Ref.TH.2809-CERN

FLUX METAMORPHOSIS

F.A. Bais
CERN - Geneva

A B S T R A C T

Some consequences of a non-Abelian fundamental group $\pi_1(G/H)$ are investigated in the context of spontaneously broken gauge theories. Vortices associated with elements of $\pi_1(G/H)$ which do not commute cannot pass through each other, without remaining attached. Various simple models featuring this phenomenon are presented. A rather general method for generating the relevant ansätze for the fields is given as well.

Ref.TH.2809-CERN
7 February 1980

1. - INTRODUCTION

It is well known that magnetic flux tubes can arise as classical solutions of spontaneously broken gauge theories. Magnetic flux is a topologically conserved quantity which corresponds to the elements of the fundamental group $\pi_1(G/H)$ of the coset space G/H , where G is the gauge groups and $H \subseteq G$ the residual exact symmetry group. In the cases considered so far $\pi_1(G/H)$ itself has been an Abelian group (typically Z_N), but from a mathematical point of view it need not be. This interesting possibility has been first considered by Poenaru and Toulouse in the context of crystal defects ¹⁾. Analogous situations may be encountered in gauge theories, leading to the appearance of "truly non-Abelian" flux tubes whose interactions differ significantly from the "Abelian" Z_N flux tubes.

In Section 2 we briefly recall some basic notions concerning magnetic flux and its topological characteristic.

A discussion of how the elements of π_1 act on each other is given in Section 3. It is argued how flux may change its identity if it passes behind another flux tube. It is this property of flux metamorphosis which leads to the remarkable property that flux tubes corresponding to non-commuting elements of $\pi_1(G/H)$ cannot pass through each other without staying connected by some other flux tube.

In Section 4, we present various models in which these effects occur some of which are surprisingly simple. Section 5 is devoted to the construction of explicit ansätze. A rather general method is presented to construct a solution of any given type. The mathematical level of our discussions is kept as elementary as possible throughout.

2. - THE TOPOLOGY OF MAGNETIC FLUX

We briefly recall the basic notion of magnetic flux as an element of the fundamental group of the vacuum manifold ²⁾. Imagine a spontaneously broken gauge theory with some Higgs field Φ . We assume that asymptotically the fields approach the vacuum, i.e., they satisfy the equations $F_{\mu\nu} = 0$, $D_\mu \Phi = 0$ and $(\delta V(\Phi))/\delta \Phi = 0$. A solution to these can be written in the form

$$\Phi(\vec{r}) = U(\vec{r}, \vec{r}', \gamma) \Phi(\vec{r}') \quad (2.1)$$

where

$$U(\vec{r}, \vec{r}', \gamma) = \mathcal{P} \exp \left[i e \int_{\vec{r}'}^{\vec{r}} \vec{A} \cdot d\vec{\gamma} \right]$$

is the path dependent phase factor from $\vec{r}'$ to $\vec{r}$ along a path parametrized by γ . For example, going around a large circle around the Z axis, one may write

$$U(\varphi) = \mathcal{P} \exp \left[i e \int_0^\varphi A_\varphi r d\varphi \right] \quad (2.2)$$

The phase factor $U(\varphi)$ maps the path in real space onto a path in the representation space of G . Thinking of $U(\varphi)$ as the gauge transformation relating $\Phi(\varphi)$ and $\Phi(0)$ we have to demand it to be single-valued, i.e., that $U(2\pi)$ leaves $\Phi(0)$ invariant or $U(2\pi) \in H$. Closed paths in configuration space therefore do not necessarily require closed paths in G , but in the space G/H , where we identify all elements of H with the identity, the paths are closed. The closed paths in G/H fall into homotopy classes labelled by the elements of the fundamental group $\pi_1(G/H)$. Two paths for which $U(2\pi) = h_1$ and h_2 , respectively, are homotopic only if h_1 and h_2 are path-wise connected in H . There is one complication we should mention here : if G is not simply connected then not all the closed paths in G are homotopic to the trivial path (which is contractible to a point). In fact, only paths which are closed in the universal covering group $\bar{G}$ of G correspond to trivial paths in G . Therefore, to classify all configurations which are not homotopic (i.e., they are not gauge equivalent hence physically different), we shall use $\pi_1(\bar{G}/\bar{H})$ rather than $\pi_1(G/H)$ (we denote the lift of any subgroup to the covering group by a bar). Standard example : in $O(3)$ which is doubly connected we should think of $U(\varphi) \in SU(2)$, i.e., acting on a spinor rather than on a vector. These previous statements are conveniently summarized by the simple statement that magnetic flux is characterized by $\pi_1(\bar{G}/\bar{H})$, and that

$$\pi_1(G/H) \simeq \pi_1(\bar{G}/\bar{H}) \simeq \pi_0(\bar{H})$$

i.e., there exists an isomorphism between $\pi_1(\bar{G}/\bar{H})$ and $\pi_0(\bar{H})$. Though $\pi_0(\bar{H})$, which just labels the components of $\bar{H}$, is not necessarily a group, it is endowed with a group structure through the isomorphism. For example, if $\bar{H}$ is a discrete group then $\pi_0(\bar{H}) \simeq \bar{H}$. So the question of a topological classification of flux boils down to the simple question of the connectedness of the residual symmetry group $\bar{H}$. Furthermore the group structure of $\pi_0(\bar{H})$ determines the way fluxes add as we now consider.

3. - FLUX METAMORPHOSIS OR THE ACTION OF π_1

Given some discrete group

$$\pi_0(\bar{H}) = \{ u_i ; i = 1, \dots, n \} \quad (3.1)$$

with $u_i u_j = u_k$, etc., what are the physical properties reflected by its structure ?

The elements of (3.1) fall into conjugacy classes. The conjugacy class $[u_i]$ is the orbit of the element u_i under the group $[u_i] = \{ u_j u_i u_j^{-1} ; v_j \}$. So a single flux tube is only defined up to conjugacy, only a conjugacy class has a gauge invariant meaning. This property leads to the so-called patching ambiguity ²⁾. Namely in the product of two classes

$$[u_i] \cdot [u_j] = \sum a_k [u_k]$$

one may find more than a single class, so that if we patch two solutions, one of the class $[u_i]$ and one of the class $[u_j]$ together, the result is not unique. This means that a choice has to be made in the patching of the solutions. For example, imagine u_i and u_i^{-1} belong to the same class, then $[u_i] \cdot [u_i]$ will contain u_i^2 as well as the identity E , which often belong to two different classes since $[E] = E$.

A somewhat different ambiguity arises if $\pi_0(H)$ is non-Abelian. In that case, there are elements for which $[u_i, u_j] \neq 0$ or

$$u_i u_j u_i^{-1} u_j^{-1} = u_k \neq 1 \quad (3.2)$$

Combining two of these solutions, α and β , we obtain in principle two different possibilities $\alpha\beta$ or $\beta\alpha \neq \alpha\beta$, depending on the order we encircle them. These, however, belong to the same class $(\alpha^{-1}(\alpha\beta)\alpha = \beta\alpha)$. What we need is some kind of gauge or a better patch fixing. This may be done by defining the paths in real and group space with respect to some fixed base point in the following way. We assign an element $\alpha \in \pi_0(\bar{H})$ to a flux tube which we represent as an oriented line by starting at the base point (chosen, for convenience, to be the observer's eye) and going around the tube in a right-hand way. The total flux associated to a number of flux tubes is defined by going around the minimal surface they span. This procedure which fixes the order in which we encircle a number of flux tubes as well as the patching is depicted in Fig. 1.

Let us now study the case of two loops α and β , one of which passes behind the other as shown in Fig. 2. The homotopy element β , associated with β after it has passed behind α , need not be equal to β in general. To see this, go back to the phase factor $U(\varphi)$. First we have $\Phi(\varphi) = U(\varphi)\Phi(0)$ with $U(2\pi) = \beta$, but when the tube β goes around α , we have

$$\Phi'(\varphi) = \alpha U(\varphi) \Phi(0)$$

so that

$$\Phi'(0) = \alpha \Phi(0)$$

and

$$\Phi'(2\pi) = \alpha \beta \Phi(0) = \alpha \beta \alpha^{-1} \Phi'(0)$$

Alternatively, we may take the loop defining β_1 and continuously deform it to obtain the loop in Fig. 3, explicitly verifying the homotopic equivalence of $\beta_1 \simeq \alpha\beta\alpha^{-1}$. So passing behind α has the effect that $\beta \rightarrow \alpha\beta\alpha^{-1}$ which is different from β if $[\alpha, \beta] \neq 0$, i.e., β has changed identity, with the striking consequence that the tube β cannot close onto itself! This simple effect of flux metamorphosis specifies the way in which elements of $\pi_1(\bar{G}/\bar{H})$ act on each other and as we will see, contains the basic non-trivial aspect of "truly non-Abelian" flux.

Consider now the configurations shown in Fig. 4, where we imagine that two non-parallel flux tubes tried to cross each other. In Fig. 4a, we have that $\beta_1 = \alpha\beta\alpha^{-1}$ and $\alpha_1 = \beta_1\alpha\beta_1^{-1} = \alpha\beta\alpha\beta^{-1}\alpha^{-1}$. Both have changed but the total flux has not : $\alpha\beta = \beta_1\alpha = \alpha_1\beta_1$. For the tubes to simply pass through each other it would be necessary to reattach β' and β in front of α but this is only possible if $\beta = \beta'$, i.e., $[\alpha, \beta] = 0$. If Fig. 4b, the tubes intersect in a different way, we would have $\alpha_1 = \beta^{-1}\alpha\beta$ and $\beta_1 = \alpha_1^{-1}\beta\alpha_1 = \beta^{-1}\alpha^{-1}\beta\alpha\beta$ which is different from Fig. 4a. It is not surprising that if tubes cannot cross we may envisage all kinds of knots whose topology can become arbitrarily complicated. Figure 5 features a simple one, where the equations for α_1, β_1 and β_2 are $\beta_1 = \beta_2\beta_2^{-1}$, $\beta_2 = \alpha^{-1}\beta_1\alpha$ and $\alpha_1 = \beta_1^{-1}\alpha\beta_1$. These are now coupled which leads to a non-linear equation for β_1 ; $\beta_1 = \alpha^{-1}\beta_1\alpha\beta\alpha^{-1}\beta_1^{-1}\alpha$. Such equations are hard to solve in general if one does not know more about $\pi_0(\bar{H})$. Note, however, that β_1 belongs to the same conjugacy class. We summarize by saying that the intersection of two flux tubes α and β may be represented by a vertex (Fig. 6) where α_1 and β_1 may be anything as long as the total flux in any direction is conserved ; $\alpha_1\beta_1 = \delta$ and $\alpha_1^{-1}\alpha = \beta_1\beta^{-1} = \gamma$, which may have different solutions depending on the knot. As we do not want to get tied up in knot theory we proceed to a discussion of models in which this type of non-Abelian fluxes occur.

4. - SOME MODELS WITH NON-ABELIAN FLUX

To determine the flux content of a particular theory one has to answer the following questions :

- 1) what are the finite and disconnected subgroups $\bar{H}$ of the compact simple gauge group $\bar{G}$?
- 2) which representations allow a symmetry breaking pattern corresponding to these subgroups $\bar{H}$?

Answers to these questions are known only in special cases, and we choose some examples from these cases, mainly to illustrate the possibilities.

The first question has been answered completely for the rotation group. Its subgroups are $SO(2)$ and the normalizer of $SO(2)$ denoted by $N(SO(2))$ which consists of all rotations around one axis and of rotations by π around any axis perpendicular to it. This group is disconnected and has two components. These are all continuous subgroups. The other subgroups are the well-known point groups [or double point groups : if we go to $\bar{G} = SU(2)$ we have to add the generator of the centre (-1) which doubles the order] :

- 1) the cyclic groups Z_N of order N ;
- 2) the dihedral groups D_n , consisting of one n fold axis and of n two-fold axes perpendicular to it. D_n is of order $2n$ (and $\bar{D}_n$ of order $4n$) ;
- 3) the tetrahedral groups T of order 12, the orthohedral group O (symmetry group of the cube) of order 24, and the icosahedral group I of order 60.

Let us list some $O(3)$ models.

- 1) The trivial case. One Higgs field in the vector representation. $H = SO(2)$ so that $\pi_0(\bar{H}) = 0$. No topologically conserved flux.
- 2) More than one Higgs field in the vector representation.
In general $\bar{H} = Z_2 = \{1, -1\}$. And consequently only one type of flux exists, which is conserved modulo 2. Larger groups H can be arranged for if one adds additional discrete symmetries between the various fields.
- 3) Φ a symmetric second rank tensor. Φ can in general be decomposed with respect to a particular $SO(2)$ subgroup of rotations about a particular vector $\vec{\eta}$ as follows ³⁾

$$\Phi_{ab} = \alpha (\hat{\eta}_a \hat{\eta}_b - \frac{1}{3} \delta_{ab}) + \beta (\hat{\mu}_a \hat{\lambda}_b + \hat{\lambda}_a \hat{\mu}_b)$$

where $\hat{\eta}$, $\hat{\mu}$ and $\hat{\lambda}$ form an orthonormal basis and α and β are constants. If $\beta = 0$ or $|\alpha/\beta| = \frac{1}{2}$ then $H = N(SO(2))$ as can easily be checked. $\pi_0(\bar{H}) = Z_2$ and again there is one non-trivial flux type. G/H is now a two-sphere S^2 with antipodal points identified so that indeed $\pi_1(\bar{G}/\bar{H}) = Z_2$. This case has been discussed by Coleman ²⁾ and Shankar ³⁾.

For values $\beta \neq 0$ and $|\alpha/\beta| \neq \frac{1}{2}$, for example if :

$$\Phi_{ab} = (\hat{x} + \hat{y})_a (\hat{x} - \hat{y})_b + (\hat{x} - \hat{y})_a (\hat{x} + \hat{y})_b \quad (4.1)$$

the residual symmetry group is D_2 . Explicitly,

$$\pi_0(\bar{H}) \simeq \bar{H} \simeq \{E, I, -\Omega_x(\pm \pi), -\Omega_y(\pm \pi), -\Omega_z(\pm \pi)\}$$

where E is the identity, I the inversion and $\Omega_n(\alpha)$ a spinor rotation by α about $\hat{n}$ (i.e., $\Omega_n(\pm\pi) = \pm i\hat{n} \cdot \vec{\sigma}$). This group has eight elements which fall into five classes: $[E]$, $[I]$, $[\Omega_x]$, $[\Omega_y]$, $[\Omega_z]$. The patching ambiguity is present because, for example, $[\Omega_x][\Omega_x] = 2[E] + 2[I]$ and hence the product of two classes is not a unique class. The group is non-Abelian and we may illustrate the phenomena discussed in the previous section, for example with $\alpha = \Omega_x(\pi)$ and $\beta = \Omega_y(\pi)$ we find that $\alpha_1 = \Omega_x(-\pi)$ and $\beta_1 = \Omega_y(-\pi)$, in fact for each of the configurations of Figs. 4a, 4b and 5, as is to be expected because there are only two elements in the corresponding conjugacy classes. This implies that the fluxes in Fig. 6 will be $\gamma = -1$ and $\delta = \Omega_z(\pi)$.

It should be pointed out, however, that the assumed type of symmetry breaking (4.1), which can easily be arranged classically, does probably not correspond to a minimum of the effective one-loop potential.

Alternatively, a residual symmetry group $H = D_2$ may be realized by taking two symmetric tensors and the potential arranged such that

$$\Phi_{ab}^i = \alpha_i (\hat{x}_a^i \hat{x}_b^i - \frac{1}{3} \delta_{ab}) \quad (i = 1, 2)$$

In fact, if it is possible to take $\alpha_1 = \alpha_2$ with a symmetric potential $V(\hat{\Phi}^1, \hat{\Phi}^2) = V(\hat{\Phi}^2, \hat{\Phi}^1)$ the symmetry would be $H = D_4$, allowing for solutions where asymptotically

$$\Phi'(2\pi) = \Phi^2(0) \quad \Phi^2(2\pi) = \Phi'(0)$$

- 4) Φ is a symmetric third rank tensor. The even potential may be written as

$$V = -\frac{\mu^2}{2} \Phi_{abc} \Phi^{abc} + \frac{\lambda}{4} (\Phi_{abc} \Phi^{abc})^2 + \\ + c \Phi_{abc} \Phi^{bcd} \Phi_{def} \Phi^{efa}$$

Ovrut ⁴⁾ has shown that this potential allows for the following symmetry breaking patterns.

- a) if $c > 0$ then $H = T$, the tetrahedral group ;
- b) if $c = 0$ then $H = SO(2)$, T , D_3 , Z_1 , Z_2 , Z_3 and again it is expected that quantum corrections will drive the minimum to $H = SO(2)$;

c) if $-\lambda/2 < c < 0$ then $H = D_3$.

Hence with a single Higgs field in a third rank irreducible representation two different realizations of non-Abelian flux are possible.

We expect that the higher order point groups will appear as the stability group of yet higher rank representations, but to our knowledge this has not been determined in general.

One may wonder what happens in gauge theories with gauge groups larger than $SU(2)$. Though many particulars are known the complete classification of finite subgroups of compact simple Lie groups is still an unsolved problem. Needless to say that even less is known about the stability groups of arbitrary representations of these groups. For $SU(3)$, the disconnected and finite subgroups have been determined⁵⁾ and we conclude this section by indicating their properties. There are basically two classes of discrete groups which are labelled $\Sigma(n\varphi)$ and $\Delta(\alpha n^2)$, where the number in parentheses gives the rank.

The groups $\Sigma(n\varphi)$ where $\varphi = 1$ or 3 are like the usual crystal groups and their double groups. The following possibilities exist : $\Sigma(60)$, $\Sigma(168)$, $\Sigma(36\varphi)$, $\Sigma(72\varphi)$, $\Sigma(216\varphi)$ and $\Sigma(360\varphi)$. The groups with $\varphi = 1$ are subgroups of $SU(3)/Z_3$, whereas those with $\varphi = 3$ are subgroups of $SU(3)$ (they contain the centre Z_3) ; and $\Sigma(60)$ and $\Sigma(168)$ are subgroups of both. $\Sigma(60)$ is isomorphic to the icosahedral group Y , the others have been only identified as subgroups of symmetric groups of a given rank.

The groups $\Delta(3n^2)$ and $\Delta(6n^2)$ are like the dihedral groups. $\Delta(3n^2)$ is a subgroup of $\Delta(6n^2)$ and both series are subgroups of $SU(3)/Z_3$ except if $n/8$ is an integer in which case they do contain the centre.

We have not undertaken the task of determining which potentials and representations in $SU(3)$ can give rise to the afore-mentioned groups as a residual symmetry. It seems reasonable to expect that these will occur with Higgs fields in large representations.

5. - SOLUTIONS

We outline a procedure for generating the appropriate ansatz for the fields which yields a single flux tube solution of a given type. The type is determined by $U(2\pi) \subseteq \pi_0(\bar{H})$ where $\bar{H}$ is determined by a particular vacuum expectation value Φ_0 . The construction proceeds in two stages : first we construct a singular solution which has the correct angular (φ) dependence, and then we will implement the non-trivial dependence on the radial variable.

The correct angular dependence may be generated by choosing a convenient path $U(\varphi)$ in group space. It is the shortest path from E ending at $U(2\varphi)$, i.e., by going on a circle around the z axis in real space we move along the geodesic line from E to $U(2\pi)$ parametrizing it uniformly by some angle α which typically runs from 0 to $2\pi/n$. This implies that we can write

$$U(\varphi) = e^{i \frac{\varphi}{n} T} \quad (5.1)$$

where T is now some properly normalized generator of G and path ordering has become trivial. The singular solution takes symbolically the form

$$U(\varphi) = U(\varphi) \Phi_0 \quad (5.2a)$$

meaning that we apply $U(\varphi)$ to all the indices of the tensor Φ . The vector field satisfying $D_\mu \Phi = 0$ is then

$$A_\mu = \frac{i}{e} U(\varphi) \partial_\mu U^{-1}(\varphi)$$

so that only the $\hat{\varphi}$ component in the T direction is non-vanishing :

$$A_\varphi = \frac{1}{ner} T \quad (5.2b)$$

This singular solution is symmetric with respect to the mixed rotation operator $J = L_z + 1/n T$, in the sense that

$$J \Phi = 0 \quad (5.3a)$$

and

$$[J, A_i] = i \epsilon_{ij} A_j \quad (i, j = 1, 2) \quad (5.3b)$$

Maintaining this J symmetry for the full r dependent ansatz severely restricts the number of independent radial functions entering the ansatz. For the vector field it is trivial ; one merely multiplies (5.2b) with an arbitrary function of r . One may expect, however, that, since Φ may be in a large representation, its radial dependence will be much more involved. This is true, but there is in most cases a simple way to determine it. This trick consists of decomposing Φ with respect to some orthonormal vector basis and then applying (5.3a) to each of the basis vectors, as we will illustrate in an explicit example shortly. Because the ansatz thus produced exhibits the J symmetry, one may substitute it directly in the Lagrangian and vary with respect to the radial functions ²⁾.

Let us, as a final stroke, illustrate the outlined procedure for the example 4a) of the previous section, where we have a Higgs field in the third rank symmetric tensor representation. Its vacuum expectation value is chosen Φ_{123} (and its permutations) = 1, all others vanish. This is equivalent to

$$\Phi_0 = [\hat{x} \hat{y} \hat{z}]_s \quad (5.4)$$

where we have written it as a symmetric product of three base vectors. The stability group is the tetrahedral group T , with four three-fold axes and three two-fold axes, so that $\pi_0(\bar{H}) = \bar{H} = \bar{T}$, explicitly

$$\begin{aligned} \bar{T} = \{ & E, I, \Omega_x(\pm\pi), \Omega_y(\pm\pi), \Omega_z(\pm\pi), \\ & \Omega_{x+y+z}(\pm\frac{2\pi n}{3}), \Omega_{x+y-z}(\pm\frac{2\pi n}{3}), \Omega_{x-y+z}(\pm\frac{2\pi n}{3}), \\ & \Omega_{x-y-z}(\pm\frac{2\pi n}{3}) \} \end{aligned} \quad (5.5)$$

where $I = \Omega(2\pi) = -1$ and $n = 1, 2$.

First we generate an ansatz for a tube with $U(2\pi) = \Omega_z(\pi)$, the others corresponding to $\Omega_{\pm}(\pm\pi)$ are basically the same. According to our prescription we choose

$$U(\varphi) = e^{i\frac{\varphi}{2}T_3} = \begin{pmatrix} \cos \varphi/2 & -\sin \varphi/2 & 0 \\ \sin \varphi/2 & \cos \varphi/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (5.6)$$

We want our ansatz to be symmetric with respect to $J = i\partial_\varphi + \frac{1}{2}T_3$. Operating with $\exp(2\pi iJ)$ on $\Phi = U(\varphi)\Phi_0$ has the effect that $\hat{x} \rightarrow -\hat{x}$, $y \rightarrow -\hat{y}$ and $\hat{z} \rightarrow \hat{z}$, so that $\Phi_{111} = \Phi_{112} = \Phi_{122} = \Phi_{133} = \Phi_{233} = 0$. Only Φ_{113} , Φ_{222} , Φ_{333} and Φ_{123} (and permutations) may be non-zero which, after imposing the trace condition $\Phi_{aa3} = 0$, reduces the total number of independent non-zero components to three. Using (5.2a), (5.4) and (5.6) the singular form of the Higgs field is

$$\Phi(\varphi) = \left[\begin{pmatrix} \cos \varphi/2 \\ \sin \varphi/2 \\ 0 \end{pmatrix} \begin{pmatrix} -\sin \varphi/2 \\ \cos \varphi/2 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right]_s \quad (5.7)$$

This fixes the angular dependence of the solution. To determine the radial dependence multiply each entry in the vectors of (5.7) with different functions of r , i.e., we have three vectors $\alpha = (b_1(r) \cos \varphi/2, b_2(r) \sin \varphi/2, 0)$, $\beta = (-b_3(r) \sin \varphi/2, b_4(r) \cos \varphi/2, 0)$ and $\gamma = (0, 0, b_5(r))$ with $\Phi = [\alpha \beta \gamma]_s$. Requiring invariance (5.3a) under J we obtain

$$J\Phi = \left[(J\alpha) \beta \gamma \right]_s + \left[\alpha (J\beta) \gamma \right]_s + \left[\alpha \beta (J\gamma) \right]_s \quad (5.8)$$

but from the given angular dependence (5.7) of $\alpha \beta$ and γ it follows that the only solutions are those where each of the terms vanishes, so that the general solution of (5.8) is found by just imposing $J\alpha = J\beta = J\gamma = 0$. This yields $b_1 = b_2$, $b_3 = b_4$ so that all the non-vanishing components of Φ depend on a single function $c(r) = b_1 b_3 b_5$; $\Phi_{113} = -c \sin \varphi$, $\Phi_{223} = c \sin \varphi$, $\Phi_{123} = c \cos \varphi$.

Let us finally turn to the case of $U(2\pi) = \Omega_{x+y+z}(2\pi/3)$ which is more involved, but the same analysis holds. The symmetry generator becomes $J = i\partial_\varphi + 1/3 \hat{T}_{x+y+z}$ where

$$T_{x+y+z} = \frac{i}{\sqrt{3}} \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

Following the same steps, we obtain $\Phi = [\alpha \beta \gamma]_S$ with

$$\alpha = (a_\alpha(r) + 2b_\alpha(r)\cos\varphi, c_\alpha(r) - d_\alpha(r)\cos\varphi + f_\alpha(r)\sqrt{3}\sin\varphi, \\ c_\alpha(r) - d_\alpha(r)\cos\varphi - f_\alpha(r)\sin\varphi)$$

and β, γ obtained from α by successive cyclic permutations of the components and replacing the subscript of the radial functions. Again we may apply J on α, β and γ separately, which yields $a_i = c_i, b_i = d_i = f_i$ ($i = \alpha, \beta, \gamma$). Applying the trace conditions leads to $a_i = a_j$ and $b_i = b_j$ so that the complete ansatz only depends on two independent radial functions.

Though ansätze for all elements of $\pi_0(\bar{H})$ may be generated this way, one does not expect that all of them give rise to stable solutions, since as long as the total flux is conserved they can decay into each other. The total number of stable flux tubes is presumably equal to the number of generators of $\pi_0(\bar{H})$. This may be studied by a more careful investigation of the solutions, and we will not discuss it any further here.

6. - DISCUSSION

We have studied the properties of magnetic vortices associated with a fundamental group $\pi_1(\bar{G}/\bar{H})$ which is non-Abelian. This non-Abelian character manifests itself by the property that flux tubes associated with non-commuting elements of π_1 cannot pass through each other without staying attached ¹⁾. A number of models featuring these phenomena are presented as well as a rather general method to generate the appropriate ansätze for explicit solutions. So far we do not see an important rôle for the models presented in particle physics. Still there are examples of grand unified models where large representations are used for the Higgs field and it is conceivable that in those cases a phase exists where this non-Abelian type of flux occurs. One may imagine that with this non-Abelian flux there is the possibility of stable localized knots. We have not considered the introduction of monopoles.

When this work was near its completion, a paper by G. McCauley⁶⁾ was brought to my attention in which similar questions are considered.

ACKNOWLEDGEMENTS

I thank R. Casalbuoni, D. Olive, P. Sorba and S.-T. Tsou for conversations and references, E. Weinberg for some critical comments, and J. Ellis for reading the manuscript.

REFERENCES

- 1) V. Poenaru and G. Toulouse - J.Physique 38 (1977) 887 ;
G. Toulouse - J.Physique Lett. 28 (1977) 67.
- 2) S. Coleman - Erice Lectures (1975).
- 3) R. Shankar - Phys.Rev. D14 (1976) 1107.
- 4) B.A. Ovrut - J.Math.Phys. 19 (1978) 418.
- 5) W.M. Fairbairn, T. Fulton and W.H. Klink - J.Math.Phys. 5 (1964) 1038.
- 6) G. McCauley - Birmingham University Preprint (1978).

FIGURE CAPTIONS

- Figure 1 The minimal surface spanned by a number of vortices defines the order in which we would encircle them and therefore assigns a unique element to the total flux.
- Figure 2 One vortex passing behind two others.
- Figure 3 Loops illustrating the homotopic equivalence of $\beta_1 = \alpha\beta\alpha^{-1}$, i.e., the phenomenon of flux metamorphosis.
- Figure 4 Crossing vortices.
- Figure 5 A simple knot.
- Figure 6 Possible configurations after crossing. The exact situation being determined by the detailed dynamics and boundary conditions.

FIG. 1

FIG. 2

FIG. 3

FIG. 4a

FIG. 4b

FIG. 5

FIG. 6