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STABILITY PROPERTIES OF CLASSICAL SOLUTIONS
TO NON - LINEAR σ MODELS

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A B S T R A C T

We show that the classical solutions of two-dimensional Euclidean $O(2p+1)$ non-linear σ models given in terms of spherical harmonics are unstable under small perturbations for $p \geq 2$. The negative and zero energy fluctuation modes are displayed.

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1. - INTRODUCTION

The two-dimensional non-linear Euclidean σ models are of special interest because they have many features in common with four-dimensional Yang-Mills theory and have the further advantage that many of their properties can be worked out explicitly. Here we will be particularly interested in finite action classical solutions of the $O(n)$ ^{1),2)} models and their stability.

We define the $O(n)$ model in terms of a Lagrangian density

$$\mathcal{L}(\varphi) = (\partial \varphi)^2 = \sum_{\mu=1}^2 \sum_{i=1}^n (\partial_{\mu} \varphi_i)^2 \quad (1.1)$$

where $q_i(x)$ is a real n component field on the two-dimensional Euclidean space E^2 such that $q^2 = 1$.

The $O(n)$ models are known to have stable instanton solutions only for $n=3$ ¹⁾ which can be found explicitly as

$$W = c \frac{\prod_{i=1}^k (x_+ - a_i)}{\prod_{i=1}^k (x_+ - b_i)} \quad (1.2)$$

where the complex field w is related to q by (the stereographic projection)

$$W = \frac{\varphi_1 + i\varphi_2}{1 + \varphi_3} \quad (1.3)$$

with $x_+ = x_1 + ix_2$ and $c, a_j \neq b_j$ are complex constants. Anti-instanton solutions arise by complex conjugation of w . The instanton number corresponding to the solution (1.2) is k and the action takes the value

$$S = 8\pi k \quad (1.4)$$

Of course, any solution of the $O(3)$ σ model is also a solution of the $O(n)$ σ model, $n > 3$. However, such solutions, as we will see later, are unstable in the sense that small fluctuations about these solutions have negative eigenvalues.

One can give reasons why the instanton solutions of the $O(3)$ model are stable : they belong to different homotopy classes and are therefore characterized by different values of a conserved topological number. For $O(n)$, $n > 3$, there is no non-trivial topological number suggesting that in the same way as the non-trivial topology guarantees the stability of the instanton solutions, the lack of it makes all non-trivial solutions unstable.

We will study this point by looking at some known non-trivial solutions of $O(2p+1)$ models which are given in terms of spherical harmonics ²⁾. It turns out that with exception of the $p=1$ case, when these solutions reduce to $O(3)$ one-instanton solutions, they are all unstable and the negative and zero energy fluctuation modes can be displayed explicitly. This discussion is given in the next section.

2. - STABILITY PROPERTIES OF $O(n)$ MODELS

The problem of finding solutions to the equations of motion corresponding to the Lagrangian (1.1) which reads

$$\partial_t^2 \mathbf{q} + \mathcal{L} \mathbf{q} = 0, \quad (2.1)$$

with $\mathcal{L} = (\partial \mathbf{q})^2$, is equivalent, in a mathematical language, to finding the harmonic maps of the theory ³⁾. Harmonic maps are by definition maps between spaces which produce a finite energy extremum of a given energy functional (i.e., the Euclidean Lagrangian $\mathcal{L}$).

Unfortunately the procedures for constructing harmonic maps are not very systematic and it is rarely known whether a set of constructed maps is complete or not. For $O(3)$ models the instantons (1.2) are harmonic maps and it is known ^{1),2)} that their set is complete. For $O(n)$ models, $n > 3$, the situation is unclear. Of course, simple embeddings

$$\begin{aligned} \mathbf{q}_i &= \tilde{\mathbf{q}}_i, & i &= 1, 2, 3 \\ \mathbf{q}_i &= 0, & i &> 3 \end{aligned} \quad (2.2)$$

where $\tilde{q}$ is a solution of the $O(3)$ model, are also solutions of (2.1). Moreover for $n=2p+1$, as will be described below, it is possible to construct further solutions in terms of spherical harmonics ²⁾⁻⁴⁾. It is unknown whether there are further classes of solutions.

To simplify the calculation, we may project E^2 stereographically on S^2 by $(x_1, x_2) \rightarrow (r_1, r_2, r_3)$ with

$$\begin{aligned} r_p &= \frac{2x_p}{1+x^2} \quad , \quad p = 1, 2 \\ r_3 &= \frac{1-x^2}{1+x^2} \end{aligned} \quad (2.3)$$

We then find that

$$\partial^2 = - \frac{4}{(1+x^2)^2} L^2 \quad (2.4)$$

where L is the angular momentum operator given in terms of the r variables. The equations of motion (2.1) take the form

$$L^2 \tilde{q} - (L^2 \tilde{q} \cdot \tilde{q}) \tilde{q} = 0 \quad (2.5)$$

Since the spherical harmonics Y_{pm} are eigenfunctions of L^2 it is easy to see that

$$\tilde{q}_i^{(0)} = \left(\frac{4\pi}{2p+1} \right)^{1/2} \sum_{m=-p}^p U_{im} Y_{pm} \quad (2.6)$$

provides a solution to (2.4).

Here the $2p+1$ valued index i runs from $-p$ to p and U is a unitary $(2p+1) \times (2p+1)$ matrix fulfilling the condition

$$U_{im} = (-1)^m U_{i,-m}^* \quad (2.7)$$

to ensure the reality of $q^{(0)}$ (this condition is, of course, compatible with the unitarity requirement). Using the addition theorem for the spherical harmonics one easily checks that in fact $(q^{(0)})^2 = 1$.

The Lagrangian density on E^2 of the solution (2.6) is $\mathcal{L}_0 = (4/(1+x^2)^2) p(p+1)$ which, on the sphere S^2 , translates into $p(p+1)$ since the factor $4/(1+x^2)^2$ gets absorbed into the measure on the sphere

$$d\Omega = \frac{4}{(1+x^2)^2} d^2x \quad . \quad (2.8)$$

We now turn to the stability properties of the embeddings (2.2) and the solution (2.6). Suppose $q^{(0)}$ is one of these solutions and consider a nearby q such that $q^2 = 1$. Then we may put

$$q = q^{(0)} (1 - \phi^2)^{1/2} + \varphi \quad (2.9)$$

where φ is a small real fluctuation which is orthogonal to $q^{(0)}$, i.e.,

$$q^{(0)} \cdot \varphi = 0 \quad . \quad (2.10)$$

Expanding the action to second order in φ we find

$$S = \int d^2x \mathcal{L}(q) = S_0 + V(\varphi) \equiv \int d^2x \mathcal{L}_0 + \int d^2x [-\varphi \square \varphi - \mathcal{L}_0 \varphi^2] \quad . \quad (2.11)$$

For the case of the solution (2.6) we have on S^2 that

$$V(\varphi) = \int d\Omega [\varphi \square \varphi - p(p+1) \varphi^2] \quad . \quad (2.12)$$

It is easy to see that the embeddings (2.2) are unstable, because the fluctuations φ_i , $i > 3$ are not constrained by the orthogonality condition (2.10). Thus choosing all φ_i , $i > 3$ to be (non-zero) constants we see from (2.11) that their contribution to $V(\varphi)$ will be negative and if they are chosen sufficiently large it will be numerically larger than the $0(3)$ (i.e., $i = 1, 2, 3$) contribution to $V(\varphi)$. This discussion of course only holds if $\mathcal{L}_0$ is not identically zero. The $S_0 = 0$ constant solutions are all stable when embedded in (2.2).

For the solutions (2.6) the situation is more complicated as it is less straightforward to analyze the implication of the constraint equation $\varphi \cdot q^{(0)} = 0$. We have found, however, that the analysis simplifies when φ is expanded in terms of spherical tensors $Y_{JM}^{\ell p}$. The spherical tensors are defined by ⁴⁾

$$Y_{JM}^{\ell p} = \sum_{m=-\ell}^{\ell} \sum_{i=-p}^p C_{\ell m p i}^{JM} Y_{\ell m} X_{pi} \quad (2.13)$$

where $C_{\ell m p i}^{JM}$ is a Clebsch-Gordan coefficient and X_{pi} are the Cartesian components of a rank p spinor, which, since p is an integer, is a $2p+1$ dimensional vector. The $Y_{JM}^{\ell p}$, of course, arise physically from adding two angular momenta ℓ and p to get an angular momentum J and thus we have the restriction

$$J = |\ell - p|, \dots, \ell + p \quad (2.14)$$

The spherical tensors are eigenfunctions of L^2 (as well as J^2).

We use the conventional definition (for integer p) of the co-variant component of $Y_{JM}^{\ell p}$:

$$(Y_{JM}^{\ell p})_i = (-1)^i C_{\ell, M+i, p, -i}^{JM} Y_{\ell, M+i} \quad (2.15)$$

which we regard as components of a $2p+1$ dimensional vector. Under complex conjugation they transform as (j integer)

$$(Y_{JM}^{\ell p})_i^* = (-1)^{j+M+\ell+p-J} (Y_{J, -M}^{\ell p})_{-i} \quad (2.16)$$

Using this property we observe that if we expand the fluctuation φ in terms of the spherical tensors

$$\varphi_i = \sum_{JML} a_{JML} U_{ij} (Y_{JM}^{\ell p})_j \quad (2.71)$$

where U_{ij} is the unitary matrix introduced before (2.7). There the reality property of φ is guaranteed if the a_{JML} satisfy

$$a_{JML}^* = a_{J,-ML} (-1)^{M+L+J}. \quad (2.18)$$

Moreover, a rather lengthy, although straightforward, calculation shows that

$$V(\varphi) = \sum_{JML} |a_{JML}|^2 \varepsilon_L^P \quad (2.19)$$

where

$$\varepsilon_L^P = L(L+1) - P(P+1) = (L-P)(L+P+1). \quad (2.20)$$

From this we see that as ε_L^P is negative for $L < P$, it is likely that $V(\varphi)$ may take on negative values. However, not all a_{JML} can be chosen independently; they have to be chosen in such a way that the constraint (2.10) is satisfied

$$\varphi \cdot \xi^{(0)} = \sum_{JMLi} a_{JML} (u \cdot Y_{JM}^{LP})_i \left(\frac{4\pi}{2P+1} \right)^{1/2} u_{im}^* Y_{Pm}^* = 0. \quad (2.21)$$

Using

$$\sum_j Y_{Pj}^* (Y_{JM}^{LP})_j = \left[\frac{(2P+1)(2L+1)}{4\pi(2J+1)} \right]^{1/2} C_{L0P0}^{J0} Y_{JM} \quad (2.22)$$

we find

$$\varphi \cdot \xi^{(0)} = \sum_{JML} a_{JML} \left[\frac{2L+1}{2J+1} \right]^{1/2} C_{L0P0}^{J0} Y_{JM} = 0 \quad (2.23)$$

As $C_{L0P0}^{J0} = 0$ if $J+L+P$ is odd, this expression is real as it should be.

As Y_{JM} are linearly independent functions (2.23) has to be satisfied for each component independently and we see that the condition of orthogonality imposes to us the restrictions

$$\sum_{L=|J-p|}^{J+p} C_{L0p0}^{J0} a_{JML} (2L+1)^{1/2} = 0 \quad (2.24)$$

which have to be imposed on a_{JML} for all J and M . Of course, a_{JML} have to satisfy also (2.18) which, due to the vanishing property of C_{L0p0}^{J0} , for the non-zero terms, becomes

$$a_{JML}^* = (-1)^M a_{J,-ML} \quad (2.25)$$

We can now consider the case of $p=1$. In this case (see Fig. 1) for a given $J>1$ $a_{JML} \neq 0$ only for $L=J-1, J, J+1$. We have a negative ϵ_L^1 only for $L=0$, $\epsilon_0^1 = -2$ which, without the constraint (2.24), would indicate the instability of the solution since $V(\varphi)$ in (2.18) is negative. However, the only coefficients a_{JMO} which appear are those with $J=1$ and we see that they enter constrained with a_{JM2} . The constraint equation (2.24) gives for them

$$C_{0010}^{10} a_{1M0} + C_{2010}^{10} a_{1M2} \sqrt{5} = 0 \quad (2.26)$$

As $C_{0010}^{10} = 1$, $C_{2010}^{10} = -\sqrt{2/5}$ we see that

$$\sqrt{2} a_{1M2} = a_{1M0} \quad (2.27)$$

Inserting this constraint into the contribution of these modes to $V(\varphi)$ we find

$$V(\varphi) = \left[-2 + 4 \cdot \frac{1}{5} \cdot \frac{1}{2/5} \right] |a_{1M0}|^2 = 0 \quad (2.28)$$

which shows that the constraint has effectively replaced the $J=1, L=0$ negative and $J=1, L=2$ positive modes by additional $J=1$ zero modes. A quick glance at Fig. 1 shows that there are six zero modes in the case $p=1$: the additional three zero modes are the unconstrained a_{11M} modes. The additional possible zero mode a_{010} is set to zero by the constraint. Thus we find that the solutions (2.6) in the case of the $O(3)$ model are stable as was already known. Fluctuations about them have no negative

eigenvalues and there are six zero modes. As the fluctuations about a one-instanton solution to the $O(3)$ model are also characterized by six zero modes it is likely that these two solutions are related ; in fact they are equivalent. The solution (2.6), in the case of $p=1$, provides a $k=1$ solution (1.4).

Next we consider the case $p=2$, i.e., an $O(5)$ model. Here, as Fig. 2 shows, we have several negative modes. First of all, clearly all the five modes a_{21M} are unconstrained. Secondly, the constraint between a_{20M} , a_{22M} and a_{24M} can be used to determine a_{22M} in terms of a_{20M} and a_{24M} . As a_{22M} for any M would be a zero mode we see that the constraint effectively eliminates a_{22M} modes, leaving a_{20M} and a_{24M} unchanged. Of these a_{20M} are negative modes, thus showing that there are at least 10 negative modes. It is a matter of algebra to show that other modes are non-negative. For the $J=1$ case the situation is very similar to the $p=1$ case. As $c_{1020}^{10} = -\sqrt{2/5}$ and $c_{3020}^{10} = 3/\sqrt{35}$ the constraint gives

$$a_{1M3} = a_{1M1} \sqrt{\frac{2}{3}} \quad (2.29)$$

which is just what is required for the resultant modes to be zero modes of $V(\varphi)$:

$$V(\varphi) = -4|a_{1M1}|^2 + 6|a_{1M1}|^2 = (-4+6)|a_{1M1}|^2 = 0 . \quad (2.30)$$

The case of $J=3$ is more complicated. Clearly there is one set of positive modes as we can always consider the case of $a_{3M1} = 0$. However, as the algebra is not too complicated, one can use the values of c_{1020}^{30} , c_{3020}^{30} and c_{5020}^{30} to find that the constraint equation reads

$$a_{3M1} \frac{3}{\sqrt{5}} - a_{3M3} \frac{2\sqrt{7}}{\sqrt{15}} + a_{3M5} \sqrt{\frac{10}{3}} = 0 \quad (2.31)$$

and then solve this equation for a_{3M1} and find that the contribution of these constrained modes is given by

$$V(\varphi) = -4|a_{3M1}|^2 + 6|a_{3M3}|^2 + 24|a_{3M5}|^2 = \frac{50}{27} |a_{3M3} - \frac{4\sqrt{14}}{5} a_{3M5}|^2 \quad (2.32)$$

which show that the constrained system has, for a given M , one positive and one zero mode. Adding up all multiplicities we find the fluctuations about a (2.6) solution in the $O(5)$ case are characterized by 10 negative modes and $3(a_{1M2}) + 3(a_{1M1} + a_{1M3}) + 7(a_{3M2}) + 7(a_{3M1} + a_{3M3} + a_{3M5}) = 20$ zero modes.

For $p > 2$ there are even more negative modes. Clearly all the unconstrained negative modes remain and guarantee the instability of the solution. Such modes are provided in a general case by, e.g., all $L < p$ a_{pM1} , $a_{p-1,M,2}$, $a_{p+1,M,2}$, etc., modes. Notice also that for J even the constraint equations involve the $L = p$ modes. In this case (like in the $p = 2$ case considered above) we can use the constraint equations to determine these modes in terms of the other modes observing that the constrained system effectively reduces to the unconstrained system without the zero modes. For J odd the constraint equations connect only positive and negative modes. We believe that the effect of the constraint, for each M , is to reduce the number of positive and negative modes by one each and to generate one zero mode. This is what happened for $J = 1$ modes in the $p = 1$ and $p = 2$ cases and also for $J = 3$ in the $p = 2$ case. In general this will be the case if the following relation holds :

$$\sum_{L=|J-p|}^{J+p} \frac{2L+1}{\xi_L^p} \left(C_{L0p0}^{J0} \right)^2 = 0 \quad (2.33)$$

where J is odd and the sum only goes over L 's for which $L+J+p$ is even. We have been able to prove (2.34) for the $J = 1$ case (any p). In this case we observe that as

$$\frac{\left(C_{p-1,0p0}^{10} \right)^2}{\left(C_{p+1,0p0}^{10} \right)^2} = \frac{p}{p+1} \frac{2p+3}{2p-1} \quad (2.34)$$

the contribution to $V(\varphi)$ of the modes a_{1Mp-1} and a_{1Mp+1} vanishes. However, so far we have been unable to find a general proof of our assumption. Such a proof will clearly rely on a specific symmetry property of C_{b0c0}^{a0} . We have, however, checked our hypothesis on some specific examples. The case $p = 3$ is described in Fig. 3. In this way we have found it to be true for the $J = 3$ and $J = 5$ modes in the $p = 3$ case and also for the $J = 5$ modes in the $p = 5$ case. Assuming that our assertion is correct we can determine the total number of negative and zero modes for arbitrary p .

They are given respectively by $p(p-1)(2p+1)$ and by $2p(1+2p)$. In determining them we have used the correct multiplicity of each mode, i.e., $2J+1$ for each J .

The solutions of $O(n)$ models discussed above will also be solutions of CP^{n-1} models when embedded trivially as real vectors in a complex vector space. The properties of these embeddings will be studied in a separate paper.

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FIGURE CAPTION

For a given J (and M) the a_{JIM} 's whose L is denoted by a circle are unconstrained, and those for which L is denoted by a cross are related to each other by the constraint (2.6). The terms with $L=p$ (i.e., when $\epsilon_L^p = 0$) are denoted by vertical lines in the figures.

		L					
		0	1	2	3	4	5
J	0		X				
	1	X	0	X			
	2		X	0	X		
	3			X	0	X	
	4				X	0	X

p = 1

Fig. 1

		L							
		0	1	2	3	4	5	6	
J	0			X					
	1		X	0	X				
	2	X	0	X	0	X			
	3		X	0	X	0	X		
	4			X	0	X	0	X	
	5				X	0	X	0	X

p = 2

Fig. 2

		L								
		0	1	2	3	4	5	6	7	
J	0				X					
	1			X	0	X				
	2		X	0	X	0	X			
	3	X	0	X	0	X	0	X		
	4		X	0	X	0	X	0	X	
	5			X	0	X	0	X	0	X
	6				X	0	X	0	X	0

p = 3

Fig. 3