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CHIRAL DYNAMICS IN THE LARGE N LIMIT

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A B S T R A C T

We construct an effective Lagrangian describing the low energy spectrum and dynamics of the pseudoscalar nonet in the large N_{colour} limit of QCD. Effects of the axial anomaly and of a possible non-zero vacuum angle are incorporated together with corrections of order $m_q N_c / \Lambda$.

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1. - INTRODUCTION AND OUTLINE OF THE PAPER

One of the long standing problems of the hadronic world is to understand the spectrum and low energy dynamics of pseudoscalar mesons.

The methods of current algebra ¹⁾ have been extensively and successfully used to compute a number of parameters, such as low energy $\pi\pi$ scattering, but the mass spectrum and interactions of the whole nonet of low-lying pseudoscalars (π, K, η, η') have remained difficult to understand. This set of problems, which has been extensively analyzed in the literature ²⁾, goes under the name of the $U(1)$ problem. It consists of the following, not unrelated, questions :

- i) why is the η mass much larger than the pion mass and of the same order of the K mass ? this is the $SU(2) \otimes SU(2)$ version of the problem ;
- ii) why is the η' mass much larger than that of the other eight pseudoscalars ? this is the $U(1)$ problem at the $SU(3) \otimes SU(3)$ level ;
- iii) why is the mixing angle between singlet and octet so small for pseudoscalars ? or, in other words, why is the OZI rule violated in this case by an anomalously large amount ?

After the discovery of the instanton solutions of the Yang-Mills theory ³⁾ and of the connection between the topological charge density and the axial anomaly ⁴⁾, it has become clear that a solution of the $U(1)$ problem must be based on a better treatment of the axial vector anomaly which may take a non-zero value when topologically non-trivial field configurations are included.

Recently the $U(1)$ problem has been analyzed in the framework of the $1/N$ expansion of QCD (N is the number of colours) where it has been shown that the anomalous Ward identities can be saturated without introducing a massless singlet Goldstone boson ⁵⁾⁻⁷⁾. In this approach the mass of the singlet is intimately related to the anomaly : they both vanish in the $N \rightarrow \infty$ limit. The inclusion of the anomaly also gives a very reasonable spectrum for the low lying pseudoscalar mesons ⁶⁾ since it justifies the use of a mass matrix of the type proposed on more phenomenological grounds by several authors ⁸⁾.

These results, that follow only from the Ward identities and from the $1/N$ expansion of QCD, have been then re-obtained from an effective Lagrangian ⁷⁾ which includes the meson fields and the topological charge density. The two-point Green's functions, computed from this Lagrangian, satisfy the anomalous Ward identities without a singlet Goldstone boson.

In this paper we generalize the approach of Ref. 7) by constructing an effective Lagrangian that describes the low energy physics of low-lying pseudoscalar mesons and that can be used to compute any n point Green's function. This Lagrangian, that has been also constructed independently by Rosenzweig, Schechter and Trahern ⁹⁾, is a generalization of the usual σ model, has the correct transformation properties under the flavour group $U(L) \otimes U(L)$ (L is the number of light fermion flavours), and includes the effect of the axial vector anomaly and of a possibly non-zero value of the vacuum angle Θ ¹⁰⁾.

The outline of the paper is as follows. The construction of the effective Lagrangian from general invariance and $1/N$ counting arguments is presented in Section 2A. Section 2B is devoted to the computation of one- and two-point functions in a general Θ vacuum and with an arbitrary (light) quark mass spectrum. In Section 3 we discuss the solution of the $U(1)$ problem for any value of Θ , showing that physical quantities are periodic in Θ with period 2π . Particular attention is devoted to $\Theta = \pi$ where Dashen's mechanism for spontaneous CP violation is seen to occur in the, probably unrealistic, case where $|m_u - m_d| m_s < m_u m_d$. Finally, in an Appendix, we give support to the correctness of the above somewhat formal arguments for deriving the effective Lagrangian by rederiving the same results through a diagrammatic S matrix approach.

2. - THE EFFECTIVE LAGRANGIAN AT LARGE N

A. - Construction of the effective Lagrangian through invariance arguments

We shall take here the point of view that QCD exhibits, in the large N limit, both confinement and dynamical breaking of chiral symmetry. Then, neglecting quark masses and effects due to the anomaly (e.g., by neglecting non-planar diagrams), the low energy dynamics of the pseudoscalar mesons can be described by a kind of linear σ model with Lagrangian ¹¹⁾ :

$$\mathcal{L} = \mathcal{L}_0 \equiv \frac{1}{2} \text{Tr}(\partial_\mu U \partial_\mu U^\dagger) - \frac{\lambda}{4} \text{Tr}[(U^\dagger U - F_\pi^2/2)^2] \quad (2.1)$$

where F_π will be indeed identified with the pion decay constant ($F_\pi \sim 95$ MeV) and $U_{ij}(x)$ is an $L \times L$ complex matrix representing scalar and pseudoscalar fields in a world with L light quark flavours. In general, U_{ij} can be written as :

$$U_{ij} = \sigma_{ij} + i \pi_{ij} = (\sigma^\alpha + i \pi^\alpha) \tau_{ij}^\alpha \quad (2.2)$$

where τ_{ij}^α are the generators of $U(L)$ in the fundamental representation with normalization :

$$\text{Tr} (\tau^\alpha \tau^\beta) = \delta_{\alpha\beta} \quad (2.3)$$

The Lagrangian (2.1) is invariant under chiral $U(L) \otimes U(L)$ transformations :

$$U \rightarrow A U B^\dagger \quad (2.4)$$

where A and $B^\dagger$ are arbitrary, unitary $L \times L$ matrices. Furthermore, because of the form of the potential term in (2.1), chiral symmetry is spontaneously broken and

$$\langle U^\dagger U \rangle = \frac{1}{2} F_\pi^2 \mathbb{1} \quad (2.5)$$

Actually, the specific form of the potential is not important : the results that follow only from current algebra (and not from the specific form of the effective Lagrangian) can be recovered, for instance, in the limit $\lambda \rightarrow \infty$ where (2.1) becomes a non-linear σ model with the operator equation $2U^\dagger U = F_\pi^2 \mathbb{1}$ [rather than its expectation value (2.5)] imposed as a constraint.

The Lagrangian (2.1) has to be modified in order to include the effect of the $U(1)$ axial anomaly. We know that, in actual QCD, because of quantum corrections, a $U(1)$ chiral transformation is equivalent to change the QCD action by

$$\int d^4x \mathcal{L}_{\text{QCD}}(x) \rightarrow \int d^4x \mathcal{L}_{\text{QCD}}(x) + L \phi \int d^4x \mathcal{Q}(x) \quad (2.6)$$

where ϕ is the parameter of the chiral transformation.

Here $\mathcal{L}_{\text{QCD}}$ depends on quark and gluon fields and the topological charge density field is expressible in terms of the latter as :

$$\begin{aligned}
 q(x) &= \frac{g^2}{64\pi^2} \epsilon_{\mu\nu\rho\sigma} F_{\mu\nu}^a F_{\rho\sigma}^a = \frac{1}{4} \epsilon_{\mu\nu\rho\sigma} F_{\mu\nu\rho\sigma} , \\
 F_{\mu\nu\rho\sigma} &= \partial_\mu A_{\nu\rho\sigma} - \partial_\sigma A_{\mu\nu\rho} + \partial_\rho A_{\sigma\mu\nu} - \partial_\nu A_{\rho\sigma\mu} \\
 A_{\nu\rho\sigma} &= \frac{g^2}{96\pi^2} \left[A_\nu^a \overleftrightarrow{\partial}_\rho A_\sigma^a - A_\rho^a \overleftrightarrow{\partial}_\nu A_\sigma^a - A_\sigma^a \overleftrightarrow{\partial}_\rho A_\nu^a + 2 C_{abc} A_\nu^a A_\rho^b A_\sigma^c \right]
 \end{aligned} \tag{2.7}$$

where A_a^ν are the usual gauge fields ($a=1, \dots, N^2-1$), C_{abc} are the $SU(N)$ structure constants and $A_{\nu\rho\sigma}$ is an Abelian totally antisymmetric gauge field ¹²⁾ transforming under colour gauge transformations

$$\delta A_\mu^a(x) = \partial_\mu \Lambda^a(x) + ig C_{abc} \Lambda^b A_\mu^c$$

as :

$$\begin{aligned}
 A_{\nu\rho\sigma} &\rightarrow A_{\nu\rho\sigma} + \partial_\nu \Lambda_{\rho\sigma} - \partial_\rho \Lambda_{\nu\sigma} - \partial_\sigma \Lambda_{\rho\nu} \\
 \Lambda_{\rho\sigma} &\propto A_\rho^a \partial_\sigma \Lambda^a - A_\sigma^a \partial_\rho \Lambda^a
 \end{aligned} \tag{2.8}$$

so that $F_{\mu\nu\rho\sigma}$ is gauge invariant.

Since the effective Lagrangian should not contain colour degrees of freedom (confinement assumption) it can contain fields such as A_μ or $F_{\mu\nu}$ only through $F_{\mu\nu\rho\sigma}$, the curl of $A_{\mu\nu\rho}$. Indeed the correct transformation under a $U(1)$ axial rotation is immediately given by a term

$$\int \mathcal{L}_{\text{anomaly}} = \frac{1}{4} \epsilon_{\mu\nu\rho\sigma} F^{\mu\nu\rho\sigma} \frac{i}{2} \text{Tr} [\log U - \log U^\dagger] \tag{2.9}$$

which, furthermore, preserves $SU(L) \otimes SU(L)$ ^{*)}. At this point, it is natural to add, in analogy with QED or QCD, a kinetic term for the field $A_{\nu\rho\sigma}$ of the usual curl form :

*) We wish to thank R.J. Crewther for repeatedly emphasizing to us his criticism of previously proposed effective Lagrangians which did not satisfy the transformation property (2.6) [see also : R.J. Crewther, CERN preprint TH. 2791 (1979)].

$$\int \mathcal{L}_{kin.} = -c F_{\mu\nu\rho\sigma} F^{\mu\nu\rho\sigma} \quad (2.10)$$

where c is a positive constant if we want positivity of the Hamiltonian in terms of the canonical field $A_{\mu\nu\rho}$.

Noticing that terms containing $A_{\mu\nu\rho}$ are not gauge invariant and that higher derivatives of $A_{\mu\nu\rho}$ and $F_{\mu\nu\rho\sigma}$ are irrelevant for the low energy regime in which we are interested, one arrives at the general structure :

$$\begin{aligned} \mathcal{L}(U, A_{\mu\nu\rho}) = & \mathcal{L}_0(U, U^\dagger) + \frac{i}{8} \epsilon_{\mu\nu\rho\sigma} F^{\mu\nu\rho\sigma} \text{Tr} [\log U - \log U^\dagger] \\ & - c F_{\mu\nu\rho\sigma} F^{\mu\nu\rho\sigma} + \sum_{k=1}^{\infty} L_k(U, U^\dagger) (\epsilon_{\mu\nu\rho\sigma} F^{\mu\nu\rho\sigma})^k \end{aligned} \quad (2.11)$$

We now impose on $\mathcal{L}$ the following invariance and simplicity requirements :

- i) $\mathcal{L}$ should be invariant under $SU(L) \otimes SU(L)$ and should transform as in (2.6) under $U_A(1)$;
- ii) except for a term proportional to the QCD vacuum angle θ , $\mathcal{L}$ should be invariant under parity ($U \rightarrow U^\dagger$) ;
- iii) $\mathcal{L}$ does not contain terms with more than two derivatives of the fields U and $A_{\mu\nu\rho}$ (low energy approximation).

The above requirements imply :

$$L_{2k+1}(U, U^\dagger) = 0, \quad k = 1, 2, \dots ; \quad (2.12)$$

$L_{2k}(U, U^\dagger)$ invariant under parity and under $U(L) \otimes U(L)$. Up to here we have made no use of the large N limit. If we now demand that the effective Lagrangian (2.11) follows from QCD in the large N limit, we find the following N dependences :

$$\begin{aligned}
 F_{\pi}^2 &\sim O(N) \\
 \langle L_0 \rangle_0 &\sim O(N) \\
 \langle L_2 \rangle_0 &\sim O(1) \\
 \langle L_{2K} \rangle_0 &\sim O(N^{2-2K})
 \end{aligned}
 \tag{2.13}$$

where a trivial vacuum energy constant of $O(N^2)$ has been subtracted. Equations (2.13) imply that terms with $K > 2$ in Eq. (2.11) can be neglected for large N . The only possible term with $K=1$ is the θ dependent piece

$$\int \mathcal{L}_{\theta} \equiv -\theta \frac{1}{4} \epsilon_{\mu\nu\rho\sigma} F^{\mu\nu\rho\sigma}
 \tag{2.14}$$

Finally, terms with $K=2$ and not of the form (2.10) can also be proved to be of higher order in $1/N$ and are therefore dropped ^{*)}.

Because of the invariance of Eq. (2.11) under the gauge transformations (2.8), quantization would require a gauge fixing term of the type ¹³⁾:

$$\begin{aligned}
 \beta (\partial_{\mu} A_{\mu\nu\rho})^2 &\propto (\partial_{\mu} K_{\nu} - \partial_{\nu} K_{\mu})^2, \\
 K_{\mu} &\equiv \epsilon_{\mu\nu\rho\sigma} A_{\nu\rho\sigma}
 \end{aligned}
 \tag{2.15}$$

However, because of the fact that only $F_{\mu\nu\rho\sigma}$, and not $A_{\nu\rho\sigma}$, appears in the Lagrangian (2.11) and that $F_{\mu\nu\rho\sigma}$ has no physical degrees of freedom, such a field can be eliminated by using its classical equations of motion (see below) and the introduction of a gauge fixing term is not really essential.

In conclusion, our effective Lagrangian in the large N limit can be written in terms of the $q(x)$ field of Eq. (2.7) simply as :

*) Although these terms are irrelevant for the questions discussed in this paper, they come into play when the leading terms in $1/N$ vanish (typically because of Adler zeros) as, for instance, in soft multi- η' amplitudes ⁵⁾.

$$\begin{aligned} \mathcal{L}(U, U^\dagger, \varrho) = & \mathcal{L}_0(U, U^\dagger) + \frac{i}{2} \varrho(x) \text{Tr} [\not{\partial}_\mu U - \not{\partial}_\mu U^\dagger] + \\ & + \frac{N}{a F_\pi^2} \varrho^2(x) - \Theta \varrho(x) + \frac{F_\pi}{2\sqrt{2}} \text{Tr} (M U + M^\dagger U^\dagger) \end{aligned} \quad (2.16)$$

with $\mathcal{L}_0(U)$ given by Eq. (2.1). Let us now comment on Eq. (2.13). First we have added a mass term that breaks explicitly chiral invariance. The mass matrix M can be taken, without loss of generality, to be of the form :

$$M_{ij} = \mu_i^2 \delta_{ij} \quad (2.17)$$

The term $-\Theta \varrho$ allows us to work in a general Θ vacuum. Only the combination

$$\bar{\Theta} \equiv \Theta + A g \det M \quad (2.18)$$

will influence physical quantities and, by our choice of M , $\Theta = \bar{\Theta}$.

Finally, we comment on the term proportional to q^2 whose positive coefficient [related to c in Eq. (2.11)] has been redefined for later convenience as $N/a F_\pi^2$. This sign of the q^2 term would be the wrong one ^{*)} for the mass term of a scalar field $q(x)$ giving a negative unbounded Hamiltonian. If we recall, however, that the basic canonical field to be used in constructing the Hamiltonian is $A_{\mu\nu\rho}$, it is easy to see that the resulting Hamiltonian contains $q^2(x)$ with a positive coefficient. It will become clear later in this section why our sign is the only one capable of solving the $U(1)$ problem.

An immediate consequence of the $q^2(x)$ term is to give :

$$\int d^4x \langle \varrho(x) \varrho(0) \rangle_{\Theta=0} = \frac{i F_\pi^2}{2N} a \quad (2.19)$$

in pure Yang-Mills (YM) theory (i.e., in the absence of quarks).

^{*)} We are grateful to E. Witten for raising this question and for illuminating remarks.

Using Eqs. (2.13) one finds that a , if not zero, is of $O(1)$ in $1/N$, hence proportional to Λ^2 ($\Lambda \sim 500$ MeV is the renormalization group invariant scale of QCD).

Of course, $a=0$ to any order of perturbation theory due to the fact that $q(x)$ is a total divergence [Eq. (2.7)].

The basic point of the recent work ⁵⁾⁻⁷⁾ on the $U(1)$ problem at large N has been to assume ⁵⁾ that a is non-zero because of possible non-trivial topologies contributing to quantum fluctuations. We shall now see that this single assumption gives us a full and explicit solution of the $U(1)$ problem

As we have already stressed, the linear σ type model contains redundant fields. One can eliminate the scalar fields by taking the limit $\lambda \rightarrow \infty$ in Eq. (2.1). In this limit the potential term becomes a constraint

$$U^\dagger U = \frac{F_\pi^2}{2} \mathbb{1} \quad (2.20)$$

and the model becomes of the non-linear type with, for instance :

$$U = \frac{F_\pi}{\sqrt{2}} e^{\frac{i\sqrt{2}}{F_\pi} (\pi_a \tau_a + S/\sqrt{L})} \quad (2.21)$$

where τ_a are the generators of $SU(L)$ and S is the $SU(L)$ singlet field. The Lagrangian obtained by inserting (2.21) in (2.16) is a generalized non-linear σ model. It has been proposed independently in a recent paper by Rosenzweig, Schechter and Taheri ⁹⁾. As already stressed, another redundant field appearing in Eq. (2.16) is $q(x)$. Since there is no kinetic term associated with it, $q(x)$ can be eliminated using its equations of motion. This is particularly simple (see Section 2C) if we neglect terms of higher order in $1/N$ because then $\mathcal{L}$ is just quadratic in $q(x)$. Finally, we remark that a Lagrangian [similar in structure to Eq. (2.16)] has been obtained ¹⁴⁾ in CP^{n-1} models, where the large n expansion can be explicitly performed.

B. - Effective Lagrangian for arbitrary θ : two-point functions

Starting from the large N effective Lagrangian that we have derived in the previous subsection, we want to study the θ dependence of Green's functions in the limit of light quark masses.

After the elimination of the field $q(x)$ by means of its classical equation of motion, one can rewrite the Lagrangian (2.16) as follows :

$$\mathcal{L} = \mathcal{L}_0 + \frac{F_\pi}{2\sqrt{2}} \text{Tr} [MU + M^\dagger U^\dagger] - \frac{a F_\pi^2}{4N} \left[\theta - \frac{i}{2} \text{Tr} (\log U - \log U^\dagger) \right]^2 \quad (2.22)$$

In order to make chiral perturbation theory around the "right" vacuum state, according to Dashen's theorem¹⁵⁾, one must find the field configuration $\langle U \rangle$ that minimizes the energy corresponding to Lagrangian (2.22). Since the potential contained in $\mathcal{L}_0(U)$ implies that $\langle UU^\dagger \rangle = (F_\pi^2/2) \mathbf{1}$, the most general form for $\langle U \rangle$ is given by

$$\langle U_{ij} \rangle = \delta_{ij} \frac{F_\pi}{\sqrt{2}} e^{-i\phi_i} \quad (2.23)$$

Inserting this expression in the energy density corresponding to the Lagrangian (2.22), one arrives at the following expression :

$$E = \frac{a F_\pi^2}{4N} \left(\theta - \sum_i \phi_i \right)^2 - \frac{F_\pi^2}{2} \sum_{i=1}^L \mu_i^2 \cos \phi_i \quad (2.24)$$

The angles ϕ_i are determined by minimizing (2.24) and therefore satisfy the equations

$$\mu_i^2 \sin \phi_i = \frac{a}{N} \left(\theta - \sum_{j=1}^L \phi_j \right) ; \quad i = 1, 2, \dots, L \quad (2.25)$$

where, if more than one solution is found, it is understood that the one with minimal energy has to be selected. Having determined the angles ϕ_i , it is convenient to rewrite the Lagrangian (2.22) in terms of a physical field $V(x)$, which has a vacuum expectation value proportional to the unit matrix. $V(x)$ is defined by a $U(L) \otimes U(L)$ transform of U [Eq. (2.4) :

$$V = A U B^\dagger ; \quad A_{\alpha\beta} = B_{\alpha\beta}^\dagger = \delta_{\alpha\beta} e^{i\phi_\alpha/2} \quad (2.26)$$

In terms of the matrix $V_{ij}(x)$ the Lagrangian (2.22) becomes

$$\begin{aligned} \mathcal{L} = & \mathcal{L}_0(V) + \frac{a F_\pi^2}{16N} \left[\text{Tr}(\log V - \log V^\dagger) \right]^2 + \\ & + \frac{F_\pi}{2\sqrt{2}} \text{Tr} \left[M(\theta) (V + V^\dagger - \sqrt{2} F_\pi) \right] + \\ & + \frac{ia}{2\sqrt{2}N} F_\pi (\theta - \sum \phi_i) \left[-\text{Tr}(V - V^\dagger) + \frac{F_\pi}{\sqrt{2}} \text{Tr}(\log V - \log V^\dagger) \right] \end{aligned} \quad (2.27)$$

where an inessential constant term has been omitted and

$$M_{ij}(\theta) = \mu_i^2(\theta) \delta_{ij} ; \mu_i^2(\theta) = \mu_i^2 \cos \phi_i \quad (2.28)$$

Since we are interested only in results that follow from "current algebra" we can make the transition to the non-linear σ model where one has :

$$V = \frac{F_\pi}{\sqrt{2}} e^{i \frac{\sqrt{2}}{F_\pi} \Phi} ; \Phi = \pi^i \tau^i + S/\sqrt{2} \quad (2.29)$$

and the Lagrangian (2.27) becomes :

$$\begin{aligned} \mathcal{L} = & \frac{1}{2} \text{Tr} (\partial_\mu V^\dagger \partial_\mu V) - \frac{a}{2N} (\text{Tr} \Phi)^2 + \\ & + \frac{F_\pi^2}{2} \text{Tr} \left[M(\theta) \left(\cos \frac{\sqrt{2}}{F_\pi} \Phi - 1 \right) \right] + \\ & + \frac{F_\pi}{\sqrt{2}N} a (\theta - \sum \phi_i) \left[\frac{F_\pi}{\sqrt{2}} \sin \left(\frac{\sqrt{2}}{F_\pi} \Phi \right) - \Phi \right] \end{aligned} \quad (2.30)$$

In conclusion the effective Lagrangian for an arbitrary value $\theta \neq 0$ has the same form as the original Lagrangian except that the mass matrix is a function of θ [see Eq. (2.28)] and in addition one gets a CP violating term containing cubic and higher order terms in the field Φ . The mass spectrum of the pseudoscalar mesons can be obtained from the quadratic part of (2.30), which is given by :

$$\mathcal{L}_2 = \frac{1}{2} \text{Tr} (\partial_\mu \Phi \partial_\mu \Phi) - \frac{a}{2N} (\text{Tr} \Phi)^2 - \frac{1}{2} \text{Tr} [M \Phi^2] \quad (2.31)$$

By decomposing

$$\Phi_{ij} = \tilde{\pi}^{\alpha\beta} \tilde{\tau}_{ij}^{\alpha\beta} + \tilde{\pi}_i \delta_{ij} \quad (2.32)$$

where the matrices $\tilde{\tau}_{ij}^{\alpha\beta}$ are the $L(L-1)$ non-diagonal generators of $U(L)$, one gets the following two-point functions :

$$\langle \tilde{\pi}^{\alpha\beta}(x) \tilde{\pi}^{\gamma\delta*}(0) \rangle_{\theta}^{\text{F.T.}} = \frac{i \int^{\alpha\gamma} \int^{\beta\delta}}{p^2 - M_{\alpha\beta}^2(\theta)} ; M_{\alpha\beta}^2(\theta) = \frac{1}{2} (\mu_{\alpha}^2(\theta) + \mu_{\beta}^2(\theta)) \quad (2.33)$$

$$\langle \tilde{\pi}_i(x) \tilde{\pi}_j(0) \rangle_{\theta}^{\text{F.T.}} = i A_{ij}^{-1}(p^2) \quad (2.34)$$

where F.T. stands for the Fourier transform and the matrix A_{ij}^{-1} is the inverse of the mass matrix :

$$A(p^2) = \text{diag.}(p^2 - \mu_i^2(\theta)) - \frac{a}{N} \begin{pmatrix} 1 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \end{pmatrix} \quad (2.35)$$

The masses of the physical states $M_i^2(\theta)$ are obtained by diagonalizing the mass matrix A_{ij} and they are given by the obvious identities :

$$\det A = \prod_{i=1}^L (p^2 - M_i^2(\theta)) = \prod_{i=1}^L (p^2 - \mu_i^2(\theta)) \left[1 - \frac{a}{N} \sum_{i=1}^L (p^2 - \mu_i^2(\theta))^{-1} \right] \quad (2.36)$$

that show clearly that the singlet is not massless in the chiral limit if $a \neq 0$. In this limit, the squared mass of the singlet is equal to L/Na (hence a must be positive !) and its behaviour with L and N is governed by the same factor L/N as the anomaly. This shows that the solution of the $U(1)$ problem is intimately related to the existence of the axial vector anomaly.

The Lagrangian (2.30) contains also information about the coupling of the pseudoscalar mesons and in particular also a CP violating term which is non-zero if $a \neq 0$ (see Section 3). It is easy to check that this CP violating term contains for instance the amplitude $\eta \rightarrow 2\pi$ which was computed using current algebra techniques ¹⁶⁾.

Another possible application of the Lagrangian (2.30) is in processes involving the η' interacting with the other pseudoscalar mesons as, for instance, the decay $\eta' \rightarrow \eta\pi\pi$. Applications of this type will be discussed elsewhere ¹⁷⁾.

We remark next an important feature of the Lagrangian (2.30), i.e., the fact that it does not contain terms linear in any of the pseudoscalar fields including the $U(1)$ particle. This ensures that the CP violating term of $\mathcal{L}$ does not create any of the light pseudoscalar on the vacuum and that, consequently, straightforward chiral perturbation theory can be carried out without having a vacuum instability.

This is in agreement with the general arguments of Dashen ¹⁵⁾, Nuyts and Bèg ¹⁸⁾.

The results obtained here with the effective Lagrangian method have been confirmed by a more direct (if less elegant) diagrammatic S matrix approach ⁶⁾, which is briefly sketched in the Appendix.

We should finally like to mention that an elegant rederivation of the non-linear Lagrangian (2.30), which does not make use of redundant fields [such as $q(x)$], has been given by Witten ¹⁹⁾. He has also discussed the consequences of such a Lagrangian both in the limit $(\mu^2 N/a) \lesssim 1$ considered in this paper and in the opposite limit $(\mu^2 N/a) \gg 1$ ($N \rightarrow \infty$ at fixed μ^2).

3. - THREE LIGHT FLAVOURS AT ANY θ : SOME PHYSICAL RESULTS

It is self-evident from the discussion of Section 2 that the $U(1)$ problem (in the sense of the existence of $L-1$ rather than L flavourless NG bosons) is solved at all values of θ if $a \neq 0$. This is obvious from Eq. (2.36) which implies that, as $\mu_1 \rightarrow 0$, $(L-1)$ flavourless mesons become massless whereas the L^{th} one has the mass ^{5),6)}

$$M_s^2 = \frac{La}{N} \neq 0 \quad (3.1)$$

One can also see that, in such a limit, the massive particle is an $SU(L)$ singlet. The fact that the pseudoscalar mixing angle away from pure $SU(3)$ singlet-octet is experimentally small, indicates that the physical world is close to having three light quarks (u , d and s) with :

$$\mu_u^2, \mu_d^2, \mu_s^2 < a/N \quad (3.2)$$

The further experimental fact that isospin is a very good symmetry of strong interactions and that $SU(3)$ is not as good implies that, furthermore :

$$\mu_u^2, \mu_d^2 \ll \mu_s^2 < a/N \quad (3.3)$$

It was argued in Ref. 6) that the experimental spectrum of pseudoscalars and their mixing angle can be explained under assumption (3.3). We wish now to discuss the consequences of a possible non-zero value of θ and in particular we want to argue that only the choice $\theta \ll 1 \pmod{2\pi}$ is consistent with the data.

For this purpose, consider the system of Eqs. (2.25) for $L=3$:

$$\mu_u^2 \sin \phi_u = \mu_d^2 \sin \phi_d = \mu_s^2 \sin \phi_s = \frac{a}{N} \left(\theta - \sum_{j=1}^3 \phi_j \right) \quad (3.4)$$

whose solution of minimal energy [as given in Eq. (2.24)] has to be selected.

Equation (2.25) [or (3.4)] immediately implies periodicity of physical quantities in θ with period 2π . Indeed, if one has a solution of the minimization problem for a certain θ ,

$$\phi_i = \phi_i^{(0)}(\theta, \mu_i, a) \quad (3.5)$$

then, for $\theta \rightarrow \theta + 2\pi$, the minimizing solution can be taken to be, e.g. :

$$\phi_u(\theta + 2\pi) = \phi_u(\theta) + 2\pi ; \quad \phi_i(\theta + 2\pi) = \phi_i(\theta) \quad (i \neq u) \quad (3.6)$$

with no effect on the physics which only depends on $e^{\pm i\phi_i}$. This does not imply, however, a trivial periodicity of each solution of Eq. (3.4) : it means, in general, that through some value of θ (typically $\theta = \pm\pi$) one

has to change from one solution to a different one. A trivial example of this is the case of two flavours with $\mu_u^2 = \mu_d^2 \ll a/N$ where one finds immediately :

$$\cos \phi_u = \cos \phi_d = +\sqrt{\frac{1+\cos \Theta}{2}} = |\cos \Theta/2| \quad (3.7)$$

with periodicity 2π (and not 4π !).

Consider now the question of CP violation. As a consequence of Eqs. (2.27) the condition for exact CP conservation is that the energy minimizing solution satisfies :

$$\left(\Theta - \sum_{j=1}^L \phi_j \right) = 0 \quad (3.8)$$

This can be realized in the following cases :

- i) $\Theta = 0 \pmod{2\pi}$. In this case all $\phi_j = 0 \pmod{2\pi}$ and the energy is minimized ;
- ii) one quark (say u) is massless. In this case the system (3.4) is satisfied with $\phi_u = \Theta$, $\phi_i = 0$ ($i \neq u$) and this can be proved to be the solution of minimal energy ;
- iii) if $\Theta \neq 0$ and $\mu_i \neq 0$ (all i), the only possibility of CP conservation is for

$$\begin{aligned} \sin \phi_i &= 0 ; \quad \phi_i = 0, \pm\pi \text{ (all } i) \\ \Theta &= \sum \phi_i = 0, \pm\pi \pmod{2\pi} \end{aligned} \quad (3.9)$$

Hence the only CP conserving case left is $\Theta = \pi \pmod{2\pi}$. Let us discuss this case in more detail in order to see if $|\Theta - \pi| \ll 1$ is experimentally ruled out or not.

Let us consider first the case of two flavours (i.e., $m_s \gg \Lambda$) and let us also neglect effects of order $\epsilon \equiv \mu_u \mu_d N/a$. Then one finds :

$$\begin{aligned} \phi_u &= \bar{\phi}_u \equiv \arctg \frac{m_d \sin \Theta}{m_u + m_d \cos \Theta} , \\ \phi_d &= \bar{\phi}_d \equiv \arctg \frac{m_u \sin \Theta}{m_d + m_u \cos \Theta} ; \quad \bar{\phi}_u + \bar{\phi}_d = \Theta \end{aligned} \quad (3.10)$$

This is about the only case which can be solved in closed form at all θ . Although simplified, it already shows some of the general features we have mentioned above.

For instance, in Fig. 1, we show the behaviour of $\bar{\phi}_u, \bar{\phi}_d$ for $0 \leq \theta \leq 2\pi$. Notice that, for m_u/m_d not too close to one (say $\sim \frac{1}{2}$), $\bar{\phi}_u$ increases smoothly from 0 to 2π whereas $\bar{\phi}_d$ reaches a maximum (at $\cos \theta = -m_u/m_d$), goes back to zero at $\theta = \pi$, and describes the symmetric curve (with $\bar{\phi}_d < 0$) between π and 2π . As m_u/m_d approaches one, $\bar{\phi}_u$ and $\bar{\phi}_d$ stay together for longer and longer then, very near to $\theta = \pi$, they depart abruptly from each other reaching $+\frac{3}{2}\pi$ and $-\frac{1}{2}\pi$, respectively, soon after $\theta = \pi$. In the opposite limit $m_u/m_d \rightarrow 0$ the curve for $\bar{\phi}_u$ becomes $\bar{\phi}_u = \theta$ whereas the one for $\bar{\phi}_d$ is squashed on the real axis. A similar behaviour is found to occur in the energy $E(\theta)$ computed at the minimizing point $\bar{\phi}_u, \bar{\phi}_d$. This turns out to be

$$E(\theta) = -\mu_u \mu_d R \equiv \bar{E}(\theta); R = + \sqrt{\frac{m_u^2 + m_d^2 + 2m_u m_d \cos \theta}{m_u m_d}} \quad (3.11)$$

Again, for $m_u \neq m_d$ $E(\theta)$ is analytic (see Fig. 2a). The solution with the opposite sign of R gives the dotted curve which never crosses the curve of real minimal energy. However, at $m_u = m_d$, the two solutions intersect and a non-analytic behaviour of $E(\theta)$ results at $\theta = \pm\pi$ (Fig. 2b).

It is interesting to compute now (still for $m_s \gg \Lambda$, i.e., $L=2$) the first order corrections to Eqs. (3.6) in the parameter

$$\epsilon \equiv \mu_u \mu_d N/a \ll 1 \quad (3.12)$$

One finds easily :

$$\begin{aligned} \phi_u &= \bar{\phi}_u - \epsilon \sin \theta \left(\frac{m_d + m_u \cos \theta}{m_u} \right) R^{-3}; \text{ same with } u \leftrightarrow d; \\ (\theta - \phi_u - \phi_d) &= \epsilon \sin \theta R^{-1}; \\ E(\theta) &= \bar{E}(\theta) \left(1 + \frac{1}{2} \epsilon \sin^2 \theta R^{-3} \right) \end{aligned} \quad (3.13)$$

where the left-hand side of the second equation is the strength of CP violation and R was defined in Eq. (3.11). We see that, as long as $m_u \neq m_d$, $(\theta - \phi_u - \phi_d)$ vanishes at $\theta = \pi$. If, however, $m_u = m_d$ and we take $\theta \sim \pi$ we find ($m_u = m_d = m, \mu_u = \mu_d = \mu$)

$$(\theta - \phi_u - \phi_d) = \frac{\mu^2 N}{a} \frac{\sin \theta}{(2+2\cos\theta)^{1/2}} \sim \frac{\mu^2 N}{a} \quad (3.14)$$

Hence, for $\theta \rightarrow \pi$ we still have a CP violating piece in the Lagrangian of the type $i m_{u,d} \sum_i \bar{\psi}_i \gamma_5 \psi_i$, which is the typical amount of CP violation for $\theta \neq 0, \pm\pi$).

We have thus rediscovered, in our context, Dashen's mechanism for spontaneous CP violation, if $m_u = m_d$.

A more realistic situation is the one with three flavours, i.e., with

$$\begin{aligned} m_u, m_d &\ll m_s \ll \Lambda/N \\ (\mu_u^2, \mu_d^2 &\ll \mu_s^2 \ll a/N) \end{aligned} \quad (3.15)$$

In this case ϕ_s cannot be set to zero. Defining now :

$$\begin{aligned} \alpha &= \pi - \phi_u - \phi_d \\ \beta &= \pi - \theta \end{aligned} \quad (3.16)$$

and using the fact that ϕ_s, α, β are all $\ll 1$, one arrives at the equations

$$(\theta - \phi_u - \phi_d - \phi_s) = (\alpha - \beta) \mu_s^2 N / a$$

$$\frac{\mu_u \mu_d \sin \alpha}{\sqrt{\mu_u^4 + \mu_d^4 - 2\mu_u^2 \mu_d^2 \cos \alpha}} = \frac{a/N \cdot \mu_s^2}{(a/N + \mu_s^2) \mu_u \mu_d} (\alpha - \beta) \simeq \frac{\mu_s^2}{\mu_u \mu_d} (\alpha - \beta) \quad (3.17)$$

If $\beta = 0$ ($\theta = \pi$), we can see by solving Eq. (3.17) graphically in α (Fig. 3) that one has one or three small α solutions according to what one takes for quark mass ratios. If we take

*) A similar result follows at $\pi - \theta = \beta \ll 1$ if $(m_u - m_d)^2 < \beta^2 m_u m_d$.

$$|m_d - m_u| m_s > m_u m_d$$

$$(|\mu_d^2 - \mu_u^2| \cdot \mu_s^2 > \mu_u^2 \mu_d^2) \quad (3.18)$$

then $\alpha = 0$ is the only solution and we have CP conservation at $\Theta = \pi$.
If, however,

$$|m_d - m_u| m_s < m_u m_d$$

$$(|\mu_d^2 - \mu_u^2| \cdot \mu_s^2 < \mu_u^2 \mu_d^2) \quad (3.19)$$

there are two more solutions (which can be shown to give lower energy) given by

$$\alpha^2 = \alpha_0^2 \approx \frac{\mu_u^2 \mu_d^2}{\mu_s^4} \left(1 - \frac{(\mu_u^2 - \mu_d^2)^2 \mu_s^4}{\mu_u^4 \mu_d^4} \right) \approx 0 \left(\frac{\mu_u^2 \mu_d^2}{\mu_s^4} \right) = 0 \left(\frac{m_u m_d}{m_s^2} \right) \quad (3.20)$$

In this case, the resulting CP violation is of the type :

$$\delta \mathcal{L}_{CP} \sim i m_{u,d} \sum_i \bar{\psi}_i \gamma_5 \psi_i \quad (3.21)$$

i.e., "large". Clearly we recover the two-flavour case by letting $m_s \rightarrow \infty$ in which case one is always in case (3.18) except at exactly $m_u = m_d$.

According to present belief, physics belongs to case (3.18) so that Dashen's phenomenon is not expected to occur : thus $\Theta = \pi$ cannot be ruled out on the basis that CP is experimentally valid to a high degree of accuracy. Another argument ¹⁶⁾, based on the experimental pseudoscalar mass ratios, can be produced, however, to rule out $\Theta \sim \pi$.

Finally, if $\beta = \pi - \Theta \neq 0$ but less than $O(m_{u,d}/m_s)$, one can see that Dashen's phenomenon can still occur if (3.19) is valid (see Fig. 3) in which case CP violation is not proportional to β . Instead it is as large as for an arbitrary $\Theta \neq 0, \pi$.

We want finally to make some remarks on isospin and SU(3) conservation at all Θ .

It can be explicitly checked that, at all θ , after a chiral rotation of each quark field ψ_i by the angles θ_i determined from Eq. (2.25), one can redefine conserved isospin [or $SU(3)$] currents

$$J_\mu^i = \bar{\psi}' \lambda^i \not{\partial}_\mu \psi' \quad (3.22)$$

whose generators annihilate the vacuum. One then finds that breaking effects coming from $m_u \neq m_d \neq m_s$ are not of $O(m_i - m_j / m_i + m_j)$ as can be naively expected, but only $^{*)}$ of order $O(m_i - m_j) / a/N$ if $\Lambda m_i \ll a/N$.

Only for $N \rightarrow \infty$, at fixed $m_i - m_j$, would one find large isospin violations. Indeed, in such a case, one recovers the planar approximation for which the quark and not the isospin $[SU(3)]$ basis is the good one to diagonalize the mass matrix. One would then find a π^+ and a π^- with

$$m_{\pi^\pm}^2 \sim \frac{1}{2} (m_u + m_d) \Lambda \quad (3.23)$$

and two $\pi^{0s}, \pi_u^0 \sim \bar{u}u$ and $\pi^0 \sim \bar{d}d$ with

$$m_{\pi_{u,d}^0}^2 \sim m_{u,d} \Lambda \quad (3.24)$$

A π^0 of mass very close to that of $\pi^\pm$ would be just missing from the Rosenfeld tables for $a/N \ll \Lambda(m_d - m_u)$!

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*) Such effects are probably the main contribution to the decay $\eta \rightarrow 3\pi$. See Ref. 17) for a new discussion of this and other η, η' processes.

APPENDIX - DIAGRAMMATIC S MATRIX DERIVATION OF THE RESULTS
OF SECTION 2

In this Appendix we rederive the results obtained in Section 2 with the effective Lagrangian approach, by saturating directly the relevant n point functions of a given topology with lowest-lying particle poles. This approximation can be justified in the large N (small L/N) chiral limit, since other intermediate states, being of mass $O(\Lambda)$, give contributions which are down by powers of m_q/Λ and/or of L/N .

We start with the planar approximation at $\theta=0$. This has L^2 light pseudoscalar mesons π_{ij} ($i, j=1, 2, \dots, L$) of definite quark content (exact OZI rule) $q_i \bar{q}_j$ and of mass

$$\mu_{ij}^2 = -F_\pi^{-2} (m_i + m_j) \langle \bar{\Psi} \Psi \rangle_{\theta=0} \quad (A.1)$$

where ψ_i is the i^{th} quark field (of mass m_i) and $\langle \bar{\Psi} \Psi \rangle_{\theta=0} = \langle \bar{\psi}_i \psi_i \rangle_{\theta=0}$ independent of i up to corrections of $O(m_i/\Lambda)$.

Let us now concentrate our attention on the L "diagonal" (flavourless) states with $i=j$. Their θ dependent masses are given in the planar approximation by

$$\mu_i^2(\theta) \equiv \mu_{ii}^2(\theta) = F_\pi^{-2} \left\{ -2m_i \langle \bar{\Psi}_i \Psi_i \rangle_\theta \right\} \quad (A.2)$$

The parameters μ_i^2 are to be identified with those introduced in Section 2. Let us also define, for simplicity of notation :

$$u_i = \bar{\Psi}_i \Psi_i \\ v_i = i \bar{\Psi}_i \gamma_5 \Psi_i \quad (A.3)$$

In order to study the θ dependence of $\langle \bar{\psi}_i \psi_i \rangle_\theta$, $\langle \bar{\psi}_i \gamma_5 \psi_i \rangle_\theta$, we now consider the two-point functions $\langle q(x) q(0) \rangle_\theta$, $\langle q(x) u_i(0) \rangle_\theta$, $\langle q(x) v_i(0) \rangle_\theta$. As in Section 2 we assume that the Fourier transform (FT) of the first one is non-vanishing at $p=0$ in pure Yang-Mills theory, i.e.

$$\left. \int d^4x T \langle q(x) q(0) \rangle \right|_{Y.M.} = \frac{ia F_\pi^2}{2N} ; \quad a = O(\Lambda^2) \quad (A.4)$$

Using then the techniques of Ref. 6), one finds in the absence of (non-minimal) quark loops

$$\int d^4x e^{-ipx} T \langle q(x) \bar{\psi}_i(0) \rangle_\theta \Big|_{1\text{-}q\text{-loop}} = \frac{ia}{N} (p^2 - \mu_i^2(\theta))^{-1} \langle u_i \rangle_\theta \quad (\text{A.5})$$

$$\int d^4x e^{-ipx} T \langle q(x) u_i(0) \rangle_\theta \Big|_{1\text{-}q\text{-loop}} = -\frac{ia}{N} (p^2 - \mu_i^2(\theta))^{-1} \langle \bar{\psi}_i \rangle_\theta \quad (\text{A.6})$$

Adding quark loops amounts ⁶⁾ to multiply the right-hand side of Eqs. (A.4) to (A.6) by the geometric series :

$$1 + \frac{a}{N} \sum_{i=1}^L (p^2 - \mu_i^2(\theta))^{-1} + \dots = \left(1 - \frac{a}{N} \sum_{i=1}^L (p^2 - \mu_i^2(\theta))^{-1} \right)^{-1} \quad (\text{A.7})$$

The final outcome is then :

$$\int d^4x e^{-ipx} T \langle q(x) q(0) \rangle_\theta = \frac{ia F_\pi^2}{2N} \frac{1}{1 - \frac{a}{N} \sum_j (p^2 - \mu_j^2(\theta))^{-1}} \quad (\text{A.4'})$$

$$\int d^4x e^{-ipx} T \langle q(x) \bar{\psi}_i(0) \rangle_\theta = \frac{ia}{N} \frac{(p^2 - \mu_i^2(\theta))^{-1}}{1 - \frac{a}{N} \sum_j (p^2 - \mu_j^2(\theta))^{-1}} \langle u_i \rangle_\theta \quad (\text{A.5'})$$

$$\int d^4x e^{-ipx} T \langle q(x) u_i(0) \rangle_\theta = -\frac{ia}{N} \frac{(p^2 - \mu_i^2(\theta))^{-1}}{1 - \frac{a}{N} \sum_j (p^2 - \mu_j^2(\theta))^{-1}} \langle \bar{\psi}_i \rangle_\theta \quad (\text{A.6'})$$

The poles appearing on the right-hand side of Eqs. (A.4') to (A.6') correspond to the physical particle masses $M_1^2(\theta)$. Clearly, they coincide with those found in Section 2, Eqs. (2.35), (2.36).

Going now to $p=0$ and using the identification

$$\frac{\partial}{\partial \theta} \langle \dots \rangle = \int d^4x T \langle q(x) \dots \rangle$$

one sees immediately that Eqs. (A.5') and (A.6') imply :

$$\frac{\partial}{\partial \theta} \left[\langle u_i \rangle_\theta^2 + \langle \bar{\psi}_i \rangle_\theta^2 \right] = 0 \quad (\text{A.8})$$

so that one can set, in analogy with Eqs. (2.23) :

$$\langle u_i \rangle_\theta = \langle u_i \rangle_{\theta=0} \cos \phi_i \quad ; \quad \langle v_i \rangle_\theta = - \langle u_i \rangle_{\theta=0} \sin \phi_i. \quad (\text{A.9})$$

with

$$\frac{\partial \phi_i}{\partial \theta} = - \frac{a}{N} \frac{(\rho^2 - \mu_i^2(\theta))^{-1}}{1 - \frac{a}{N} \sum_j (\rho^2 - \mu_j^2(\theta))^{-1}} \bigg|_{\rho=0} = \frac{a}{N} \frac{\prod_{j \neq i} \mu_j^2(\theta)}{\prod_j \mu_j^2(\theta)}$$

$$\frac{\partial}{\partial \theta} (\theta - \sum_i \phi_i) = \left(1 + \frac{a}{N} \sum_j \mu_j^2(\theta) \right)^{-1} \quad (\text{A.10})$$

Using now Eqs. (A.5') and (A.9), it is straightforward to see that the energy minimizing conditions (2.25) follow.

The physical interpretation of the above result is that, once $\theta \neq 0$, a set of chiral rotations of angles $\phi_i/2$ has to be performed so that, in terms of the rotated fields ($\psi'_i = e^{i(\gamma_5/2)\phi_i}\psi_i$) :

$$\langle u'_i \rangle_\theta = \langle u_i \rangle_{\theta=0} \quad ; \quad \langle v'_i \rangle_\theta = 0 \quad (\text{A.11})$$

It is only in terms of these rotated fields that the naive vector currents $(\bar{\psi}' \gamma_\mu \lambda \psi')$ define vacuum annihilating charges for the symmetry group $SU(L)_{\text{vector}}$. It is then easy to see that such a symmetry is broken only at $O((m_1 - m_j)N/\Lambda)$.

We close this Appendix by rederiving directly in QCD language the result of Section 2B according to which, after the chiral rotation (A.11) is performed, the new CP violating part of $\mathcal{L}$ does not create light pseudo-scalars [the $U(1)$ particle included] on the vacuum. Indeed, taking into account the anomaly :

$$\begin{aligned} \mathcal{L}_{\text{QCD}} &= \mathcal{L}_{\text{inv}}(\psi, \bar{\psi}, A_\mu) - \sum_i m_i \bar{\psi}_i \psi_i - \theta \mathcal{Q}(x) \rightarrow \\ &\rightarrow \mathcal{L}_{\text{inv}}(\psi', \bar{\psi}', A_\mu) - \sum_i m_i \cos \phi_i \bar{\psi}'_i \psi'_i + \\ &+ \sum_i m_i \sin \phi_i \bar{\psi}'_i i \gamma_5 \psi'_i - (\theta - \sum \phi_i) \mathcal{Q}(x) \end{aligned} \quad (\text{A.12})$$

However, from Eqs. (2.25) and (A.2) one finds

$$m_i \sin \phi_i = \frac{a}{N} \frac{m_i}{\mu_i^2} (\theta - \sum \phi_i) \quad (\text{A.13})$$

so that

$$\begin{aligned} \mathcal{L}_{\text{QCD}}(\psi_i', \bar{\psi}_i', A_\mu) &= \mathcal{L}_{\text{inv}} - \sum_i m_i \cos \phi_i \bar{\psi}_i' \psi_i' \\ &\quad - (\theta - \sum \phi_i) \left(q(x) - i \frac{m_i}{\mu_i^2} \frac{a}{N} \sum_k \bar{\psi}_k' \gamma_5 \psi_k' \right) \end{aligned} \quad (\text{A.14})$$

Notice that the CP violating interaction in (A.14) is a flavour singlet so that it does not create genuine Goldstone bosons on the vacuum. Furthermore, using Eq. (A.5) and a simple generalization of it for

$\langle v_i' \sum_k \bar{\psi}_k' \gamma_5 \psi_k' \rangle$, one finds easily

$$\begin{aligned} \langle 0 | \delta \mathcal{L}_{\text{CP}}^{(x)} v_j'(0) | 0 \rangle &\simeq \langle 0 | \left\{ q(x) - i \frac{m_i}{\mu_i^2} \frac{a}{N} \sum_k \bar{\psi}_k' \gamma_5 \psi_k' \right\} v_j'(0) | 0 \rangle = \\ &= 0 ; \quad (\text{any } j=1, 2, \dots, L) \end{aligned} \quad (\text{A.15})$$

The decoupling of each of the L pseudoscalar mesons from $\delta \mathcal{L}_{\text{CP}}'$ follows from the absence of poles in the two point function (A.15).

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FIGURE CAPTIONS

Figure 1 Solutions of the energy minimizing conditions (2.25) for $L=2$ and $\mu_i^2 \ll a/N$. Straight lines refer to $m_u = m_d$, solid curve lines to $m_d = 2m_u$ and solid dotted lines to $m_d = 1.1 m_u$.

Figure 2 Minimized energy $E(\theta)$ as a function of θ , for $L=2$ and $\mu_i^2 \ll a/N$:
(a) $m_u \neq m_d$ ($m_d = 1.1 m_u$) ;
(b) $m_u = m_d$.

Figure 3 Energy minimizing solutions for $L=3$, $\theta=\pi$ are found as intersections of straight lines of slope $m_s (m_u m_d)^{-1/2}$ with oscillating curves passing through the origin with slope $(m_u m_d)^{1/2} |m_d - m_u|^{-1}$. For $\theta \neq \pi$ the straight lines are displaced by $\pi - \theta$.

FIG. 1

FIG. 2

FIG. 3