Ref.TH.2777-CERN

MASS GENERATION IN THE $O(3)$ NON-LINEAR σ MODEL

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ABSTRACT

We study the two point function of the azimuthal angle, $G^{(\phi)}(x) = \langle e^{i\phi(x)} e^{-i\phi(0)} \rangle_{\text{inst.}}$ [$\phi = \arg(q^1 + iq^2)$, where q^a is a three component unit vector field], in the dense instanton gas approximation for the two-dimensional $O(3)$ non-linear σ model. We find that $G^{(\phi)}(x)$ decreases exponentially as $|x| \rightarrow \infty$. This suggests that the dense instanton gas may generate a mass gap in the $O(3)$ non-linear σ model. The physical mechanism of this mass generation is also discussed.

1. INTRODUCTION

The $O(3)$ non-linear σ model in two-space time dimensions (or equivalently the continuous limit of the classical Heisenberg ferromagnet in two-space dimension) has many properties in common with the four dimensional Yang-Mills theories. At the classical level, this model is conformally invariant, exhibits a non-trivial topological structure and admits instanton solutions. At the quantum level, it is asymptotically free and has a non-trivial vacuum structure characterized by a parameter θ . These facts together with the relative simplicity of the $O(3)$ non-linear σ model have made it into an ideal testing ground of any speculations about the role of instantons in QCD.

In 1975, it has been conjectured¹⁾ that instantons may play a very important role in this model. They may generate a finite mass gap (finite correlation length) and thus provide us with the desired mechanism which forbids the appearance of a phase transition in this two-dimensional system.

For many years after that, the only known method for taking instanton effects into account has been the infra-red divergent dilute gas approximation. It is well known, however, that this approximation scheme fails to provide us with the desired mass gap (finite correlation length) in the $O(3)$ non-linear σ model.

Fortunately, recent developments^{2),3)} on instantons have led us to perhaps a better approximation scheme. In particular, the determinant of quantum fluctuations around the exact multi-instanton solutions has been computed. On the basis of this computation, an infra-red finite dense instanton gas approximation for the $O(3)$ non-linear σ model has been constructed.

In this note, we will discuss the question whether this dense instanton gas generates a mass gap in this theory. Our results, without providing a rigorous proof, suggest that a generation of a mass gap does take place.

It is important to point out that the dense instanton gas approximation, despite the fact that it is infra-red finite, is not free of difficulties. It is by now clear that it does not fulfil the cluster decomposition theorem and that it does not match with the far more systematic $1/n$ expansion in the two dimensional CP^{n-1} models. One expects that the inclusion (in some sense) of mixed instanton-anti-instanton configurations is very important for the cluster decomposition property. Moreover, one can argue that the exact multi-instanton solutions are only a minor part of the fluctuation spectrum of the non-linear σ models. Therefore, the dense instanton gas approximation can only provide a hint of how mass generation may occur.

This note is organized as follows. In Section 2 we summarize the dense instanton gas approximation for later use. In Section 3 we discuss the spin correlation functions in this approximation and describe the mass generation mechanism. Finally, in Section 4 we draw our conclusions.

2. THE DENSE INSTANTON GAS

The Euclidean action of the two-dimensional $O(3)$ non-linear σ model is given by

$$S[q^a] = \frac{1}{2f} \int d^2x \partial_\mu q^a(x) \partial_\mu q^a(x) , \quad (2.1)$$

where $q^a(x)$ ($a = 1, 2, 3$) is a three component unit vector field ($q^a(x)q^a(x) = 1$). Obviously, $S[q^a]$ is invariant under the global transformations

$$q^a(x) \longrightarrow T^a_b q^b(x) , \quad (2.2)$$

where $T \in O(3)$.

The multi-instanton solutions of this model are most conveniently described if one uses the complex variable $z = x_1 - ix_2$ to label space time points and the new field

$$w = \frac{q^1 + iq^2}{1 + q^3} \quad (2.3)$$

obtained from the field $q^a \in S^2$ by stereographic projection. The q^a 's are given in terms of the complex valued field w by

$$q_+ \equiv q^1 + iq^2 = \frac{2w}{1+|w|^2} , \quad q_- = \bar{q}_+ , \quad q^3 = \frac{1-|w|^2}{1+|w|^2} . \quad (2.4)$$

Then, almost all instanton fields with topological charge k ($k \geq 0$) can be written as follows^{1),2)}

$$w(z) = c \prod_{i=1}^k \frac{(z - a_i)}{(z - b_i)} \equiv v_k(z) , \quad (2.5)$$

where c, a_i and b_i ($i = 1, 2, \dots, k$) are arbitrary complex parameters. The anti-instantons with topological charge $-k$ ($k \geq 0$) are obtained from Eq. (2.5) by complex conjugation.

The expectation value of any observable $O[q^a] \equiv \hat{O}[w]$ in the instanton approximation is given^{2),3)} by

$$\langle O[q^a] \rangle_{\text{inst.}} = Z^{-1} \sum_{k=0}^{\infty} \frac{\lambda^{2k}}{(k!)^2} \int \frac{d^2 c}{\pi(1+|c|^2)^2} \prod_{i=1}^k d^2 a_i d^2 b_i \hat{O}_k(c, a_i, b_i) e^{-U_k(a_i, b_i)}, \quad (2.6)$$

where $\lambda = (4e^{-1}/f)M$ ($M = \mu \exp(-\pi/f_R(\mu))$ is the renormalization group invariant mass), $\hat{O}_k(c, a_i, b_i) = \hat{O}[w = v_k]$ and

$$U_k(a_i, b_i) = - \sum_{0 \leq i < j \leq k} \ln |a_i - a_j|^2 - \sum_{0 \leq i < j \leq k} \ln |b_i - b_j|^2 + \sum_{0 \leq i, j \leq k} \ln |a_i - b_j|^2. \quad (2.7)$$

$U_k(a_i, b_i)$ is the Coulomb interaction energy of k positive and k negative charges (called instanton quarks) sitting at the points a_i and b_i respectively. Thus, the instanton gas in the $O(3)$ non-linear σ model is equivalent to the two-dimensional neutral classical Coulomb gas (CCG), whose partition function in a volume V at a temperature β^{-1} (in our case $\beta^{-1} = 1$) and with fugacity λ is

$$Z_V(\lambda, \beta^{-1}) = \sum_{k=0}^{\infty} \frac{\lambda^{2k}}{(k!)^2} \int \prod_{i=1}^k d^2 a_i d^2 b_i e^{-\beta U_k(a_i, b_i)} \quad (2.8)$$

Equation (2.6) can, then, be written as follows

$$\langle O[q^a] \rangle_{\text{inst.}} = \lim_{\beta^{-1} \rightarrow 1^+} \left\langle \int \frac{d^2 c}{\pi(1+|c|^2)^2} \hat{O}_k(c, a_i, b_i) \right\rangle_{\text{CCG}(\lambda, \beta^{-1})}. \quad (2.9)$$

It has been proved by Fröhlich⁴⁾ that the thermodynamic limit of the pressure of this gas exists provided $\beta^{-1} > 1$, where the Coulombic system is in a plasma phase. At $\beta^{-1} = 1$, the pressure and the density of the gas diverge. This is due to the ultra-violet divergence of the partition function $Z_V(\lambda, \beta^{-1} = 1)$

whenever a positive and a negative charge come together ($a_i \rightarrow b_j$). Thus, as $\beta^{-1} \rightarrow 1^+$, the charges of the gas collapse to form neutral dipoles which can, then, be densely packed.

Despite the fact that $Z_V(\lambda, \beta^{-1})$ diverges, as $\beta^{-1} \rightarrow 1^+$, the expectation values, $\langle 0 \rangle_{\text{inst.}}$, of various observables, 0, usually remain finite, i.e., the ultra-violet divergences factorize and cancel in the ratio (2.6). This follows from the fact that the two-dimensional neutral classical Coulomb gas at a temperature $\beta^{-1} = 1$ is equivalent to the free massive Dirac field⁵⁾.

3. SPIN CORRELATION FUNCTIONS

The spin field $q^a(x)$ can be expressed in terms of the polar angles ϕ and θ

$$(q^1, q^2, q^3) \equiv (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta). \quad (3.1)$$

Equation (2.4), then, implies

$$e^{i\phi} = \frac{q_+}{|q_+|} = \frac{w}{|w|} = e^{i \arg w}. \quad (3.2)$$

We will now study the instanton approximation for the two point function of the azimuthal angle ϕ ,

$$G^{(\phi)}(x) \equiv \langle e^{i\phi(x)} e^{-i\phi(0)} \rangle_{\text{inst.}} = \langle e^{i(\arg w(x) - \arg w(0))} \rangle_{\text{inst.}} \quad (3.3)$$

Using Eq. (2.9), we can write this two point function as

$$G^{(\phi)}(x) = \lim_{\beta^{-1} \rightarrow 1^+} G_{\beta}^{(\phi)}(x), \quad (3.4)$$

$$G_{\beta}^{(\phi)}(x) = \langle e^{i(\arg v_k(x) - \arg v_k(0))} \rangle_{CCG(\lambda, \beta^{-1})}.$$

The exact calculation of $G_{\beta}^{(\phi)}(x)$ at $\beta^{-1} = 1$ is rather difficult. So, we will first present a high temperature ($\beta^{-1} \gg 1$) approximation to this function. The mass gap, which we find in this approximation, persists for all temperatures $\beta^{-1} > 1$, where the Coulomb system is in a plasma phase. The large β^{-1} approximation is qualitatively very instructive, but it does not give a good estimate for the value of the mass gap. This can be obtained by other methods, as we will see later.

Equation (2.5) implies that $v_k(z)$ is a meromorphic function of z and, therefore, satisfies the equation

$$\partial_\mu \arg v_k(x) = -\epsilon_{\mu\nu} \partial_\nu \ln |v_k(x)|, \quad \epsilon_{12} = +1. \quad (3.5)$$

Then, Eq. (3.4) becomes

$$\begin{aligned} G_\beta^{(\phi)}(x) &= \langle e^{-i \int_0^x dy_\mu \epsilon_{\mu\nu} \partial_\nu \ln |v_k(y)|} \rangle_{CCG(\lambda, \beta^{-1})} \\ &\equiv \langle e^{-\beta \mathcal{E}_k(a_i, b_i)} \rangle_{CCG(\lambda, \beta^{-1})} \end{aligned} \quad (3.6)$$

One can now give a statistical interpretation for $G_\beta^{(\phi)}(x)$. From the electrostatic potential at x produced by a positive charge sitting at y , $V(x, y) = -\ln(x - y)^2$ [see Eq. (2.7)], we can obtain the electrostatic potential at x produced by an elementary electric dipole of dipole moment e_ν sitting at y

$$V^{(e_\nu)}(x, y) = \frac{2e_\nu (x-y)_\nu}{(x-y)^2} \quad (3.7)$$

$\mathcal{E}_k(a_i, b_i)$ in Eq. (3.6) is equal to

$$\mathcal{E}_k(a_i, b_i) = i\beta^{-1} \int_0^x dy_\mu \epsilon_{\mu\nu} \sum_{i=1}^k \left[\frac{(y-a_i)_\nu}{(y-a_i)^2} - \frac{(y-b_i)_\nu}{(y-b_i)^2} \right] \quad (3.8)$$

and, therefore, is the interaction energy of the charges of the gas with a string of imaginary electric dipoles arranged along any curve from 0 to x . The dipole moment of an element dx_μ of this string, $de_\nu = -i/2 dx_\mu \epsilon_{\mu\nu} \beta^{-1}$, is always perpendicular to the string. $G_\beta^{(\phi)}(x)$ can be written as

$$G_\beta^{(\phi)}(x) = e^{-\Delta\Omega}, \quad (3.9)$$

where $\Delta\Omega$ is the change in the thermodynamic potential, Ω , of the classical Coulomb gas caused by the introduction of this string of imaginary electric dipoles.

It is well known⁴⁾ that the two-dimensional neutral classical Coulomb gas with fugacity λ and at a temperature $\beta^{-1} > 1$ is equivalent to the sine-Gordon theory (SGT) described by the Euclidean Lagrangian density

$$L_{SG} = \frac{1}{2} (\partial_\mu \varphi)^2 + \frac{m^2}{4\pi\beta} : (\cos(2\sqrt{\pi\beta} \varphi) - 1) :_{m^2} \quad (3.10)$$

Here, $:_{m^2}$ is the normal ordering with respect to the bare but finite mass, m , of the sine-Gordon field. m is related to the fugacity λ by

$$\lambda = \frac{m^2}{8\pi\beta} \left(\frac{1}{2} e^{\gamma} m \right)^{-\beta}, \quad (3.11)$$

where γ is the Euler constant.

The change in the thermodynamic potential, Ω , of the Coulomb gas caused by the introduction of any neutral charge distribution $\beta^{-1}J(y)$ into the system can be expressed in terms of the SGT

$$e^{-\Delta\Omega_J} = \left\langle e^{-\int d^2y J(y) \sum_{i=1}^k [\ln(y-b_i)^2 - \ln(y-a_i)^2]} \right\rangle_{CG(\lambda, \beta^{-1})} \quad (3.12)$$

$$= e^{-2\pi\beta^{-1} \int d^2x d^2y J(x) \hat{D}_m(x-y) J(y)} \left\langle : e^{-2i\sqrt{\pi\beta^{-1}} \int d^2x J(x) \varphi(x)} :_{m^2} \right\rangle_{SGT},$$

where

$$\hat{D}_m(x) = D_m(x) + \frac{1}{4\pi} \ln\left(\frac{e^{2\gamma} m^2 x^2}{4}\right), \quad (3.13)$$

with $D_m(x)$ being the massive propagator

$$D_m(x) = \frac{1}{2\pi} K_0(m|x|), \quad (\partial_\mu \partial_\mu - m^2) D_m(x) = -\delta(x). \quad (3.14)$$

Equation (3.6), then, implies

$$G_\beta^{(\phi)}(x) = e^{-2\pi\beta^{-1} \int J_0 \hat{D}_m J_0} \left\langle : e^{-2i\sqrt{\pi\beta^{-1}} \int J_0 \varphi} :_{m^2} \right\rangle_{SGT}, \quad (3.15)$$

where

$$\beta^{-1} J_0(y) = -\frac{i}{2} \beta^{-1} \int_0^{\lambda} ds_\mu \epsilon_{\mu\nu} \partial_\nu \delta(s-y) \quad (3.16)$$

is the external charge distribution which corresponds to the string of the imaginary electric dipoles.

For $\beta \rightarrow \infty$ and λ fixed, Eq. (3.12) becomes

$$e^{-\Delta\Omega_J} \cong e^{-2\pi\beta^{-1} \int J \hat{D}_m J} \quad (3.17)$$

or

$$\Delta\Omega_J \cong 2\pi\beta^{-1} \int \frac{d^2p}{(2\pi)^2} \tilde{J}(-p) \tilde{J}(p) \left[\frac{1}{p^2+m^2} - \frac{1}{p^2} \right], \quad (3.18)$$

where $\tilde{J}(p)$ is the Fourier transform of $J(y)$

$$\tilde{J}(p) = \int d^2y e^{-ip\cdot y} J(y) \quad (3.19)$$

For a real charge distribution, $\beta^{-1} J(y)$, we have $\tilde{J}(-p) = \tilde{J}(p)$ and Eq. (3.18) gives $\Delta\Omega_J < 0$. This is due to the fact that the external source is screened by attracting an opposite charge distribution from the gas and, therefore, the mean energy of the system decreases. In our case, however, the external source $\beta^{-1} J_0(y)$ [see Eq. (3.16)] is a string of imaginary electric dipoles. Thus, $\Delta\Omega$ in Eq. (3.9) is positive and a sort of antiscreening takes place. This implies that

$$G_\beta^{(\phi)}(x) = e^{-\Delta\Omega} < 1, \quad (3.20)$$

for large temperature β^{-1} and fixed fugacity λ .

We now turn to the actual computation of the two point function, $G_\beta^{(\phi)}(x)$, of the azimuthal angle for large temperature β^{-1} . For simplicity, we take the dipole string along the x_1 axis from 0 to $x = (\ell, 0)$. Then, Eq. (3.16) gives

$$\tilde{J}_0(p) = \frac{p_2}{2ip_1} (e^{-ip_1\ell} - 1) \quad (3.21)$$

and $\Delta\Omega$ becomes [see Eq. (3.18)]

$$\Delta\Omega = \pi\beta^{-1} \int \frac{d^2p}{(2\pi)^2} \frac{p_2^2}{p_1^2} (1 - \cos(p_1\ell)) \left[\frac{1}{p^2} - \frac{1}{p^2 + m^2} \right]. \quad (3.22)$$

Performing the p_1 integration by the residue theorem, we obtain

$$\Delta\Omega = \pi\beta^{-1} \int_0^\infty \frac{dp}{2\pi} p^2 \left[\frac{1}{\omega^3} (1 - e^{-\omega\ell}) - \frac{1}{p^3} (1 - e^{-p\ell}) + \ell \left(\frac{1}{p^2} - \frac{1}{\omega^2} \right) \right], \quad (3.23)$$

where $\omega = \sqrt{p^2 + m^2}$. For $\ell = |x| \rightarrow \infty$,

$$\Delta\Omega = \frac{m\pi|x|}{4\beta} - \frac{1}{2\beta} \ln(m\ell) + O(1). \quad (3.24)$$

This implies that

$$G_\beta^{(\phi)}(x) \cong \text{const.} (m|x|)^{\frac{1}{2\beta}} e^{-\frac{m\pi|x|}{4\beta}}, \quad \text{as } |x| \rightarrow \infty, \quad (3.25)$$

for β^{-1} large and λ fixed. Thus, due to the antiscreening of the external imaginary dipole string, $G_\beta^{(\phi)}(x)$ (for large temperature β^{-1} and fixed fugacity λ) decreases exponentially as $|x| \rightarrow \infty$. Notice that the correlation length $r = 4\beta/m\pi$ becomes zero at infinite temperature β^{-1} . The existence of a finite correlation length must persist for all $\beta^{-1} > 1$, where the Coulomb gas is in a plasma phase. Moreover, it seems that r remains finite even at $\beta^{-1} = 1$ (see below) and consequently the two point function of the azimuthal angle $G^{(\phi)}(x)$ decreases exponentially as $|x| \rightarrow \infty$.

We will now turn to a brief discussion of a different approximation scheme for $G_\beta^{(\phi)}(x)$. Equation (3.15) gives

$$G_\beta^{(\phi)}(x) \propto \langle e^{-2i\sqrt{\pi\beta^{-1}} \int \mathcal{J}\varphi} \rangle_{SGT} \equiv \Xi \quad (3.26)$$

Expressing Ξ as a functional integral and using Eq. (3.16), we obtain

$$\Xi = Z_{sq}^{-1} \int (D\varphi) \exp \int \left[\sqrt{\pi\beta} \theta_{0\ell}(y_1) \delta'(y_2) \varphi(y) - \frac{1}{2} (\partial_\mu \varphi)^2 - \frac{m^2}{4\pi\beta} \cos(2\sqrt{\pi\beta} \varphi) \right] dy, \quad (3.27)$$

where $\theta_{0\ell}(y_1) = 1$ for $0 \leq y_1 \leq \ell$ and $\theta_{0\ell}(y_1) = 0$ otherwise. The functional integral in Eq. (3.27) can be calculated by the saddle point approximation. The relevant classical equation of motion is

$$\partial_\mu \partial_\mu \varphi(y) = - \frac{m^2}{2\sqrt{\pi\beta}} \sin(2\sqrt{\pi\beta} \varphi(y)) - \frac{1}{2\sqrt{\pi\beta}} 2\pi \theta_{0\ell}(y_1) \delta'(y_2). \quad (3.28)$$

For ℓ large and y being far from the ends of the dipole string $(0,0)$ and $(\ell,0)$, $\varphi(y)$ depends essentially only on y_2 . Thus, Eq. (3.28) becomes

$$\varphi''(y_2) = - \frac{m^2}{2\sqrt{\pi\beta}} \sin(2\sqrt{\pi\beta} \varphi(y_2)) - \frac{1}{2\sqrt{\pi\beta}} 2\pi \theta_{0\ell}(y_1) \delta'(y_2). \quad (3.29)$$

This equation admits the following solution, satisfying the boundary condition $\varphi(y) \rightarrow 0$ as $|y| \rightarrow \infty$,

$$\varphi_{cl}(y) = - \frac{1}{2\sqrt{\pi\beta}} \text{sign}(y_2) \theta_{0\ell}(y_1) 4 \tan^{-1}(e^{-m|y_2|}). \quad (3.30)$$

$\varphi_{cl}(y)$ is an approximate solution of Eq. (3.28) provided that $|x| = \ell \rightarrow \infty$ and y lies outside two circles of finite radius around $(0,0)$ and $(\ell,0)$.

$$\chi_{cl}(y) \equiv 2i\sqrt{\pi\beta}^{-1} \varphi_{cl}(y)$$

is the imaginary mean electrostatic potential of the system in the presence of the external imaginary dipole string. One can, then, easily convince oneself that the form of $\varphi_{cl}(y)$ in Eq. (3.30) corresponds to the antiscreening of this dipole string. The "imaginary mean charge density" is

$$\rho_{cl}(y) \equiv \beta^{-1} J_{cl}(y) = - \frac{\partial_\mu \partial_\mu \chi_{cl}}{4\pi} = i \frac{m^2}{2\pi\beta} \frac{\tanh(my_2)}{\cosh(my_2)} + \frac{i}{2\beta} \theta_{0\ell}(y_1) \delta'(y_2). \quad (3.31)$$

Moreover, $\varphi_{c\ell}(y_2)$ in Eq. (3.30), for $0 \leq y_1 \leq \ell$, is the soliton solution of the classical sine-Gordon theory with a step discontinuity at $y_2 = 0$. Thus, computing the functional integral in Eq. (3.27) in the semi-classical approximation around the saddle point (3.30) we obtain

$$-\ln G_p^{(\phi)}(x) \cong M_s |x|, \text{ as } |x| \rightarrow \infty, \quad (3.32)$$

where M_s is the sine-Gordon soliton mass and is given by⁶⁾

$$M_s = \frac{m}{\pi\beta} (2 - \beta). \quad (3.33)$$

For β^{-1} large, $M_s \approx 2m/\pi\beta = (8/\pi^2) \cdot r^{-1}$. Therefore, the mass gap in Eq. (3.32) is very close to the mass gap obtained in Eq. (3.25) provided that the temperature β^{-1} is sufficiently large. M_s is positive for all temperatures $\beta^{-1} > \frac{1}{2}$. In particular, for $\beta^{-1} = 1$, we obtain

$$-\ln G^{(\phi)}(x) \cong \frac{m}{\pi} |x|, \text{ for } |x| \gg 1, \quad (3.34)$$

which shows that the two-point function of the azimuthal angle, $G^{(\phi)}(x)$, decreases exponentially as $|x| \rightarrow \infty$ with a correlation radius $r' = \pi/m$.

Another way to obtain information about the function $G^{(\phi)}(x)$ is to exploit the equivalence of the classical Coulomb system at a temperature $\beta^{-1} = 1$ and the free massive Dirac field⁴⁾. To this end, we consider the function

$$\Theta_p(x) \equiv \left\langle |v_k(0)|^{-1} |v_k(x)| e^{-i \int_0^x dy_\mu \epsilon_{\mu\nu} \partial_\nu \ln |v_k(y)|} \right\rangle_{ccq(1, \beta^{-1})}. \quad (3.35)$$

$-\ln \Theta_p(x)$ is the change in the thermodynamic potential of the gas caused by the introduction of an imaginary dipole string as well as two real external charges $-\frac{1}{2}\beta^{-1}$ and $\frac{1}{2}\beta^{-1}$ at 0 and x respectively. The leading behaviour of $-\ln \Theta_p(x)$ as $|x| \rightarrow \infty$ is solely determined by the presence of the dipole string, since the two external real charges are screened by the gas and produce changes essentially only in the interior of two circles around them with radius equal to the Debye length. Thus, $-\ln \Theta_p(x)$ and $-\ln G_p^{(\phi)}(x)$ have the same leading asymptotic behaviour as $|x| \rightarrow \infty$.

It follows from Eq. (3.12) that

$$\Theta_{\beta=1}(x) = \frac{\langle e^{-i\sqrt{\pi}\varphi(x) + i\sqrt{\pi}\varphi(0) - \sqrt{\pi}\int_0^x dy_\mu \epsilon_{\mu\nu} \partial_\nu \varphi(y)} \rangle_{S\&T}}{\langle e^{-i\sqrt{\pi}\varphi(x) + i\sqrt{\pi}\varphi(0) - \sqrt{\pi}\int_0^x dy_\mu \epsilon_{\mu\nu} \partial_\nu \varphi(y)} \rangle_0}, \quad (3.36)$$

where $\langle \rangle_0$ is the expectation value in the free massless field φ . It is well known⁵⁾ that the sine-Gordon theory at $\beta = 1$ is equivalent to the free Dirac field with mass $M_D = m/2eY$. Using the correspondence between these theories⁷⁾

$$\psi_2(x) \propto e^{\sqrt{\pi}\int_0^x dy_\mu \epsilon_{\mu\nu} \partial_\nu \varphi(y) + i\sqrt{\pi}\varphi(x)}$$

$$\text{and } \psi_2^\dagger(x) \propto e^{-\sqrt{\pi}\int_0^x dy_\mu \epsilon_{\mu\nu} \partial_\nu \varphi(y) - i\sqrt{\pi}\varphi(x)}, \quad (3.37)$$

we obtain

$$\Theta_{\beta=1}(x) = \frac{\langle \psi_2(0) \psi_2^\dagger(x) \rangle_{M_D}}{\langle \psi_2(0) \psi_2^\dagger(x) \rangle_{M=0}} \quad (3.38)$$

where $\langle \rangle_{M_D}$ is the expectation value in the free Dirac field with mass M_D . Equation (3.38), then, gives

$$\Theta_{\beta=1}(x) \underset{|x| \rightarrow \infty}{\sim} \left(\frac{\pi}{2}\right)^{1/2} (M_D |x|)^{1/2} e^{-M_D |x|} \quad (3.39)$$

or

$$-\ln G_{\beta=1}^{(\phi)}(x) \cong M_D |x|, \text{ as } |x| \rightarrow \infty \quad (3.40)$$

Notice that $M_D = m/2eY$ is very close to the mass gap m/π which appears in Eq. (3.34). Presumably, if we could calculate the soliton mass, M_s , exactly, we would find a complete agreement between Eqs (3.34) and (3.40).

So far, we have studied only the two point function of the azimuthal angle and we have argued that it decreases exponentially as $|x| \rightarrow \infty$. This already

suggests that the spin correlation function $\langle q^a(x) q^a(0) \rangle_{\text{inst.}}$ may also decrease exponentially as $|x| \rightarrow \infty$ and, therefore, the dense instanton gas may generate the desired finite correlation length in the two-dimensional classical Heisenberg ferromagnet.

Unfortunately, the actual computation of $\langle q^a(x) q^a(0) \rangle_{\text{inst.}}$ is technically much more difficult. The high temperature approximation, for example, does not apply in this case, since it violates $O(3)$ invariance. This is due to the fact that

$$d\mu_{\beta}^k(a_i, b_i) = \prod_{i=1}^k d^2 a_i d^2 b_i e^{-\beta U_k(a_i, b_i)} \quad (3.41)$$

is not a $O(3)$ invariant measure unless $\beta = 1$. Therefore, any reliable calculation of $\langle q^a(x) q^a(0) \rangle_{\text{inst.}}$ must be done exactly at $\beta^{-1} = 1$, which is rather difficult. Fortunately, $d\mu_{\beta}$ is still invariant under $O(2)$ rotations in the (q^1, q^2) plane for any β^{-1} . Thus, the two point function of the azimuthal angle in the large β^{-1} approximation does not violate the symmetry of the model.

4. CONCLUSIONS

We have studied the dense instanton gas approximation for the two point function of the azimuthal angle, $G^{(\phi)}(x) = \langle e^{i \arg q_+(x)} e^{-i \arg q_+(0)} \rangle_{\text{inst.}}$, in the two dimensional $O(3)$ non-linear σ model (or the continuous classical Heisenberg ferromagnet in two space dimensions). The dense instanton gas in this model coincides with the two-dimensional neutral classical Coulomb gas at a temperature $\beta^{-1} = 1$ which is further equivalent to the sine-Gordon theory. We showed that $-\ln G^{(\phi)}(x)$ is equal to the change in the thermodynamic potential produced by introducing into the Coulomb gas an external imaginary dipole string along any curve connecting the points 0 and x . Due to a sort of anti-screening of this imaginary dipole string by the Coulomb system, $-\ln G^{(\phi)}(x)$ grows linearly with $|x|$ in the leading approximation. This implies that the correlation function $G^{(\phi)}(x)$ decreases exponentially as $|x| \rightarrow \infty$.

We also showed that the physical picture of the antiscreening of the imaginary dipole string can be described by using the semi-classical approximation for the sine-Gordon theory. The mean imaginary electrostatic potential in the system along any direction perpendicular to the string is proportional to the sine-Gordon soliton with a step discontinuity across this string.

The asymptotic behaviour of $G^{(\phi)}(x)$ can also be estimated by using the equivalence of the classical Coulomb gas at temperature $\beta^{-1} = 1$ and the free massive Dirac field.

Our analysis suggests that it is quite plausible that $\langle q^a(x) q^a(0) \rangle_{\text{inst.}}$ also decreases exponentially as $|x| \rightarrow \infty$ and, therefore, the dense instanton gas may generate a finite correlation length in the two-dimensional Heisenberg ferromagnet. Unfortunately, we have not been able, so far, to estimate $\langle q^a(x) q^a(0) \rangle_{\text{inst.}}$.

We thank Martin Lüscher for discussions and Werner Nahm for reading the manuscript.

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