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DERIVATION FROM FIRST PRINCIPLES OF THE INFRA-RED STRUCTURE
OF QUANTUM ELECTRODYNAMICS

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ABSTRACT

It is shown how a contribution to the action from a surface at spatial infinity determines the infra-red structure of quantum electrodynamics.

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Several authors have emphasized the importance of dynamical variables at infinity both in gravitation ¹⁾ and in Yang-Mills theories ²⁾. Basically these degrees of freedom arise when the relevant fields decrease too slowly to allow the neglect of surface contributions at large distance. Such is the case, in particular, in quantum electrodynamics if the total charge is non-zero as one sees immediately from Gauss's law ²⁾.

These degrees of freedom at infinity govern the large distance physics and hence are obviously relevant for infra-red problems. In the present letter we explore such a connection in quantum electrodynamics. As we shall see, the dynamical treatment of large distance phenomena introduced in Ref. 2) determines in a very natural way the infra-red structure of photon asymptotic states and substantiates earlier treatments of this problem ³⁾. In this letter we only explain the basic ideas leaving detailed discussions to forthcoming publications where we shall also consider the non-Abelian case.

In the Gupta-Bleuler formalism, the first order Lagrangian density reads, with standard notation,

$$\begin{aligned} \mathcal{L} = & -\frac{1}{2} F^{\mu\nu} (\partial_\mu A_\nu - \partial_\nu A_\mu) + \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \\ & - B \partial_\mu A^\mu + \frac{1}{2} \lambda B^2 + \bar{\psi} (\not{\partial} - e \not{A} - m) \psi. \end{aligned} \quad (1)$$

The need of surface variables arises because one wishes to establish a formalism consistent with the existence of the following finite limits

$$\begin{aligned} A_\mu^{as}(x) & \equiv \lim_{\lambda \rightarrow \infty} \lambda A_\mu(\lambda x) \\ F_{\mu\nu}^{as}(x) & \equiv \lim_{\lambda \rightarrow \infty} \lambda^2 F_{\mu\nu}(\lambda x) \\ B^{as}(x) & \equiv \lim_{\lambda \rightarrow \infty} \lambda^2 B(\lambda x) \\ J_\mu^{as}(x) & \equiv \lim_{\lambda \rightarrow \infty} \lambda^3 J_\mu(\lambda x). \end{aligned} \quad (2)$$

The asymptotic fields so defined are really functions of the direction of x only. It is convenient to consider them as homogeneous functions with degrees determined by (2). In fact the assumed behaviour follows from the

engineering dimensions of the fields if one considers x^2 as the only scale parameter at large distance. It includes the possibility of a finite total electric charge as Gauss's law indicates.

At this stage it is already instructive to illustrate the meaning of Eq. (2) for the free field

$$A_\mu(x) = (2\pi)^{-3/2} \int [e^{ik \cdot x} \alpha(\omega, \hat{k}) + h.c.] (2\omega)^{-1} d^3k; \\ \omega = |\underline{k}| \quad \hat{k} = \underline{k}/\omega. \quad (3)$$

After rewriting A_μ as

$$A_\mu(x) = \lim_{\varepsilon \rightarrow 0} \frac{1}{2} (2\pi)^{-3/2} \int d\hat{k} \int_0^\infty d\omega [\omega \alpha(\omega, \hat{k}) e^{i\omega(\hat{k} \cdot \underline{x} - t) - \varepsilon t} + h.c.], \quad (4)$$

one obtains by standard arguments

$$\lim_{\lambda \rightarrow \infty} \lambda A_\mu(\lambda x) = \frac{1}{2} \int \frac{d\hat{k}}{(2\pi)^{3/2}} \left[\frac{\beta(\hat{k})}{\varepsilon + i(t - \hat{k} \cdot \underline{x})} + h.c. \right] \quad (5)$$

$$\beta(\hat{k}) = \lim_{\omega \rightarrow 0} \omega \alpha(\omega, \hat{k}). \quad (6)$$

Hence, for free fields, the assumed behaviour implies that the infra-red limit (6) depends upon the direction of $\hat{k}$ in general, although $\omega = 0$ lies on the tip of the light-cone $k_\mu k^\mu = 0$. This feature has been repeatedly used in previous discussions of infra-red problems by one of us ³⁾.

For later purpose we note that, for arbitrary $\alpha(\omega, \hat{k})$, A_μ satisfies $\square A = 0$. Hence, for arbitrary $\beta(\hat{k})$, the right member of (5) has a vanishing D'Alembertian also.

In practice, new dynamical variables at infinity come into play because, with the assumed behaviour [Eq. (2)], one cannot neglect the surface term at infinity in the partial integration necessary to derive the field equation from the action (1). These degrees of freedom have a dynamics described by an action of the surface at infinity which is derived from the need to recover the field equations at finite distance without breaking gauge invariance apart from gauge fixing terms ²⁾.

With the above motivations we define the volume action as the limit

$$S_{vol} = \lim_{\lambda \rightarrow \infty} \int_{x^2 < \lambda^2} \mathcal{L} d^4x \quad (7)$$

and add a surface action at spacelike infinity

$$S_{surf} = x^2 \int d\Omega \left[e A_\mu^{as} - \partial_\mu \Theta \right] s_\mu + e \int d\Omega \left[\partial_\mu l A_\mu^{as} + \frac{1}{2} f l^2 \right]; \quad (8)$$

Θ, s_μ, l are the new dynamical variables at spacelike infinity which we define as homogeneous functions of x of degrees 0, -1, and 0, respectively, s_μ is taken such that $s_\mu x^\mu = 0$, and $d\Omega$ is the integration measure over the angles of x^μ . The homogeneity conditions are such that $S_{surf.}$ can be computed on any hyperboloid $x^2 = \text{const.} > 0$.

There is no need for a surface action at timelike infinity since the behaviour of the fields in this limit is determined by the scattering process. We shall indeed derive this behaviour from the field equations with the asymptotic currents of particles moving at constant velocity in the far past or distance future.

The surface action (8) follows Ref. 2) but we choose the surface at infinity to be the hyperboloid rather than a space-time cylinder. This choice is Lorentz invariant. It is motivated by the natural splitting of the asymptotic Lienard-Wiechert potentials into a radial part and a tangential part which is a pure gauge

$$\frac{p_\mu}{[(p \cdot x)^2 - p^2 x^2]^{1/2}} = \frac{x_\mu x \cdot p}{x^2 [(p \cdot x)^2 - p^2 x^2]^{1/2}} + \partial_\mu f \quad x_\mu \partial_\mu f = 0.$$

With the assumed behaviour of the fields, it is natural to consider gauge transformations $\Lambda(x)$ with the finite limit

$$\Lambda^{as}(x) = \lim_{\lambda \rightarrow \infty} \Lambda(\lambda x). \quad (9)$$

The first term of S_{surf} [Eq. (8)] is invariant provided $\Theta \rightarrow \Theta + \Lambda^{\text{as}}$. The second term is the corresponding Gupta-Bleuler term (ζ is an arbitrary parameter). It is invariant under restricted transformations ($\square \Lambda^{\text{as}} = 0$).

The main point of our discussion is to show that the asymptotic space of infra-red photons is completely specified by the surface action (8). For this purpose one applies the principle of least action at spacelike infinity with $J^{\text{as}} = 0$ obtaining

$$A_{\mu}^{\text{as}} = -1/2 x_{\mu} (x^2)^{-1} l + e^{-1} \partial_{\mu} \Theta, \quad (10)$$

$$\square l = 0, \quad (11)$$

$$l = x^2 \square \Theta, \quad (12)$$

$$F_{\mu\nu}^{\text{as}} = 1/2 (x_{\mu} \partial_{\nu} - x_{\nu} \partial_{\mu}) (x^2)^{-1} l; \quad (13)$$

l is only a function of the angles and (11) can be solved by partial wave expansion. The general solution can be written as

$$l(x) = \frac{1}{4\pi} \left[Q \frac{2t}{r} \right] - \frac{x^2}{(2\pi)^{3/2}} \frac{\partial}{\partial t} \int d\hat{k} \left[\frac{c(\hat{k})}{\varepsilon - i(\hat{k} \cdot \underline{x} - t)} + h.c. \right] \quad (14)$$

where $c(\hat{k})$ is arbitrary ($c^* \neq c$) and Q is the total charge.

Next the fundamental point is that because of local commutativity, $F_{\mu\nu}^{\text{as}}$ in spacelike directions commutes with all local observables. Moreover, by canonical quantization of the action (7) and (8) and $\partial^2 l = 0$, l commutes with itself everywhere (like $\partial.A$ in the Gupta-Bleuler method). In an irreducible representation of local observables, l will be a c -number or classical function according to Schur's lemma. Hence Q and $c(\hat{k})$ are c -numbers and define superselection sectors of the S -matrix⁴⁾.

We now turn to timelike infinity. In this limit J_{μ}^{as} is the current of the scattered particles. These are in uniform rectilinear motion with four-velocity u_{μ} ($u^2 = -1$) such that impact parameters and Coulombic corrections are negligible compared to vt for very large times. The general form of J_{μ}^{as} is

$$J_{\mu}^{as}(x) = \int \frac{d^3 u}{u^0} \left[\rho^{in}(u) \theta(-x^0) + \rho^{out}(u) \theta(x^0) \right] \times \int_{-\infty}^{\infty} d\tau u_{\mu} \delta(x - u\tau) \quad (15)$$

ρ^{in} and ρ^{out} act on the in and out scattering spaces of charged particles according to

$$\rho^{in} = \sum_{\alpha} e_{\alpha} b_{\alpha}^{in\dagger} b_{\alpha}^{in} \quad (16)$$

and $in \rightarrow out$. The index α runs over the different types of charged particles.

The standard Lienard-Wiechert potentials provide solutions of Maxwell's equations with the above external currents. Indeed one can see that

$$A_{\mu}^{\pm} = \frac{\theta(-x^2) \theta(\pm x^0)}{4\pi} \int \frac{d^3 u}{u^0} \rho^{out}(u) \frac{u_{\mu}}{[x^2 + (u \cdot x)^2]^{1/2}} \quad (17)$$

is a solution of

$$\square A_{\mu}^{\pm}(x) = - J_{\mu}^{as}(x) \theta(\pm x^0). \quad (18)$$

Since $A_{\mu}^{\pm}$ vanishes at space-like infinity, one can obtain asymptotic equations without currents in all directions by subtracting $A_{\mu}^{\pm}$ from A_{μ}^{as} . This determines, after some calculations, A_{μ}^{as} for all x :

$$A_{\mu}^{as} = A_{\mu}^{+} + A_{\mu}^{-} + \frac{1}{(2\pi)^{3/2}} \int d\hat{k} \left[\frac{a_{\mu}^h(\hat{k})}{\varepsilon - i(\hat{k} \cdot x - t)} + h.c. \right] \quad (19)$$

$$a_{\mu}^h(\hat{k}) = \frac{1}{(2\pi)^{3/2}} Q \tau_{\mu} + \omega \frac{\partial}{\partial k^{\mu}} c(\hat{k}) + \text{gauge terms} \quad (20)$$

where $k^{\mu} = (\omega \hat{k}, k^0)$ and $\omega \partial / \partial k^{\mu} c(\hat{k})$ is independent of ω . τ_{μ} is the unit vector $(1, \vec{0})$. The corresponding non-covariant looking term is the Coulombic contribution.

Next we deduce the in and out asymptotic fields from the standard Yang-Feldman equation

$$A_m^{\text{in}}(x) = A_m(x) - \int d^4y \Delta^{\text{ret}}(x-y) J_m(y), \quad (21)$$

and $\text{in} \rightarrow \text{out}$, $\text{ret} \rightarrow \text{ad}$. From (15) one deduces

$$\begin{aligned} A_m^{\text{in}, \infty}(x) &\equiv \lim_{\lambda \rightarrow \infty} \lambda A_m^{\text{in}}(\lambda x) \\ &= \frac{1}{(2\pi)^{3/2}} \int d\hat{k} \left[\frac{a_m^h(\hat{k})}{\varepsilon - i(\hat{k} \cdot \underline{x} - t)} + \text{h.c.} \right] \\ &\quad - \frac{\theta(x^2)}{4\pi} \int \frac{d^3u}{u^0} \rho^{\text{in}}(\underline{u}) \frac{u_m}{[x^2 + (\underline{u} \cdot \underline{x})^2]^{1/2}}, \end{aligned} \quad (22)$$

and $\text{in} \rightarrow \text{out}$. Since A^{in} is a free field one can rewrite its asymptotic limit in the general form of Eq. (5). Indeed one has

$$-\frac{\theta(x^2)}{4\pi} \frac{1}{[x^2 + (\underline{u} \cdot \underline{x})^2]^{1/2}} = \frac{1}{(2\pi)^3} \frac{1}{2} \int \frac{d\hat{k}}{\underline{u} \cdot \hat{k} - u^0} \left(\frac{1}{\varepsilon + i(\hat{k} \cdot \underline{x} - t)} + \text{h.c.} \right). \quad (23)$$

From the above discussion about free fields one finally concludes that if we define

$$A_m^{\text{in}}(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{2\omega} \left[a_m^{\text{in}}(\omega, \hat{k}) e^{ik \cdot x} + \text{h.c.} \right] \quad (24)$$

we have

$$\begin{aligned} \lim_{\omega \rightarrow 0} \omega a_m^{\text{in}}(\omega, \hat{k}) &= \frac{1}{(2\pi)^{3/2}} \int \frac{d^3u}{u^0} \rho^{\text{in}}(\underline{u}) \left[\frac{u_m}{\underline{u} \cdot \hat{k} - u^0} + \tau_m \right] \\ &\quad + \omega \partial_\delta k^\mu C(\hat{k}) + \text{gauge terms}, \end{aligned} \quad (25)$$

and $\text{in} \rightarrow \text{out}$.

The first term on the right-hand side of (25) corresponds to the familiar coherence condition on low frequency photons³⁾. The last two terms follow from the surface action⁴⁾. Altogether Eq. (25) is compatible with the transversality condition,

$$k_n a_n^{\text{in}}(k) | > = 0, \quad (26)$$

while the first term alone is not. One sees that the introduction of surface variables solves in a very natural way the long standing contradiction between infra-red coherence condition and transversality condition for charged states. There have been numerous discussions of this problem where this contradiction was resolved by various devices⁵⁾ including the addition of c-number terms as we have obtained here. The present discussion has the advantage of deriving the modified infra-red coherence condition from first principles.

Finally, we would like to comment on the physical significance of the result (25) which states that both in and out states are coherent at zero photon frequency whereas, in fact, in calculating cross-sections, incoming coherent soft photons are safely ignored and infra-red divergences are eliminated by the sum over final (coherent) states. However, cross-sections are characteristic of scattering processes which have the particularity that the initial (charged) particles have well defined momenta⁶⁾ and consequently are completely unlocalized (and unnormalizable). For infra-red divergences to cancel out of transition probabilities in the more general case of initial wave packet or localized states of charged particles, the incident coherent soft photon must be included, as has been shown previously⁷⁾. In this case S-matrix elements are also infra-red finite. That such incident photons are present in Nature is assured by an extension of the uncertainty principle to charged particles. Let a charged particle (initially, say, in a plane wave state with no photons) be localized by passing through a slit. Momentum will be uncontrollably exchanged, so charge is accelerated and radiation inevitably occurs, the more, the greater the degree of localization⁷⁾. The one-particle plane wave state ceases to exist and is replaced, not by a localized state of a single particle, but by a coherent state which describes localized charge. The localized state is, in fact, generic, the plane wave being a convenient limiting case, and the result (25) correctly describes the zero frequency limit of such states.

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- 6) Or are a statistical mixture of momentum eigenstates, but not a superposition of such states.
- 7) This may be thought of as a manifestation of the gauge principle.