}\tilde{n}' &= \tilde{n} \\ g' &= \tilde{n} - g\end{aligned}\tag{4.7}$$

There is one fundamental monopole which can be oriented so that $q = 0$ (the $\vec{\lambda}$ imbedding) or so that $q = 1$ (the $\vec{\lambda}'$ imbedding). A configuration with two of these monopoles will have $\tilde{n} = 2$ and $q = 0, 1$, or 2 , depending on the orientation of the monopoles. The configurations with $q = 0$ or 2 are gauge-equivalent to each other, but not to the $q = 1$ configuration.

How many parameters are needed to describe an arbitrary solution? As has already been mentioned, the argument in Appendix B that $\mathcal{G}$ has no continuum contribution fails when G' is non-Abelian; indeed in the explicit examples we consider the number of normalizable zero modes is different from 29. (Note that $\mathcal{G}$ must be obtained from Eq. (3.29) rather than from Eq. (3.33), since the intervening steps use the assumption that $\vec{\alpha} \cdot \vec{h}$ is non-zero for all α .) We can, however, gain some insight by considering spherically symmetric solutions. The equations for the zero modes about such solutions simplify considerably, and it is possible to count their normalizable solutions.

We begin with solutions containing a single fundamental monopole. In Appendix C we study in detail the zero-mode equations about an imbedding of the $SU(2)$ monopole of the form of Eq. (2.20). If the imbedding is via a simple root, we find only four normalizable zero modes; just as before, a fundamental monopole has only the

three translation modes and the U(1) gauge mode which it had as an SU(2) solution^{*)}. To describe a configuration with many fundamental monopoles, one would therefore need four parameters for each, less the number of zero modes corresponding to over-all gauge transformations. This would give

$$\tilde{p} = 4 \sum_{a=1}^{\tilde{l}} \tilde{n}_a - \tilde{l} + k \quad (4.8)$$

continuously varying parameters, with k being defined as in Eq. (3.34).

Let us now consider SU(2) imbeddings using non-simple roots, and see if the number of zero modes is consistent with a multimonopole interpretation. For the sake of simplicity we take the gauge group to be SU(N). As explained in Appendix A, it is most convenient to view the roots as vectors in an N-dimensional space, even though the rank of the group is N - 1. The roots, which lie in the hyperplane orthogonal to the vector $\sum_{i=1}^N \vec{e}_i$ (the $\vec{e}_i$ are an orthonormal set of basis vectors), are the vectors $\vec{e}_i - \vec{e}_j$, $1 \leq i, j \leq N$. If we choose a basis such that the N components of $\vec{h}$ obey $h_1 > h_2 \geq \dots \geq h_N$, then the simple roots are

$$\vec{\beta}^{(a)} = \vec{e}_a - \vec{e}_{a+1}, \quad a = 1, 2, \dots, N-1 \quad (4.9)$$

If $h_a = h_{a+1}$, $\vec{\beta}^{(a)}$ is among the $\vec{\gamma}^a$, otherwise it is one of the $\vec{\lambda}^a$. Any other positive root will be of the form $\vec{e}_i - \vec{e}_j$ with $i < j$; its dual may be decomposed as

$$\vec{\delta}^* = (\vec{e}_i - \vec{e}_j)^* = \sum_{a=i}^{j-1} \vec{\beta}^{(a)*} \quad (4.10)$$

*) One might have expected that a non-Abelian unbroken gauge symmetry would lead to additional gauge modes. These are absent because it is only the asymptotic value of the scalar field which is invariant under the non-Abelian group; there are terms of order $1/r$ in the scalar field which do not have this larger invariance group, and these are sufficient to eliminate the additional modes.

For a monopole solution obtained by imbedding via this root, $\sum \tilde{n}_a$ will be the number of $\vec{\lambda}^{a*}$ appearing on the right-hand side of this equation. With respect to the SU(2) defined by this root, the remaining roots belong either to singlets or to doublets of the form $(\vec{e}_i - \vec{e}_k, \vec{e}_j - \vec{e}_k)$ or $(\vec{e}_k - \vec{e}_j, \vec{e}_k - \vec{e}_i)$. In Appendix C we show that in addition to the four SU(2) zero modes there are two Yang-Mills (or one Dirac) zero modes for every doublet with "hypercharge" satisfying $|y| < \frac{1}{2}$, where the hypercharge of a root $\vec{\alpha}$ is

$$y = \frac{\vec{h} \cdot \vec{\alpha}}{\vec{h} \cdot \vec{\delta}} - \frac{\vec{\delta} \cdot \vec{\alpha}}{\vec{\delta} \cdot \vec{\delta}} \quad (4.11)$$

If $\vec{\alpha}$ is of the form $\pm(\vec{e}_i - \vec{e}_k)$ ($k \neq j$), then

$$y = \pm \left(\frac{h_i - h_k}{h_i - h_j} - \frac{1}{2} \right) = \pm \left(\frac{h_j - h_k}{h_i - h_j} + \frac{1}{2} \right) \quad (4.12)$$

satisfies $|y| < \frac{1}{2}$ if $i < k < j$ and if $h_i - h_k$ and $h_j - h_k$ are non-zero. Comparison with Eq. (4.10) shows that if none of the h_k between h_i and h_j are degenerate, there will be $4 \sum \tilde{n}_a$ zero modes, just as Eq. (4.8) would predict for a multimonopole configuration. This will not be so if some of these h_k are equal. However, in this case the corresponding zero modes are gauge-equivalent and so do not lead to distinct physical parameters; the number of gauge-inequivalent zero modes is again $4 \sum \tilde{n}_a$.

Using the information in Appendices A and C one can carry out the corresponding analysis for the remaining classical groups and obtain similar results. On the basis of this evidence and the analogy with the case of maximal symmetry breaking, we conjecture that Eq. (4.8) gives the number of parameters for all Prasad-Sommerfield solutions, and that all such solutions should be interpreted as multimonopole solutions.

5. CONCLUSION

We conclude with the following remarks:

- 1) Although our calculations have been done in terms of purely magnetic solutions, with $Q_E = 0$, the results apply equally well to the electrically charged dyon solutions. In general there will be many "electric" charges, one for each $U(1)$ factor of the unbroken gauge group. However, these cannot be varied independently; Eq. (2.17) implies that a static solution is obtained only if corresponding electric and magnetic charges have a fixed ratio. Similarly, each dyon in a multidyon solution must have the same ratio of electric to magnetic charge. Indeed, only if this is the case can the repulsive electric and magnetic forces cancel the attractive scalar force.
- 2) Semiclassical arguments indicate that the quantized version of the theory contains states corresponding to the solutions of the classical field equations. Thus there should be a series of particles (with quantized electric charges) corresponding to each of the fundamental monopole solutions and its dyon generalizations. When the unbroken gauge symmetry is non-Abelian, some of these solutions occur with several discrete gauge orientations; corresponding to these will be degenerate multiplets of particles. Matters should be somewhat different for the solutions with higher topological charges. As we have seen, both the parameter counting and the energy formula strongly suggest that all such solutions, including those obtained by imbedding the $SU(2)$ monopole, be viewed as being composed of a number of non-interacting monopoles or dyons. Their quantum counterparts would then not be new particles, but simply multiparticle states.

Montonen and Olive¹³⁾ have argued that many properties of the $SU(2)$ theory in the Prasad-Sommerfield limit suggest the existence of a dual theory in which the roles of electric and magnetic charges are interchanged. In this dual theory the monopoles would appear as elementary particles, while the charged vector bosons would be topological solitons. Bais¹⁰⁾ has shown that these arguments can be extended to an arbitrary compact gauge group, with the counterpart of the charged

boson corresponding to E_{α} being the imbedding of the $SU(2)$ monopole via the $SU(2)$ defined by the root $\vec{\alpha}$. But we have seen that if $\vec{\alpha}$ is not simple (and is not related to a simple root by a Weyl group transformation of the form of Eq. (4.1)) these solutions should be interpreted as multiparticle states. Consequently, there will be some massive vector bosons which one would treat, at least in perturbation theory, as elementary, which will not have "elementary" dual partners. We note that every such boson will be degenerate in both mass and $U(1)$ charges with some multiparticle state containing lighter bosons.

3) By exploiting spherically symmetric ansatzes, one can obtain solutions of various topological charges without invoking the Prasad-Sommerfield limit. Analogy with our results would suggest that those with minimal topological charges are fundamental, while those with higher topological charges are multimonopole solutions. Since the scalar field is no longer massless there will not be an exact cancellation of the magnetic repulsion between monopoles, so the latter solutions should have only the zero modes arising from translation and gauge symmetry; we expect that they will be unstable under infinitesimal perturbations, decaying into their component monopoles.

4) Although the parameter counting suggests that solutions with higher topological charges should be understood as multimonopole configurations, this interpretation of the parameters can only be confirmed by constructing families of such solutions. In the $SU(2)$ theory no solutions with multiple magnetic charge are known, although some progress has been made in this direction by superposition methods¹⁴⁾. In contrast, for larger gauge groups there are, in addition to the $SU(2)$ imbeddings used in this paper, a set of spherically symmetric $SU(N)$ solutions of remarkable simplicity¹⁵⁾. These may well provide a starting point for obtaining families of multimonopole solutions and for gaining further insight into the structure of these field theories.

I would like to thank Alexander Bais and David Olive for helpful discussions and to thank the CERN Theory Division for their hospitality.

APPENDIX A

In this appendix we specify in detail our choice of simple roots for the four series of classical groups. We begin with the groups $O(2\ell)$. The roots are all the vectors of the form $\pm \vec{e}_i \pm \vec{e}_j$ ($i \neq j$), where the $\vec{e}_i$ are an orthonormal set of basis vectors for the ℓ -dimensional Euclidean space. They can be chosen so that the components of $\vec{h}$ satisfy $h_1 \geq h_2 \geq \dots \geq h_\ell \geq 0$; we then take as our set of simple roots $\vec{\beta}^{(a)} = \vec{e}_a - \vec{e}_{a+1}$ ($a < \ell$) and $\vec{\beta}^{(\ell)} = \vec{e}_{\ell-1} + \vec{e}_\ell$. The roots of $O(2\ell+1)$ are those of $O(2\ell)$ plus the vectors $\pm \vec{e}_i$. With $\vec{h}$ as above, the first $\ell - 1$ simple roots are the same as for $O(2\ell)$, while $\vec{\beta}^{(\ell)} = \vec{e}_\ell$. The roots of $Sp(\ell)$ are those of $O(2\ell)$ plus the vector $\pm 2\vec{e}_i$. With $\vec{h}$ as before, the first $\ell - 1$ simple roots are the same as for $O(2\ell)$ and $\vec{\beta}^{(\ell)} = 2\vec{e}_\ell$.

Although $SU(N)$ is of rank $N - 1$, it is most convenient to take the roots to be N -dimensional vectors lying in the hyperplane orthogonal to the vector $\sum_{i=1}^N \vec{e}_i$; they are all vectors of the form $\vec{e}_i - \vec{e}_j$. The basis vectors can be ordered so that $h_1 \geq h_2 \geq \dots \geq h_N$; unlike the previous cases, some of the h_i must be negative. With this choice of basis we take the simple roots to be $\vec{\beta}^{(a)} = \vec{e}_a - \vec{e}_{a+1}$ ($a < N$). In the fundamental representation of $SU(N)$ we may take H to be the set of diagonal matrices; the h_i are the diagonal elements of ϕ_0 . In terms of the topological quantum numbers the elements of G_0 are then

$$(G_0)_{ij} = 2\pi \delta_{ij} (n_i - n_{i-1}) \quad (A.1)$$

with $n_0 = n_\ell = 0$.

APPENDIX B

In this appendix we discuss the contribution to ρ from the continuum portions of the spectra. These will be of the form

$$\rho^{\text{cont}} = \lim_{M^2 \rightarrow 0} \int_{|k| < \epsilon} \frac{d^3 k}{(2\pi)^3} \frac{M^2}{k^2 + M^2} \left[\rho_{\mathcal{D}^* \mathcal{D}}(k^2) - \rho_{\mathcal{D} \mathcal{D}^*}(k^2) \right] \quad (\text{B.1})$$

where $\rho_0(k^2)$ is the density of continuum eigenvalues of $\mathcal{O}$, and ϵ is an arbitrary positive number. Clearly ρ^{cont} will vanish unless the ρ_0 are sufficiently singular.

The eigenfunctions of $\mathcal{D}^* \mathcal{D}$ and $\mathcal{D} \mathcal{D}^*$ satisfy

$$(-D_j^2 - \bar{\Phi}^2 - 2is\sigma_j \hat{B}_j) \psi = k^2 \psi \quad (\text{B.2})$$

where s is one for $\mathcal{D}^* \mathcal{D}$ and zero for $\mathcal{D} \mathcal{D}^*$, and D and Φ are defined by Eq. (3.3).

This equation is most conveniently studied in a gauge where ϕ is in the subalgebra $\mathcal{H}^*$. From Eq. (3.25) we see that Φ^2 will then be diagonal and negative semidefinite, with the diagonal entries corresponding to the generators in $\mathcal{H}$ vanishing; the remaining diagonal elements will all be non-zero at spatial infinity if there is maximal symmetry breaking, but not otherwise. Let us decompose the vector potential A_i into parts contained in and orthogonal to $\mathcal{H}$:

$$A_i = A_i^{\mathcal{H}} + A_i^{\perp} \quad (\text{B.3})$$

*) Imposing this requirement everywhere will introduce string singularities. These may be avoided by imposing it only outside a sphere of sufficiently large radius and only for a solid angle of less than 4π . Since the solid angle chosen is arbitrary, one obtains the same results.

If there is maximal symmetry breaking, all components of $A_i^\perp$ are massive and decrease at spatial infinity as e^{-mr} ; if the symmetry breaking is not maximal, some components are massless and need only fall like $1/r$. In either case A_i^H is massless.

Let us decompose ψ in a similar fashion:

$$\psi = \psi^H + \psi^\perp \quad (B.4)$$

and consider separately the components of Eq. (B.2) perpendicular to and contained in H. Further, let us assume maximal symmetry breaking. Since ψ^H enters the perpendicular components of Eq. (B.2) only through terms involving the massive field $A_i^\perp$, we have

$$[-\nabla^2 - \Phi^2 + O(1/r)]\psi^\perp + O(e^{-mr}) = k^2 \psi^\perp \quad (B.5)$$

This clearly requires that $\psi^\perp$ fall exponentially at spatial infinity. In the remaining components, A_i^H only enters through terms involving $\psi^\perp$, while the relevant components of Φ^2 vanish, so we have

$$[-\nabla^2 + O(e^{-mr})]\psi^H + O(e^{-mr}) = k^2 \psi^H \quad (B.6)$$

This is similar to the Schrödinger equation for a non-relativistic particle in an exponentially vanishing potential. Such a potential gives a non-singular density of scattering states, so we expect $\rho_{\mathcal{D}\mathcal{D}}^*$ and $\rho_{\mathcal{D}\mathcal{D}}$ to be non-singular and that $g^{\text{cont}} = 0$.

If, on the other hand, the symmetry breaking is not maximal, we obtain equations with $1/r^2$ potentials and can make no conclusions concerning the vanishing of g^{cont} . An example with such a potential for which $g^{\text{cont}} \neq 0$ has been given by Kiskis¹⁶).

APPENDIX C

In this appendix we study the equations for the zero modes about a solution obtained by using a root $\vec{\beta}$ to imbed the SU(2) monopole as in Eq. (2.20). We will be able to check Eq. (3.33) for the case of maximal symmetry breaking and to obtain an indication of the appropriate extension to the case of a non-Abelian unbroken gauge symmetry.

We begin by noting that the solution (2.20) defines a natural "isospin" $\vec{t}$ and "hypercharge" y with which to categorize the modes. Specifically, the generators belonging to H are isospin singlets with $y = 0$, while those corresponding to the roots have t_3 and y given by

$$\begin{aligned} t_3 E_\alpha &= [t_3, E_\alpha] = \left[\frac{\vec{\beta} \cdot \vec{H}}{\vec{\beta}^2}, E_\alpha \right] \\ &= \frac{\vec{\beta} \cdot \vec{\alpha}}{\vec{\beta}^2} E_\alpha \\ y E_\alpha &= \frac{1}{\vec{h} \cdot \vec{\beta}} \left[\left(\vec{h} - \frac{\vec{h} \cdot \vec{\beta}}{\vec{\beta}^2} \vec{\beta} \right) \cdot \vec{H}, E_\alpha \right] \\ &= \left(\frac{\vec{h} \cdot \vec{\alpha}}{\vec{h} \cdot \vec{\beta}} - t_3 \right) E_\alpha \end{aligned} \tag{C.1}$$

(The reason for normalizing y in this manner will become clear shortly.) We can therefore decompose the zero mode Eq. (3.6) into a number of equations of the form

$$0 = \left\{ -i \sigma_j \partial_j - \sigma_j A_j^s(\vec{r}; \lambda) t^s + i \phi^s(\vec{r}; \lambda) t^s + i \lambda y \right\} \psi \tag{C.2}$$

with

$$\lambda = v \vec{h} \cdot \vec{\beta}$$

The SU(2) solution³⁾ is given explicitly by

$$\begin{aligned}
 A_j^s(\vec{r}; \lambda) &= \epsilon_{j s m} \hat{r}_m A(r; \lambda) \\
 \phi^s(\vec{r}; \lambda) &= \hat{r}^s H(r; \lambda) \\
 A(r; \lambda) &= \frac{\lambda}{\sinh \lambda r} - \frac{1}{r} \\
 H(r; \lambda) &= \lambda \coth \lambda r - \frac{1}{r}
 \end{aligned} \tag{C.3}$$

If we substitute this into Eq. (C.2) and then rescale lengths by a factor of λ , we obtain

$$0 = \left\{ -i \vec{\sigma} \cdot \vec{\nabla} - \vec{\sigma} \times \vec{t} \cdot \hat{r} A(r) + i \hat{r} \cdot \vec{t} H(r) + i \gamma \right\} \psi \tag{C.4}$$

where $A(r) = A(r; 1)$ and $H(r) = H(r; 1)$. It is clear that the normalizable solutions for y and $-y$ are in one-to-one correspondence. Before proceeding further, let us see which values of t and y occur and how many zero modes should be expected for each. The SU(2) structure is determined solely by the group and the length of the root $\vec{\beta}$. In addition to $t = 0$, which clearly gives no normalizable zero modes, we can have $t = \frac{1}{2}$, 1, or $\frac{3}{2}$; the last occurs only for the exceptional group G_2 . The possible values of y depend also on the vector $\vec{h}$ and on whether or not $\vec{\beta}$ is a simple root. If $\vec{\beta}$ is simple, then it is not difficult to see that except for the $y = 0$ triplet used for the imbedding, all multiplets contain either all positive roots or all negative roots. In the former case, consideration of the root with $t_3 = -t$ leads to

$$y = \frac{\vec{h} \cdot \vec{\alpha}}{\vec{h} \cdot \vec{\beta}} + t \geq t$$

while in the latter case we find that $y \leq -t$. In either case equality occurs only if $\vec{h}$ is orthogonal to a root; i.e. if there is non-maximal symmetry breaking. It is clear that $\vec{h}$ can be chosen to give any $|y| > t$. In Section 3 we found that

about a fundamental monopole there are only the two Dirac (or four Yang-Mills) zero modes found within the $SU(2)$. This implies that Eq. (C.4) has no normalizable solutions for $|y| > t$.

We next consider some examples where the imbedding is not via a simple root, beginning with $O(5)$, whose root diagram is shown in Fig. 2. The roots marked $\vec{\beta}^{(1)}$ and $\vec{\beta}^{(2)}$ are simple and will have positive scalar product with $\vec{h}$ as long as the latter lies between $\vec{\alpha}_1$ and $\vec{\alpha}_2$. Imbedding the monopole via $\vec{\alpha}_1$ leads to two doublets of opposite hypercharge, with $0 < |y| < \frac{1}{2}$. Since

$$\vec{\alpha}_1^* = \vec{\beta}^{(1)*} + \vec{\beta}^{(2)*}$$

$$2 \sum_{a=1}^2 n_a = 4$$

there are four Dirac modes, of which two are in the $SU(2)$ triplet used for the imbedding and one is in each of the doublets. If the imbedding is via $\vec{\alpha}_2$, then in addition to the $y = 0$ triplet used in the imbedding there are two others, with $0 < |y| < 1$. Because

$$\vec{\alpha}_2^* = \vec{\beta}^{(1)*} + 2\vec{\beta}^{(2)*}$$

$$2 \sum_{a=1}^2 n_a = 6$$

we conclude that there are two Dirac modes from each triplet.

To obtain $t = \frac{3}{2}$ modes we must use G_2 , whose root diagram is given in Fig. 3. Again the roots marked $\vec{\beta}^{(1)}$ and $\vec{\beta}^{(2)}$ are simple and have positive scalar product with $\vec{h}$ if the latter lies between $\vec{\alpha}_1$ and $\vec{\alpha}_2$. Proceeding as with $O(5)$, we find that imbedding via $\vec{\alpha}_1$ gives two quadruplets with $0 < |y| < \frac{1}{2}$, and that each should have four modes. If the imbedding is via $\vec{\alpha}_3$ there are again two quadruplets, but with $\frac{1}{2} < |y| < \frac{3}{2}$; these should each have three modes.

Summarizing these results, we find that the number of normalizable solutions of Eq. (C.4) must be as follows:

$$\begin{array}{lll}
 t = \frac{1}{2}, & 0 < |y| < \frac{1}{2} & \text{one} \\
 & \frac{1}{2} < |y| & \text{none} \\
 \\
 t = 1, & 0 \leq |y| < 1 & \text{two} \\
 & 1 < |y| & \text{none} \\
 \\
 t = \frac{3}{2}, & 0 < |y| < \frac{1}{2} & \text{four} \\
 & \frac{1}{2} < |y| < \frac{3}{2} & \text{three} \\
 & \frac{3}{2} < |y| & \text{none}
 \end{array} \tag{C.5}$$

We can make no prediction for $|y| = t$, as this occurs only if the unbroken gauge group is non-Abelian.

Returning to Eq. (C.4), we note that the identity

$$\begin{aligned}
 -i \vec{\sigma} \cdot \vec{\nabla} &= -i \vec{\sigma} \cdot \hat{r} \left[\hat{r} \cdot \vec{\nabla} + \frac{\vec{\sigma} \cdot \vec{L}}{r} \right] \\
 &= -i \vec{\sigma} \cdot \hat{r} \left[\hat{r} \cdot \vec{\nabla} + \frac{(\vec{L} + \vec{S})^2 - \vec{L}^2 - \vec{S}^2}{r} \right] \tag{C.6}
 \end{aligned}$$

allows us to rewrite this equation as

$$\begin{aligned}
 0 = \left\{ \frac{d}{dr} - \frac{(\vec{J}^2 - \vec{L}^2 - \vec{S}^2)}{r} + [2 \vec{S} \cdot \vec{L} - 2(\vec{S} \cdot \hat{r})(\vec{L} \cdot \hat{r})] A(r) \right. \\
 \left. - 2(\vec{S} \cdot \hat{r})(\vec{L} \cdot \hat{r}) H(r) - 2\gamma(\vec{S} \cdot \hat{r}) \right\} \psi \tag{C.7}
 \end{aligned}$$

where we have defined $\vec{j} = \vec{L} + \vec{S}$. This equation is invariant under the action of $\vec{J} = \vec{j} + \vec{t}$, so we may expand ψ in eigenfunctions of J^2 and J_3 :

$$\psi = \sum_{J J_3} \sum_j \frac{1}{\sqrt{j+1/2}} \left\{ C_{Jj}(r) y_{j j_3}^+(\Omega) + D_{Jj}(r) y_{j j_3}^-(\Omega) \right\} \times |t_3\rangle \langle J J_3 | j t j_3 t_3 \rangle \quad (C.8)$$

Here

$$y_{j j_3}^{\pm}(\Omega) = \pm \sqrt{\frac{j \mp 1/2 \pm j_3 \mp 1/2}{(2j \mp 1) + 1}} Y_{j \mp 1/2, j_3 \mp 1/2}(\Omega) |s_3 = \pm 1/2\rangle + \sqrt{\frac{j \mp 1/2 \mp j_3 \pm 1/2}{(2j \mp 1) + 1}} Y_{j \mp 1/2, j_3 \pm 1/2}(\Omega) |s_3 = \mp 1/2\rangle \quad (C.9)$$

are eigenfunctions of L^2 , s^2 , j^2 , and j_3 with $j = L \pm 1/2$. Note that along the positive z-axis the $y_{jm}^{\pm}(\Omega)$ are given by

$$y_{jm}^{\pm}(\theta=0) = \sqrt{\frac{j+1/2}{4\pi}} \left\{ \pm \delta_{m, 1/2} |s_3 = 1/2\rangle + \delta_{m, -1/2} |s_3 = -1/2\rangle \right\} \quad (C.10)$$

If we substitute Eq. (C.8) into Eq. (C.7) and evaluate at $\theta = 0$, we obtain

$$0 = \sum_j \left\{ \left[\left(\frac{d}{dr} - \frac{(j-1/2)}{r} - t_3 H - \gamma \right) C_{Jj} - \left(\frac{d}{dr} + \frac{(j+1/2)}{r} - t_3 H - \gamma \right) D_{Jj} \right] \times \langle J J_3 | j t 1/2 t_3 \rangle + \sqrt{t(t+1) - t_3(t_3+1)} A(C_{Jj} + D_{Jj}) \times \langle J J_3 | j t -1/2 (t_3+1) \rangle \right\} \quad (C.11a)$$

$$0 = \sum_j \left\{ \left[\left(\frac{d}{dr} - \frac{(j-1/2)}{r} + t_3 H + \gamma \right) C_{Jj} + \left(\frac{d}{dr} + \frac{(j+1/2)}{r} + t_3 H + \gamma \right) D_{Jj} \right] \times \langle J J_3 | j t -1/2 t_3 \rangle + \sqrt{t(t+1) - t_3(t_3-1)} A(C_{Jj} - D_{Jj}) \times \langle J J_3 | j t 1/2 (t_3-1) \rangle \right\} \quad (C.11b)$$

If $J \geq t + \frac{1}{2}$, all values of t_3 from $-t$ to t appear and we have $4t + 2$ equations for the $2t + 1$ C_{Jj} and $2t + 1$ D_{Jj} . If $J < t + \frac{1}{2}$, there are only $(2J + 1)$ each of the C_{Jj} and the D_{Jj} , while t_3 runs from $-J - \frac{1}{2}$ to $J - \frac{1}{2}$ in Eq. (C.11a) and from $-J + \frac{1}{2}$ to $J + \frac{1}{2}$ in Eq. (C.11b).

Near $r = 0$ this set of equations has solutions behaving like $r^{k-\frac{1}{2}}$ and $r^{-k-\frac{3}{2}}$ with $k = J + t, J + t - 1, \dots, |J - t|$. Half of these are regular while half are singular and non-normalizable. As $r \rightarrow \infty$ there are solutions behaving like e^{y+p} and e^{-y+p} , where

$$p = \begin{cases} t, t-1, \dots, -t & , J \geq t + \frac{1}{2} \\ J - \frac{1}{2}, J - \frac{3}{2}, \dots, -J - \frac{1}{2} & , J < t + \frac{1}{2} \end{cases}$$

Thus if $J \geq t + \frac{1}{2}$, half the solutions will be exponentially decreasing and half will be exponentially increasing; in general it will not be possible to choose a linear combination of solutions which is regular both at the origin and as $r \rightarrow \infty$. The same is true for $J < t + \frac{1}{2}$ if $|y| \geq J + \frac{1}{2}$. However, if $J < t + \frac{1}{2}$ and $|y| < J + \frac{1}{2}$, there will be $2J + 2$ exponentially decreasing and only $2J$ exponentially increasing solutions as $r \rightarrow \infty$. There will then always be one linear combination of exponentially decreasing solutions which is regular at the origin. Taking into account the multiplicity $2J + 1$, we see that this accounts for all the modes predicted in (C.5).

These arguments indicate that the normalizable modes which exist for $|y| < t$ do not occur for $|y| = t$, where the methods of Section 3 make no prediction. To verify that there are no linear combinations of solutions that are regular both at the origin and at infinity, we now explicitly solve Eqs. (C.11) for the appropriate values of J for $t = \frac{1}{2}$ and $t = 1$. We omit the case of $t = \frac{3}{2}$, which applies only to G_2 .

For $t = \frac{1}{2}$ there is a normalizable $J = 0$ mode of $|y| < \frac{1}{2}$. Setting $J = 0$ and $y = \frac{1}{2}$ in Eqs. (C.11) leads to

$$0 = \left(\frac{d}{dr} + \frac{1}{2} H - A \right) C + \frac{1}{2} D$$

$$0 = \left(\frac{d}{dr} + \frac{2}{r} + \frac{1}{2} H + A \right) D + \frac{1}{2} C$$
(C.12)

where $C = C_{\frac{1}{2}\frac{1}{2}}(r)$ and $D = D_{\frac{1}{2}\frac{1}{2}}(r)$. These imply

$$0 = \left[\frac{d^2}{dr^2} + \left(\frac{1}{r} + \frac{\cosh r}{\sinh r} \right) \frac{d}{dr} + \frac{1}{4} \left(\frac{4 \cosh r - 5}{\sinh^2 r} + \frac{2 \cosh r}{r \sinh r} - \frac{1}{r^2} \right) \right] C$$
(C.13)

The general solution of this equation is

$$C = a_1 \frac{1}{\sqrt{r \sinh r}} \frac{1}{\cosh r/2} + a_2 \frac{1}{\sqrt{r \sinh r}} \left(\sinh r/2 + \frac{r/2}{\cosh r/2} \right)$$
(C.14)

To obtain a solution normalizable at infinity we must set $a_2 = 0$. Near the origin C will then behave like $1/r$; Eq. (C.12) then implies that D behaves like $1/r^2$, which is a non-normalizable singularity. There are thus no normalizable solutions.

For $t = 1$ there is a normalizable $J = \frac{1}{2}$ solution for $|y| < 1$. Setting $J = \frac{1}{2}$ and $y = 1$ in Eqs. (C.11) leads to

$$0 = \left(\frac{d}{dr} + \frac{2}{r} \right) F_1 - 2 \left(A + \frac{1}{r} \right) F_3 + F_2$$

$$0 = \frac{dF_2}{dr} + 2 \left(A + \frac{1}{r} \right) F_4 + F_1$$

$$0 = \left(\frac{d}{dr} + H + \frac{1}{r} \right) F_3 - \left(A + \frac{1}{r} \right) F_1 + F_4$$

$$0 = \left(\frac{d}{dr} + H + \frac{1}{r} \right) F_4 + \left(A + \frac{1}{r} \right) F_2 + F_3$$
(C.15)

where

$$F_1 = C_{\frac{1}{2}\frac{1}{2}}(r) + D_{\frac{1}{2}\frac{3}{2}}(r)$$

$$F_2 = D_{\frac{1}{2}\frac{1}{2}}(r) + C_{\frac{1}{2}\frac{3}{2}}(r)$$

$$\begin{aligned} F_3 &= C_{1/2 \ 1/2}(r) - \frac{1}{2} D_{1/2 \ 3/2}(r) \\ F_4 &= D_{1/2 \ 1/2}(r) - \frac{1}{2} C_{1/2 \ 3/2}(r) \end{aligned} \quad (C.16)$$

F_1 and F_3 can be eliminated from Eqs. (C.15) to give

$$0 = \left[\frac{d^2}{dr^2} + \frac{2 \cosh r}{\sinh r} \frac{d}{dr} - \frac{2}{\sinh^2 r} \right] F_4 \quad (C.17a)$$

$$\begin{aligned} 0 = & \left[\frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} - \left(\frac{2}{\sinh^2 r} + 1 \right) \right] F_2 \\ & + 4 \left[\frac{1}{r \sinh r} - \frac{\cosh r}{\sinh^2 r} \right] F_4 \end{aligned} \quad (C.17b)$$

The general solution of Eq. (C.17a) is

$$F_4 = a_1 \left(\frac{1}{\sinh^2 r} \right) + a_2 \left(\frac{\cosh r}{\sinh r} - \frac{r}{\sinh^2 r} \right) \quad (C.18)$$

This is non-normalizable for any non-zero choice of a_1 and a_2 , so we set $F_4 = 0$ in Eq. (C.17b) and obtain

$$F_2 = a_3 \left(\frac{1}{r \sinh r} \right) + a_4 \left(\frac{\cosh r}{r} - \frac{1}{\sinh r} \right) \quad (C.19)$$

This also is non-normalizable for any non-zero a_3 and a_4 , so we must also set $F_2 = 0$. Equations (C.15) then require that F_1 and F_3 also vanish, so again there are no normalizable modes.

Thus for all groups (with the possible exception of G_2) we find only four normalizable zero modes about the fundamental monopole solutions, even when the unbroken gauge symmetry is non-Abelian.

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Figure captions

Fig. 1 : Root diagram for $SU(3)$ with the direction of the symmetry breaking indicated:

- a) unbroken $U(1) \times U(1)$ symmetry;
- b) unbroken $SU(2) \times U(1)$ symmetry.

Fig. 2 : Root diagram for $O(5)$.

Fig. 3 : Root diagram for G_2 .

FIG. 1